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A Quasi Sujatha Distribution

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Abstract In this paper a quasi Sujatha distribution (QSD), of which Sujatha distribution of Shanker (2016 a) is a particular case, has been proposed. Its moment generating function, moments and other related properties have been studied. Important mathematical and statistical properties including hazard rate and mean residual life functions, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have also been discussed. Method of maximum likelihood and the method of moments have been discussed for estimating its parameters. A numerical example has been presented to test its goodness of fit and the fit is compared with other lifetime distributions.

Keywords Sujatha distribution, Moments, Hazard rate function, Mean residual life function, Stochastic ordering, Stress-strength reliability, Estimation of parameters, goodness of fit

1. Introduction

The Sujatha distribution, introduced by Shanker (2016 a), is defined by its probability density function (p.d.f.)

$$f(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x} \quad ;x>0,\,\theta>0\,. \tag{1.1}$$

Shanker (2016 a) has introduced this distribution for modeling lifetime data from engineering and biomedical science and discussed its applications for several lifetime data. It has been shown that it gives better fit than exponential and Lindley (1958) distributions. This distribution can be easily expressed as a mixture of exponential (θ), a gamma $(2,\theta)$ and a gamma $(3,\theta)$ distributions with their mixing proportions $\frac{\theta^2}{\theta^2+\theta+2}$, $\frac{\theta}{\theta^2+\theta+2}$ and $\frac{2}{\theta^2+\theta+2}$ respectively.

The cumulative distribution function (c.d.f.) of Sujatha distribution (1.1) obtained by Shanker (2016 a) is given by

$$F(x;\theta)=1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x};\;\; x>0,\theta>0 \tag{1.2}$$

The first four moments about origin of Sujatha distribution (1.1) are thus obtained as

$$\mu_1'=\frac{\theta^2+2\theta+6}{\theta\left(\theta^2+\theta+2\right)},\qquad \mu_2'=\frac{2\left(\theta^2+3\theta+12\right)}{\theta^2\left(\theta^2+\theta+2\right)},$$

$$\mu_3'=\frac{6\left(\theta^2+4\theta+20\right)}{\theta^3\left(\theta^2+\theta+2\right)},\qquad \mu_4'=\frac{24\left(\theta^2+5\theta+30\right)}{\theta^4\left(\theta^2+\theta+2\right)}$$

Using the relationship between moments about mean and the moments about origin, the moments about mean of the Sujatha distribution (1.1) are obtained as

$$\mu_2=\frac{\theta^4+4\theta^3+18\theta^2+12\theta+12}{\theta^2\left(\theta^2+\theta+2\right)^2}$$

$$\mu_3=\frac{2\left(\theta^6+6\theta^5+36\theta^4+44\theta^3+54\theta^2+36\theta+24\right)}{\theta^3\left(\theta^2+\theta+2\right)^3}$$

$$\mu_4=\frac{3\begin{pmatrix}3\theta^8+24\theta^7+172\theta^6+376\theta^5+736\theta^4\\+864\theta^3+912\theta^2+480\theta+240\end{pmatrix}}{\theta^4\left(\theta^2+\theta+2\right)^4}$$

Shanker (2016 a) studied some of its important properties including skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, stress-strength reliability. Shanker (2016 b) has also obtained a Poisson mixture of Sujatha distribution named, “Poisson-Sujatha distribution (PSD)” and discussed its various properties, estimation of parameter and applications for counts data. Further, Shanker and Hagos (2016 a, 2015) have obtained the size-biased and zero-truncated version of PSD, discussed their statistical

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properties, estimation of their parameter, and applications for modeling data which structurally excludes zero-counts. Shanker and Hagos (2016 b) have detailed study about applications of PSD for modeling count data. Shanker and Hagos (2016 c) have detailed and critical study on applications of zero-truncated Poisson, Poisson-Lindley and Poisson-Sujatha distributions.

In this paper, a two - parameter quasi Sujatha distribution, of which one parameter Sujatha distribution introduced by Shanker (2016 a) is a particular case, has been proposed. Its raw moments and central moments have been obtained and coefficients of variation, skewness, kurtosis and index of dispersion have been discussed. Some of its important mathematical and statistical properties including hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, stress-strength reliability have also been discussed. The estimation of the parameters has been discussed using both the maximum likelihood estimation and the method of moments. A numerical example has been given to test the goodness of fit of the distribution and the fit has been compared with other well known distributions.

