

Estimators of Population Mode Using New Parametric Relationship for Mode

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Abstract In this paper, a new parametric relationship for population mode (M_o) has been established after involving an unknown constant 'k'. The optimum value of 'k' is obtained by minimizing the MSE, upto terms of order n^{-1} , of consistent estimator of mode (\hat{M}_{o1}). Further, for bivariate populations, the ratio-type and product type estimators of M_o have been proposed. At optimum value of 'k', the expressions of biases and MSE's of proposed estimators have been obtained and compared with each other. Also, the conditions for ratio-type and product-type estimators of M_o have been found out for which these estimators are more efficient than \hat{M}_{o1} . The authenticity of new parametric relation of mode has also been verified for some theoretical univariate and bivariate distributions. Empirically, the values of minimum MSE's of proposed estimators have been obtained for some specific populations and also the authenticity of the relationship has been verified for these populations.

Keywords Estimation, Mode, Skewness, Coefficients of skewness, Univariate and bivariate normal distributions, Poisson distribution, Chi-square distribution, Ratio estimator, Product estimator, Bias, Mean squared error

1. Introduction

In sample surveys, researchers generally estimate the population mean as a measure of location. Many estimators like mean per unit estimator, usual ratio estimator, usual product estimator etc are available in literature to estimate population mean under SRS. These estimators are used because mean is generally more stable measure of location as compared to the others. But, in some situations, median and mode are also more important than the mean. Median is often used as the measure of location for skewed distribution because it is insensitive to extreme values. In this direction, Gross (1980) for the first time proposed sample median as a consistent estimator of population median. Further, Kuk and Mak (1989) suggested ratio estimator for population median. Moreover, a number of estimators for population median have been proposed by the researchers in last two decades.

Mode is another measure of location which has been neglected so far in sample surveys. But in case of unimodal distribution, unique mode represents most likely value of the population. Due to this property, mode sometimes becomes more appropriate measure of location as compared to mean and median. It is especially useful in finding the most popular size in studies related to marketing, trade and

industry etc. For example, in the manufacturing of shoes or ready-made garments, one is interested to know which size of shoe or garment is most popular. Also, in the medical field, a medical researcher would normally be more interested in the most probable value of the heart rate than in the total of heart rate measurements divided by the number of measurements.

In literature, very few researchers paid the attention for estimation of population mode, all using different techniques. Initially, Parzan (1962) estimated the mode first estimating the probability density function of the population and then mode. Bickel (2010) estimated the mode first transforming the data to the approximate normal and then estimated the mean which is equals to the mode for the transformed data. Chen et al. (2014) defined the sequence of (k, ρ) -modes where a point is (k, ρ) -mode iff its probability density is higher than all the points within distance k under a distance metric ρ . Recently Sedory and Singh (2014), first estimated the median using technique of Kuk and Mak (1989) then using the empirical relationship for mean, median and mode, defined the estimator of mode.

In this paper, we develop the new simple and efficient technique of estimating the population mode by using well known measures of skewness (Karl Pearson and based upon central moments). Here we first establish the new parametric relationship for population mode involving population parameters and an unknown constant k . By proposing the conventional consistent estimator (\hat{M}_{o1}) of population mode M_o , we find the approximate value of k by minimizing the $MSE(\hat{M}_{o1})$, upto terms of order n^{-1} .

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Further, for the bivariate (Y, X) –population, we have also proposed ratio-type (\hat{M}_{02}) and product-type (\hat{M}_{03}) estimators of M_o . We have checked authenticity of the new parametric relationship for M_o . Upto terms of order n^{-1} , the optimum biases and minimum MSE's of the proposed estimators are obtained and compared with each other. Empirically, we have also checked the authenticity of the parametric relationship by showing that for the given population established new parametric relation is more accurate than the empirical relationship for mean, median and mode, and obtained the values of minimum MSE's of the proposed estimators.

2. Notations and Expectations

Suppose a simple random sample of size n is drawn from a finite population of size N without replacement and observations on both study variables y and auxiliary variable x are taken. Let the values of variable y and x be denoted by Y_i and X_i respectively on the i^{th} unit of the population $i = 1, 2, \dots, N$ and the corresponding small letters y_i and x_i denote the values corresponding to the i^{th} unit in the sample.