2. A Quasi Sujatha Distribution

A two - parameter quasi Sujatha distribution (QSD) having parameters θ and α is defined by its probability density function (p.d.f.)

$$f(x;\theta,\alpha)=\frac{\theta^2}{\alpha\theta+\theta+2}\left(\alpha+\theta x+\theta x^2\right)e^{-\theta x}\ ;x>0,\ \theta>0\ \alpha>0. \quad (2.1)$$

It can be easily verified that at $\alpha=\theta$, (2.1) reduces to the Sujatha distribution (1.1) and at $\alpha=0$, it reduces to the size-biased Lindley distribution (SBLD) given by its p.d.f.

$$f(x;\theta)=\frac{\theta^3}{\theta+2}x(1+x)e^{-\theta x}\ ;x>0,\theta>0 \quad (2.2)$$

It can be easily verified that QSD (2.1) is a three-component mixture of exponential (θ) gamma $(2,\theta)$ and gamma $(3,\theta)$ distributions. We have

$$f(x;\theta,\alpha)=p_1 f_1(x;\theta)+p_2 f_2(x;\theta,2)+p_3 f_3(x;\theta,3) \quad (2.3)$$

where

$$p_1=\frac{\alpha\theta}{\alpha\theta+\theta+2},\ p_2=\frac{\theta}{\alpha\theta+\theta+2},\text{ and } p_3=\frac{2}{\alpha\theta+\theta+2}$$

$$f_1(x;\theta)=\theta e^{-\theta x};x>0,\theta>0$$

$$f_2(x;\theta,2)=\frac{\theta^2}{\Gamma(2)}e^{-\theta x}x^{2-1};x>0,\theta>0$$

$$f_3(x;\theta,3)=\frac{\theta^3}{\Gamma(3)}e^{-\theta x}x^{3-1};x>0,\theta>0\,.$$

The corresponding cumulative distribution function of QSD (2.1) can be obtained as

$$F(x;\theta,\alpha)=1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\alpha\theta+\theta+2}\right]e^{-\theta x};\ \ x>0,\theta>0,\alpha>0 \quad (2.4)$$

3. Moments and Related Measures

The moment generating function of QSD (2.1) are obtained as

$$M_X(t)=\frac{\theta^2}{\alpha\theta+\theta+2}\int_0^\infty e^{-(\theta-t)x}\left(\alpha+\theta x+\theta x^2\right)dx$$

$$=\frac{\theta^2}{\alpha\theta+\theta+2}\left[\frac{\alpha}{\theta-t}+\frac{\theta}{(\theta-t)^2}+\frac{2\theta}{(\theta-t)^3}\right]$$

$$=\frac{\theta^2}{\alpha\theta+\theta+2}\left[\frac{\alpha}{\theta}\sum_{k=0}^\infty\left(\frac{t}{\theta}\right)^k+\frac{1}{\theta}\sum_{k=0}^\infty\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^2}\sum_{k=0}^\infty\binom{k+2}{k}\left(\frac{t}{\theta}\right)^k\right]$$

$$=\sum_{k=0}^\infty\frac{\alpha\theta+(k+1)\theta+(k+1)(k+2)}{\alpha\theta+\theta+2}\left(\frac{t}{\theta}\right)^k$$

Thus, the r th moment about origin of QSD obtained as the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ is given by

$$\mu_r'=\frac{r!\left[\alpha\theta+(r+1)\theta+(r+1)(r+2)\right]}{\theta^r(\alpha\theta+\theta+2)};r=1,2,3,... \tag{3.1}$$

The first four moments about origin of QSD are given by

$$\mu_1'=\frac{\alpha\theta+2\theta+6}{\theta(\alpha\theta+\theta+2)},\qquad \mu_2'=\frac{2(\alpha\theta+3\theta+12)}{\theta^2(\alpha\theta+\theta+2)},$$

$$\mu_3'=\frac{6(\alpha\theta+4\theta+20)}{\theta^3(\alpha\theta+\theta+2)},\qquad \mu_4'=\frac{24(\alpha\theta+5\theta+30)}{\theta^4(\alpha\theta+\theta+2)}$$

The moments about mean of QSD are thus obtained as

$$\mu_2=\frac{\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12(\theta+1)}{\theta^2(\alpha\theta+\theta+2)^2}$$

$$\mu_3=\frac{2\left[\theta^3\left(\alpha^3+6\alpha^2+6\alpha+2\right)+6\theta^2\left(5\alpha^2+7\alpha+3\right)+36\theta\alpha+12(3\theta+2)\right]}{\theta^3(\alpha\theta+\theta+2)^3}$$

$$\mu_4=\frac{3\begin{bmatrix}\theta^4\left(3\alpha^4+24\alpha^3+44\alpha^2+32\alpha+8\right)+8\theta^3\left(16\alpha^3+43\alpha^2+40\alpha+12\right)\\+24\theta^2\left(17\alpha^2+32\alpha+14\right)+576\theta\alpha+240(2\theta+1)\end{bmatrix}}{\theta^4(\alpha\theta+\theta+2)^4}$$

The coefficient of variation $(C.V)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2) and index of dispersion (γ) of QSD are given by

$$C.V=\frac{\sigma}{\mu_1'}=\frac{\sqrt{\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12(\theta+1)}}{\alpha\theta+2\theta+6}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{2\left[\theta^3\left(\alpha^3+6\alpha^2+6\alpha+2\right)+6\theta^2\left(5\alpha^2+7\alpha+3\right)+36\theta\alpha+12(3\theta+2)\right]}{\left[\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12(\theta+1)\right]^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{3\left[\begin{array}{l}\theta^4\left(3\alpha^4+24\alpha^3+44\alpha^2+32\alpha+8\right)+8\theta^3\left(16\alpha^3+43\alpha^2+40\alpha+12\right)\\+24\theta^2\left(17\alpha^2+32\alpha+14\right)+576\theta\alpha+240\left(2\theta+1\right)\end{array}\right]}{\left[\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12\left(\theta+1\right)\right]^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^2\left(\alpha^2+4\alpha+2\right)+16\theta\alpha+12\left(\theta+1\right)}{\theta\left(\alpha\theta+\theta+2\right)\left(\alpha\theta+2\theta+6\right)}.$$

To study the nature of C.V, $\sqrt{\beta_1}$, β_2 , and γ of QSD, the numerical values of these constants have been computed for varing values of the parameters and presented in tables 3.1, 3.2, 3.3, and 3.4

Table 3.1. CV of QSD for varying values of θ and α

α \ θ	0.5	1	2	3	4	5
0.5	0.662392	0.708329	0.755148	0.777778	0.790569	0.798556
1	0.702377	0.761739	0.816497	0.840635	0.853461	0.861102
2	0.765466	0.83666	0.892143	0.912871	0.922627	0.927872
3	0.812957	0.886072	0.935414	0.95119	0.957685	0.960727
4	0.849837	0.920447	0.96225	0.97361	0.977525	0.978945
5	0.879121	0.945247	0.979796	0.987577	0.989565	0.989835

It is clear from table 3.1 that for a given value of $\alpha\left(\theta\right)$, C.V increases as the value of $\theta\left(\alpha\right)$ increases.

Table 3.2. $\sqrt{\beta_1}$ of QSD for varying values of θ and α

α \ θ	0.5	1	2	3	4	5
0.5	1.145839	1.201582	1.294584	1.353418	1.391402	1.417143
1	1.154381	1.247611	1.377838	1.451434	1.496069	1.525066
2	1.21277	1.365976	1.535588	1.617472	1.662624	1.689961
3	1.285766	1.475249	1.656065	1.733747	1.773301	1.79573
4	1.357727	1.567307	1.745907	1.815305	1.848046	1.865362
5	1.424397	1.643745	1.813728	1.873881	1.900102	1.912879

It is obvious from the table 3.2 that for a given value of $\alpha\left(\theta\right)$, $\sqrt{\beta_1}$ increases as the value of $\theta\left(\alpha\right)$ increases.

Table 3.3. β_2 of QSD for varying values of θ and α

α \ θ	0.5	1	2	3	4	5
0.5	4.944566	5.082378	5.380592	5.602897	5.76	5.872844
1	4.924032	5.170213	5.625	5.931292	6.135306	6.275987
2	5.022933	5.510204	6.21499	6.620247	6.865586	7.023417
3	5.210301	5.903269	6.757653	7.193906	7.438127	7.585918
4	5.431953	6.283795	7.2144	7.644246	7.868405	7.996219
5	5.663815	6.633262	7.591146	7.995389	8.192249	8.297711

Table 3.3 reveals that for a given value of α, β_2 increases as the value of θ increases. For $0<\theta\le 0.5$ and $0<\alpha\le 1$, β_2 decreases. Again for a given value of $\theta\ge 1$, β_2 increases as the value of α increases.