Taking,

$$\begin{aligned}\bar{Y} &= \frac{1}{N} \sum_{i=1}^N Y_i, \\ \bar{X} &= \frac{1}{N} \sum_{i=1}^N X_i \\ S_y^2 &= \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2, \\ S_x^2 &= \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})^2 \\ \mu_{rs} &= \frac{1}{N} \sum_{i=1}^N (Y_i - \bar{Y})^r (X_i - \bar{X})^s, \\ \lambda_{rs} &= \frac{\mu_{rs}}{\mu_{20}^{r/2} \mu_{02}^{s/2}} \\ m_{30} &= \frac{n}{(n-1)(n-2)} \sum_{i=1}^n (y_i - \bar{y})^3,\end{aligned}$$

Obviously

$$\begin{aligned}\lambda_{11} &= \rho_{xy} = \rho(\text{Correlation between } x \text{ and } y), \\ \lambda_{30} &= \beta_{1y} \text{ (Coefficient of skewness of } y) \\ \lambda_{40} &= \beta_{2y} \text{ (Coefficient of kurtosis of } y).\end{aligned}$$

Defining,

$$\begin{aligned}\delta_0 &= \frac{\bar{y}}{\bar{Y}} - 1, & \delta &= \frac{s_y^2}{S_y^2} - 1 \\ \epsilon &= \frac{\bar{x}}{\bar{X}} - 1, & \eta_1 &= \frac{m_{30}}{\mu_{30}} - 1\end{aligned}$$

For the sake of simplicity, assume that N is large enough as compares to n so that finite population correction (fpc) terms are ignored throughout.

For the given SRSWOR, we have the following expectations,

$$E(\delta_0) = E(\delta) = E(\epsilon) = E(\eta_1) = 0$$

$$E(\delta_0^2) = \frac{1}{n} C_y^2$$

$$E(\epsilon^2) = \frac{1}{n} C_x^2,$$

$$E(\delta_0 \epsilon) = \frac{1}{n} C_{yx}$$

$$E(\epsilon \delta) = \frac{1}{n} \lambda_{21} C_x$$

$$E(\delta_0 \delta) = \frac{1}{n} \lambda_{30} C_y = \frac{1}{n} \beta_{1y} C_y$$

and up to terms of order n^{-1}

$$E(\delta^2) = \frac{1}{n} (\lambda_{40} - 1) = \frac{1}{n} (\beta_{2y} - 1),$$

$$E(\eta_1^2) = \frac{1}{n} \frac{(\lambda_{60} - 6\lambda_{40} - \lambda_{30}^2 + 9)}{\lambda_{30}^2}$$

$$= \frac{1}{n} \frac{(\lambda_{60} - 6\beta_{2y} - \beta_{1y}^2 + 9)}{\beta_{1y}^2},$$

$$E(\delta_0 \eta_1) = \frac{1}{n} \frac{(\lambda_{40} - 3)}{\lambda_{30}} C_y = \frac{1}{n} \frac{(\beta_{2y} - 3)}{\beta_{1y}} C_y,$$

$$E(\delta \eta_1) = \frac{1}{n} \frac{(\lambda_{50} - 3\lambda_{30})}{\lambda_{30}} = \frac{1}{n} \frac{(\lambda_{50} - 3\beta_{1y})}{\beta_{1y}},$$

$$E(\epsilon \eta_1) = \frac{1}{n} \frac{(\lambda_{31} - 3\rho)}{\lambda_{30}} C_x = \frac{1}{n} \frac{(\lambda_{31} - 3\rho)}{\beta_{1y}} C_x.$$

3. Proposed Estimators and Their Biases and MSE's

Karl Pearson coefficient of skewness and coefficient of skewness based upon moments are respectively given by

$$S_k = \frac{\text{Mean-Mode}}{S.D.} = \frac{\bar{Y} - M_o}{S_y} \quad (3.1)$$

and

$$\beta_1 = \frac{\mu_{30}}{\mu_{20}^{3/2}} = \frac{\mu_{30}}{S_y^3}. \quad (3.2)$$

Without loss of generality, we can take

$$S_k = k\beta_1$$

where k is unknown constant to be determined.

Therefore,

$$\frac{\bar{Y} - M_o}{S_y} = k\beta_1,$$

That implies

$$M_o = \bar{Y} - kS_y\beta_1 = \bar{Y} - k \frac{\mu_{30}}{S_y^2}, \quad (3.3)$$

which is a new parametric functional relationship for population mode for any population.