Table 3.4. γ of QSD for varying values of θ and α

α \ θ	0.5	1	2	3	4	5
0.5	2.31348	1.218487	0.627273	0.418803	0.3125	0.248364
1	2.466667	1.305556	0.666667	0.441667	0.327778	0.259524
2	2.678571	1.4	0.696429	0.454545	0.334416	0.263348
3	2.808824	1.439394	0.7	0.452381	0.331197	0.260117
4	2.888889	1.452381	0.694444	0.446078	0.325758	0.255556
5	2.936842	1.451923	0.685714	0.438889	0.320136	0.251067

It is obvious from table 3.4 that for a given value of $\alpha(\theta)$, γ decreases (increases) as the value of $\theta(\alpha)$ increases.

4. Hazard Rate Function and Mean Residual Life Function

Let X be a continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. The hazard rate function (also known as the failure rate function) and the mean residual life function of X are respectively defined as

$$h(x)=\lim_{\Delta x\to 0}\frac{P\left(X<x+\Delta x|X>x\right)}{\Delta x}=\frac{f(x)}{1-F(x)} \tag{4.1}$$

and

$$m(x)=E\left[X-x|X>x\right]=\frac{1}{1-F(x)}\int_x^\infty\left[1-F(t)\right]\,dt \tag{4.2}$$

The corresponding hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of QSD (2.1) are obtained as

$$h(x)=\frac{\theta^2\left(\alpha+\theta x+\theta x^2\right)}{\theta^2x^2+\theta(\theta+2)x++(\alpha\theta+\theta+2)} \tag{4.3}$$

and

$$m(x)=\frac{1}{\left[\theta^2x^2+\theta(\theta+2)x+(\alpha\theta+\theta+2)\right]e^{-\theta x}}\int_x^\infty\left[\theta^2t^2+\theta(\theta+2)t+(\alpha\theta+\theta+2)\right]e^{-\theta t}dt$$

$$=\frac{\theta^2\left(x^2+x\right)+2\theta(2x+1)+(\alpha\theta+6)}{\theta\left[\theta^2x^2+\theta(\theta+2)x+(\alpha\theta+\theta+2)\right]} \tag{4.4}$$

It can be easily verified that $h(0)=\dfrac{\alpha\theta^2}{\alpha\theta+\theta+2}=f(0)$ and $m(0)=\dfrac{\alpha\theta+2\theta+6}{\theta(\alpha\theta+\theta+2)}=\mu_1'$.

5. Stochastic Orderings

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \le_{st} Y\right)$ if $F_X\left(x\right) \ge F_Y\left(x\right)$ for all x

(ii) hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X\left(x\right) \ge h_Y\left(x\right)$ for all x

(iii) mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X\left(x\right) \le m_Y\left(x\right)$ for all x

(iv) likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\dfrac{f_X\left(x\right)}{f_Y\left(x\right)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \tag{5.1}$$

$$\Downarrow$$

$$X \le_{st} Y$$

The QSD is ordered with respect to the strongest 'likelihood ratio ordering' as shown in the following theorem:

Theorem: Let X ~ QSD$\left(\theta_1,\alpha_1\right)$ and Y ~ QSD$\left(\theta_2,\alpha_2\right)$. If $\alpha_1 = \alpha_2$ and $\theta_1 > \theta_2$ or $\theta_1 = \theta_2$ and $\alpha_1 > \alpha_2$, then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^2\left(\alpha_2\,\theta_2+\theta_2+2\right)}{\theta_2^2\left(\alpha_1\,\theta_1+\theta_1+2\right)}\left(\frac{\alpha_1+\theta_1 x+\theta_1 x^2}{\alpha_2+\theta_2 x+\theta_2 x^2}\right)e^{-\left(\theta_1-\theta_2\right)x}\ ;\ x>0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^2\left(\alpha_2\,\theta_2+\theta_2+2\right)}{\theta_2^2\left(\alpha_1\,\theta_1+\theta_1+2\right)}\right]+\log\left(\frac{\alpha_1+\theta_1 x+\theta_1 x^2}{\alpha_2+\theta_2 x+\theta_2 x^2}\right)-\left(\theta_1-\theta_2\right)x.$$

This gives

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{\left(\alpha_2\theta_1-\alpha_1\theta_2\right)+2\left(\alpha_2\theta_1-\alpha_1\theta_2\right)x}{\left(\alpha_1+\theta_1 x+\theta_1 x^2\right)\left(\alpha_2+\theta_2 x+\theta_2 x^2\right)}-\left(\theta_1-\theta_2\right).$$

Thus if $\alpha_1=\alpha_2$ and $\theta_1>\theta_2$ or $\theta_1=\theta_2$ and $\alpha_1>\alpha_2$, $\dfrac{d}{dx}\log\dfrac{f_X(x)}{f_Y(x)}<0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$. This shows flexibility over Sujatha distribution introduced by Shanker (2016 a).

6. Mean Deviations

The amount of scatter in a population is measured to some extent by the totality of deviations usually from mean and median. These are known as the mean deviation about the mean and the mean deviation about the median defined by $\delta_1\left(X\right)=\int_0^\infty \left|x-\mu\right| f\left(x\right)dx$ and $\delta_2\left(X\right)=\int_0^\infty \left|x-M\right| f\left(x\right)dx$, respectively, where $\mu=E\left(X\right)$ and $M=$ Median $\left(X\right)$. The measures $\delta_1\left(X\right)$ and $\delta_2\left(X\right)$ can be calculated using the following simplified relationships

$$\begin{aligned}\delta_1\left(X\right) &= \int_0^\mu \left(\mu-x\right)f\left(x\right)dx+\int_\mu^\infty \left(x-\mu\right)f\left(x\right)dx\\ &= \mu F\left(\mu\right)-\int_0^\mu x\,f\left(x\right)dx-\mu\left[1-F\left(\mu\right)\right]+\int_\mu^\infty x\,f\left(x\right)dx\\ &= 2\mu F\left(\mu\right)-2\mu+2\int_\mu^\infty x\,f\left(x\right)dx\end{aligned}$$

$$= 2\mu F(\mu) - 2\int_0^{\mu} x f(x)\,dx \tag{6.1}$$

and

$$\delta_2(X) = \int_0^{M} (M-x) f(x)\,dx + \int_M^{\infty} (x-M) f(x)\,dx$$

$$= M F(M) - \int_0^{M} x f(x)\,dx - M\left[1 - F(M)\right] + \int_M^{\infty} x f(x)\,dx$$

$$= -\mu + 2\int_M^{\infty} x f(x)\,dx$$

$$= \mu - 2\int_0^{M} x f(x)\,dx \tag{6.2}$$

Using p.d.f. (2.1) and expression for the mean of QSD, we get

$$\int_0^{\mu} x f(x)\,dx = \mu - \frac{\left\{\theta^3\left(\mu^3+\mu^2\right)+\theta^2\left(3\mu^3+2\mu+\alpha\mu\right)+\theta(\alpha+2)+6(\theta\mu+1)\right\}e^{-\theta\mu}}{\theta(\alpha\theta+\theta+2)} \tag{6.3}$$

$$\int_0^{M} x f(x)\,dx = \mu - \frac{\left\{\theta^3\left(M^3+M^2\right)+\theta^2\left(3M^3+2M+\alpha M\right)+\theta(\alpha+2)+6(\theta M+1)\right\}e^{-\theta M}}{\theta(\alpha\theta+\theta+2)} \tag{6.4}$$

Using expressions from (6.1), (6.2), (6.3), and (6.4), the mean deviation about mean, $\delta_1(X)$ and the mean deviation about median, $\delta_2(X)$ of QSD are obtained as

$$\delta_1(X) = \frac{2\left\{\theta^2\left(\mu^2+\mu\right)+\theta(4\mu+\alpha+2)+6\right\}e^{-\theta\mu}}{\theta(\alpha\theta+\theta+2)} \tag{6.5}$$

$$\delta_2(X) = \frac{2\left\{\theta^3\left(M^3+M^2\right)+\theta^2\left(3M^2+2M+\alpha M\right)+\theta(\alpha+2)+6(\theta M+1)\right\}e^{-\theta M}}{\theta(\alpha\theta+\theta+2)} - \mu \tag{6.6}$$

7. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^{q} x f(x)\,dx = \frac{1}{p\mu}\left[\int_0^{\infty} x f(x)\,dx - \int_q^{\infty} x f(x)\,dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^{\infty} x f(x)\,dx\right] \tag{7.1}$$

and

$$L(p) = \frac{1}{\mu}\int_0^{q} x f(x)\,dx = \frac{1}{\mu}\left[\int_0^{\infty} x f(x)\,dx - \int_q^{\infty} x f(x)\,dx\right] = \frac{1}{\mu}\left[\mu - \int_q^{\infty} x f(x)\,dx\right] \tag{7.2}$$

respectively or equivalently

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \tag{7.3}$$

and

$$L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \tag{7.4}$$

respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B=1-\int_0^1 B(p)\,dp \tag{7.5}$$

and

$$G=1-2\int_0^1 L(p)\,dp \tag{7.6}$$

respectively.