We here propose three estimators of M_o under the three different situations as

$$\hat{M}_{o1} = \bar{y} - k_1 \frac{m_{30}}{s_y^2}, \quad (3.4)$$

$$\hat{M}_{o2} = \bar{y} \frac{\bar{x}}{\bar{x}} - k_2 \frac{m_{30}}{s_y^2} \frac{\bar{x}}{\bar{x}}, \quad (3.5)$$

$$\text{and } \hat{M}_{o3} = \bar{y} \frac{\bar{x}}{\bar{x}} - k_3 \frac{m_{30}}{s_y^2} \frac{\bar{x}}{\bar{x}}, \quad (3.6)$$

where k_1 , k_2 and k_3 are unknown constants, whose values are determined by minimizing the MSE's of respective estimators \hat{M}_{o1} , \hat{M}_{o2} and \hat{M}_{o3} . Here the estimator \hat{M}_{o1} uses no information on auxiliary variable x which is highly correlated with y , whereas \hat{M}_{o2} and \hat{M}_{o3} uses the known information of \bar{X} , which are of ratio and product type estimators respectively.

To find the biases and MSE's of \hat{M}_{o1} , \hat{M}_{o2} and \hat{M}_{o3} , we expand them in terms of ϵ 's, η 's, δ 's and then taking the expectations as given in section 2, upto terms of order n^{-1} , we get,

$$B(\hat{M}_{o1}) = \frac{k_1}{n} \bar{Y} C_y [\lambda_{50} - \beta_{1y}(\beta_{2y} + 3)],$$

$$B(\hat{M}_{o2}) = \frac{1}{n} \bar{Y} [(C_x^2 - C_{yx}) - k_2 C_y (\beta_{1y}(C_x^2 + \beta_{2y} + 3) + B - \lambda_{50})],$$

$$B(\hat{M}_{o3}) = \frac{1}{n} \bar{Y} [C_{yx} - k_3 C_y (\beta_{1y}(\beta_{2y} + 3) - B - \lambda_{50})],$$

and

$$MSE(\hat{M}_{o1}) = \frac{1}{n} \bar{Y}^2 C_y^2 [1 + k_1^2 (\lambda_{60} - 6\beta_y + \beta_{0y}) - 2k_1 \beta_y],$$

$$MSE(\hat{M}_{o2}) = \frac{1}{n} \bar{Y}^2 [(C_y^2 + C_x^2 - 2C_{yx}) + k_2^2 C_y^2 (\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 + 2B)) - 2k_2 C_y (\beta_y C_y + \beta_{1y}(C_x^2 - C_{yx}) + B)],$$

$$MSE(\hat{M}_{o3}) = \frac{1}{n} \bar{Y}^2 [(C_y^2 + C_x^2 + 2C_{yx}) + k_3^2 C_y^2 (\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 - 2B)) - 2k_3 C_y (\beta_y C_y + \beta_{1y}(C_x^2 + C_{yx}) - B)].$$

The above MSE's of \hat{M}_{o1} , \hat{M}_{o2} and \hat{M}_{o3} , are minimised for

$$k_{1opt} = \frac{\beta_y}{\lambda_{60} - 6\beta_y + \beta_{0y}},$$

$$k_{2opt} = \frac{\beta_y C_y + \beta_{1y}(C_x^2 - C_{yx}) + B}{C_y (\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 + 2B))},$$

$$k_{3opt} = \frac{\beta_y C_y + \beta_{1y}(C_x^2 + C_{yx}) - B}{C_y (\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 - 2B))},$$

and the Biases and minimum MSE's are given as,

$$B_{k_{1opt}}(\hat{M}_{o1}) = \frac{1}{n} \bar{Y} C_y \frac{\beta_y (\lambda_{05} - \beta_{1y}(\beta_{2y} + 3))}{(\lambda_{60} - 6\beta_y + \beta_{0y})},$$

$$B_{k_{2opt}}(\hat{M}_{o2}) = \frac{1}{n} \bar{Y} \left[(C_x^2 - C_{yx}) - \frac{(\beta_{1y}(C_x^2 + \beta_{2y} + 3) + B - \lambda_{50})(\beta_y C_y + \beta_{1y}(C_x^2 - C_{yx}) + B)}{(\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 + 2B))} \right],$$

$$B_{k_{3opt}}(\hat{M}_{o3}) = \frac{1}{n} \bar{Y} \left[C_{yx} - \frac{(\beta_{1y}(\beta_{2y} + 3) - B - \lambda_{50})(\beta_y C_y + \beta_{1y}(C_x^2 + C_{yx}) - B)}{(\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 - 2B))} \right],$$