Using p.d.f. of QSD (2.1), we get

$$\int_q^\infty x\,f(x)\,dx=\frac{\left\{\theta^3\left(q^3+q^2\right)+\theta^2\left(3q^3+2q+\alpha q\right)+\theta(\alpha+2)+6(\theta q+1)\right\}e^{-\theta q}}{\theta(\alpha\theta+\theta+2)} \tag{7.7}$$

Now using equation (7.7) in (7.1) and (7.2), we get

$$B(p)=\frac{1}{p}\left[1-\frac{\left\{\theta^3\left(q^3+q^2\right)+\theta^2\left(3q^3+2q+\alpha q\right)+\theta(\alpha+2)+6(\theta q+1)\right\}e^{-\theta q}}{\alpha\theta+\theta+2}\right] \tag{7.8}$$

and

$$L(p)=1-\frac{\left\{\theta^3\left(q^3+q^2\right)+\theta^2\left(3q^3+2q+\alpha q\right)+\theta(\alpha+2)+6(\theta q+1)\right\}e^{-\theta q}}{\alpha\theta+\theta+2} \tag{7.9}$$

Now using equations (7.8) and (7.9) in (7.5) and (7.6), the Bonferroni and Gini indices of QSD are thus obtained as

$$B=1-\frac{\left\{\theta^3\left(q^3+q^2\right)+\theta^2\left(3q^3+2q+\alpha q\right)+\theta(\alpha+2)+6(\theta q+1)\right\}e^{-\theta q}}{\alpha\theta+\theta+2} \tag{7.10}$$

$$G=-1+\frac{2\left\{\theta^3\left(q^3+q^2\right)+\theta^2\left(3q^3+2q+\alpha q\right)+\theta(\alpha+2)+6(\theta q+1)\right\}e^{-\theta q}}{\alpha\theta+\theta+2} \tag{7.11}$$

8. Estimation of Parameters

8.1. Maximum Likelihood Estimates (MLE)

Let $(x_1,x_2,x_3,\ldots,x_n)$ be a random sample from QSD (2.1)). The likelihood function, L of (2.1) is given by

$$L=\left(\frac{\theta^2}{\alpha\theta+\theta+2}\right)^n\prod_{i=1}^n\left(\alpha+\theta x_i+\theta x_i^{\,2}\right)e^{-n\theta\bar{x}}$$

and so its natural log likelihood function is thus obtained as

$$\ln L = n\ln\left(\frac{\theta^2}{\alpha\theta+\theta+2}\right)+\sum_{i=1}^{n}\ln\left(\alpha+\theta x_i+\theta x_i^2\right)-n\theta\overline{x} \tag{8.1.1}$$

The maximum likelihood estimates (MLEs) $\hat{\theta}$ and $\hat{\alpha}$ of θ and α are then the solutions of the following non-linear equations

$$\frac{\partial \ln L}{\partial\theta}=\frac{2n}{\theta}-\frac{n(\alpha+1)}{\alpha\theta+\theta+2}+\sum_{i=1}^{n}\frac{x_i+x_i^2}{\alpha+\theta x_i+\theta x_i^2}-n\overline{x}=0$$

$$\frac{\partial \ln L}{\partial\alpha}=-\frac{n\theta}{\alpha\theta+\theta+2}+\sum_{i=1}^{n}\frac{1}{\alpha+\theta x_i+\theta x_i^2}=0$$

where $\overline{x}$ is the sample mean.

These two natural log likelihood equations do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \ln L}{\partial\theta^2}=-\frac{2n}{\theta^2}+\frac{n(\alpha+1)^2}{(\alpha\theta+\theta+2)^2}-\sum_{i=1}^{n}\frac{\left(x_i+x_i^2\right)^2}{\left(\alpha+\theta x_i+\theta x_i^2\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial\alpha^2}=\frac{n\theta^2}{(\alpha\theta+\theta+2)^2}-\sum_{i=1}^{n}\frac{1}{\left(\alpha+\theta x_i+\theta x_i^2\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial\theta\partial\alpha}=-\frac{2n}{(\alpha\theta+\theta+2)^2}-\sum_{i=1}^{n}\frac{\left(x_i+x_i^2\right)}{\left(\alpha+\theta x_i+\theta x_i^2\right)^2}$$

The following equations can be solved for MLEs $\hat{\theta}$ and $\hat{\alpha}$ of θ and α of QSD

$$\begin{bmatrix}\frac{\partial^2 \ln L}{\partial\theta^2} & \frac{\partial^2 \ln L}{\partial\theta\partial\alpha}\\ \frac{\partial^2 \ln L}{\partial\theta\partial\alpha} & \frac{\partial^2 \ln L}{\partial\alpha^2}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0}}\begin{bmatrix}\hat{\theta}-\theta_0\\ \hat{\alpha}-\alpha_0\end{bmatrix}=\begin{bmatrix}\frac{\partial \ln L}{\partial\theta}\\ \frac{\partial \ln L}{\partial\alpha}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α, respectively. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

8.2. Method of Moments Estimates (MOME)

Since the QSD (2.1) has two parameters to be estimated, the first two moments about origin are required to estimate its parameters. Equating the population mean to the sample mean, we have

$$\overline{x}=\frac{\alpha\theta+2\theta+6}{\theta(\alpha\theta+\theta+2)}=\frac{\alpha\theta+\theta+2}{\theta(\alpha\theta+\theta+2)}+\frac{\theta+4}{\theta(\alpha\theta+\theta+2)}$$

$$\overline{x}=\frac{1}{\theta}+\frac{\theta+4}{\theta(\alpha\theta+\theta+2)}$$

$$\alpha\theta+\theta+2=\frac{\theta+4}{\theta\overline{x}-1} \tag{8.2.1}$$

Again replacing the second population moment with the corresponding sample moment, we have

$$m_2' = \frac{2(\alpha\theta+3\theta+12)}{\theta^2(\alpha\theta+\theta+2)} = \frac{2(\alpha\theta+\theta+2)}{\theta^2(\alpha\theta+\theta+2)} + \frac{4(\theta+5)}{\theta^2(\alpha\theta+\theta+2)} = \frac{2}{\theta^2} + \frac{4(\theta+5)}{\theta^2(\alpha\theta+\theta+2)}$$

$$\alpha\theta+\theta+2 = \frac{4(\theta+5)}{\theta^2 m_2' - 2} \qquad (8.2.2)$$

Equations (8.2.1) and (8.2.2) gives the following cubic equation in θ

$$m_2'\theta^3 + 4\left(m_2' - \overline{x}\right)\theta^2 - 2\left(10\overline{x} - 1\right)\theta + 12 = 0 \qquad (8.2.3)$$

Solving equation (8.2.3) using any iterative methods such as Newton-Raphson method, Regula -Falsi method or Bisection method, method of moment estimate (MOME) $\tilde{\theta}$ of θ can be obtained and substituting the value of $\tilde{\theta}$ in equation (8.2.1), MOME $\tilde{\alpha}$ of α can be obtained as

$$\tilde{\alpha} = \frac{-\overline{x}\left(\tilde{\theta}\right)^2 + 2\left(1-\overline{x}\right)\tilde{\theta} + 6}{\tilde{\theta}\left(\tilde{\theta}\overline{x} - 1\right)} \qquad (8.2.4)$$

9. Stress-Strength Reliability

The stress- strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y. When the stress applied to it exceeds the strength, the component fails instantly and the component will function satisfactorily till $X > Y$. Therefore, $R = P(Y < X)$ is a measure of component reliability and in statistical literature it is known as stress-strength parameter. It has wide applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc.

Let X and Y be independent strength and stress random variables having QSD (2.1) with parameter (θ_1, α_1) and (θ_2, α_2) respectively. Then the stress-strength reliability R can be obtained as

$$R = P(Y < X) = \int_0^\infty P(Y < X \mid X = x) f_X(x)\, dx$$

$$= \int_0^\infty f(x;\theta_1,\alpha_1)\, F(x;\theta_2,\alpha_2)\, dx$$

$$= 1 - \frac{\theta_1^2 \begin{bmatrix} \alpha_1(\alpha_2+1)\theta_2^5 + (\theta_1\alpha_2 + 4\theta_1\alpha_1\alpha_2 + 3\theta_1\alpha_1 + \alpha_1\alpha_2 + 6\alpha_1 + 3\theta_1)\theta_2^4 \\ +\theta_1(6\theta_1\alpha_1\alpha_2 + 4\alpha_1\alpha_2 + 3\theta_1\alpha_1 + 3\theta_1\alpha_2 + 7\theta_1 + 2\alpha_2 + 18\alpha_1 + 20)\theta_2^3 \\ +\theta_1\left(6\theta_1\alpha_1\alpha_2 + 3\theta_1^2\alpha_2 + 4\theta_1\theta_2 + 4\theta_1^2\alpha_1\alpha_2 + 20\theta_1\alpha_1 + \alpha_1\theta_1^2 + 5\theta_1^2 + 30\theta_1 + 40\right)\theta_2^2 \\ +\theta_1^2\left(\theta_1^2\alpha_1\alpha_2 + 4\theta_1\alpha_1\alpha_2 + 10\theta_1\alpha_1 + \theta_1^2\alpha_2 + \theta_1^2 + 2\theta_1\alpha_2 + 12\theta_1 + 20\right)\theta_2 \\ +(2\theta_1\alpha_1 + \theta_1\alpha_1\alpha_2 + 2\theta_1 + 4)\theta_1^3 \end{bmatrix}}{(\alpha_1\theta_1+\theta_1+2)(\alpha_2\theta_2+\theta_2+2)(\theta_1+\theta_2)^5}.$$

It can be easily verified that above expression reduces to the corresponding expression for Sujatha distribution (1.1) at $\alpha_1 = \theta_1$ and $\alpha_2 = \theta_2$.

10. Illustrative Example

The following data set represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test, Lawless (2003, pp.204)

1.4 5.1 6.3 10.8 12.1 18.5 19.7 22.2 23.0 30.6 37.3 46.3 53.9 59.8 66.2

For this data set, QSD has been fitted along with one parameter exponential, Lindley and Akash distributions and two - parameter weighted Lindley (WLD), Weibull, Gamma and Lognormal distributions. The probability density function (p.d.f.), and the cumulative distribution function (c.d.f.) of weighted Lindley, Weibull, Gamma, Lognormal, Lindley and exponential distributions are presented in table 10.5. The ML estimates, values of $-2\ln L$ and K-S statistics of the fitted distributions are presented in table 10.6. Recall that the best distribution corresponds to the lower values of $-2\ln L$ and K-S.

Table 10.5. The p.d.f. and the c.d.f. of fitted distributions

Distribution	p.d.f.	c.d.f.
WLD	$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{(\theta+\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(1+x)e^{-\theta x}$	$F(x;\theta,\alpha)=1-\frac{\left[\begin{array}{c}(\theta+\alpha)\Gamma(\alpha,\theta x)\\+(\theta x)^{\alpha}e^{-\theta x}\end{array}\right]}{(\theta+\alpha)\Gamma(\alpha)}$
Weibull	$f(x;\theta,\alpha)=\theta\alpha x^{\alpha-1}e^{-\theta x^{\alpha}}$	$F(x;\theta,\alpha)=1-e^{-\theta x^{\alpha}}$
Gamma	$f(x;\theta,\alpha)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\theta x}$	$F(x;\theta,\alpha)=1-\frac{\Gamma(\alpha,\theta x)}{\Gamma(\alpha)}$
Lognormal	$f(x;\theta,\alpha)=\frac{1}{\sqrt{2\pi}\alpha x}e^{-\frac{1}{2}\left(\frac{\log x-\theta}{\alpha}\right)^2}$	$F(x;\theta,\alpha)=\phi\left(\frac{\log x-\theta}{\alpha}\right)$
Lindley	$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$	$F(x;\theta)=1-\left[\frac{\theta+1+\theta x}{\theta+1}\right]e^{-\theta x}$
Exponential	$f(x;\theta)=\theta e^{-\theta x}$	$F(x;\theta)=1-e^{-\theta x}$

Table 10.6. MLE's, - 2ln Land K-S Statistics of the fitted distributions

Distribution	ML Estimates		$-2\ln L$	K-S Statistics
	$\hat{\theta}$	$\hat{\alpha}$		
QSD	0.084	12.211	128.018	0.081
WLD	0.059	0.704	128.405	0.698
Weibull	0.034	1.306	128.040	0.451
Gamma	0.052	1.442	128.372	0.100
Lognormal	2.931	1.061	131.234	0.161
Sujatha	0.106		132.860	0.177
Lindley	0.070		128.810	0.110
Exponential	0.036		129.480	0.155

11. Concluding Remarks

A two - parameter quasi Sujatha distribution (QSD) which includes one parameter Sujatha distribution as a particular case has been proposed and studied. Its mathematical properties including moments, coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have been discussed. The method of moments and the method of maximum likelihood estimation have also been discussed for estimating its parameters. Finally, a numerical example of real lifetime data set has been presented to test the goodness of fit of the QSD over exponential, Lindley, Sujatha, Lognormal, Gamma, Weibull and WLD.

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