and

$$MSE_{min}(\hat{M}_{o1}) = \frac{1}{n} \bar{Y}^2 C_y^2 \left[1 - \frac{\beta_y^2}{(\lambda_{60} - 6\beta_y + \beta_{0y})} \right],$$

$$MSE_{min}(\hat{M}_{o2}) = \frac{1}{n} \bar{Y}^2 \left[(C_y^2 + C_x^2 - 2C_{yx}) - \frac{(\beta_y C_y + \beta_{1y}(C_x^2 - C_{yx}) + B)^2}{(\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 + 2B))} \right],$$

$$MSE_{min}(\hat{M}_{o3}) = \frac{1}{n} \bar{Y}^2 \left[(C_y^2 + C_x^2 + 2C_{yx}) - \frac{(\beta_y C_y + \beta_{1y}(C_x^2 + C_{yx}) - B)^2}{(\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 - 2B))} \right].$$

Where,

$$\beta_y = \beta_{2y} - \beta_{1y}^2 - 3,$$

$$\beta_{0y} = \beta_{1y}^2 \beta_{2y} - 2\beta_{1y} \lambda_{50} - 9,$$

$$B = \beta_{1y} \lambda_{21} C_x - \lambda_{31} C_x + 3\rho C_x.$$

Srivastava and Jhaji (1983) have shown that if we replace the parameters involved in the optimum value of the unknown constant by their consistent estimators then upto terms of order n^{-1} , the MSE remains the same. Using this, if we replace the parameters involved in k_{1opt} , k_{2opt} and k_{3opt} by their consistent estimators then their estimators say become \hat{k}_{1opt} , \hat{k}_{2opt} and \hat{k}_{3opt} respectively. Then our estimators reduce to

$$\hat{M}'_{o1} = \bar{y} - \hat{k}_{1opt} \frac{m_{30}}{s_y^2},$$

$$\hat{M}'_{o2} = \bar{y} \frac{\bar{X}}{\bar{x}} - \hat{k}_{2opt} \frac{m_{30}}{s_y^2} \frac{\bar{X}}{\bar{x}},$$

and

$$\hat{M}'_{03} = \bar{y} \frac{\bar{x}}{\bar{X}} - \hat{k}_{3opt} \frac{m_{30}}{s_y^2} \frac{\bar{x}}{\bar{X}}$$

Among the three estimators, the most efficient is \hat{M}_{02} iff

$$\rho \frac{C_y}{C_x} > \frac{1}{2} + \frac{1}{2C_x^2} \left[\frac{\beta_y^2 C_y^2}{(\lambda_{60} - 6\beta_y + \beta_{0y})} - \frac{(\beta_y C_y + \beta_{1y}(C_x^2 + C_{yx}) - B)^2}{(\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 - 2B))} \right] \quad (3.7)$$

\hat{M}_{03} iff

$$\rho \frac{C_y}{C_x} < -\frac{1}{2} - \frac{1}{2C_x^2} \left[\frac{\beta_y^2 C_y^2}{(\lambda_{60} - 6\beta_y + \beta_{0y})} - \frac{(\beta_y C_y + \beta_{1y}(C_x^2 + C_{yx}) - B)^2}{(\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 - 2B))} \right] \quad (3.8)$$

\hat{M}_{01} iff

$$-\frac{1}{2} - \frac{1}{2C_x^2} \left[\frac{\beta_y^2 C_y^2}{(\lambda_{60} - 6\beta_y + \beta_{0y})} - \frac{(\beta_y C_y + \beta_{1y}(C_x^2 + C_{yx}) - B)^2}{(\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 - 2B))} \right] < \rho \frac{C_y}{C_x} < \frac{1}{2} + \frac{1}{2C_x^2} \left[\frac{\beta_y^2 C_y^2}{(\lambda_{60} - 6\beta_y + \beta_{0y})} - \frac{(\beta_y C_y + \beta_{1y}(C_x^2 - C_{yx}) + B)^2}{(\lambda_{60} - 6\beta_y + \beta_{0y} + \beta_{1y}(\beta_{1y} C_x^2 + 2B))} \right] \quad (3.9)$$

Above conditions (3.7), (3.8) and (3.9) represent the optimum situations for \hat{M}_{02} , \hat{M}_{03} and \hat{M}_{01} respectively.

4. Authenticity of Parametric Functional Relationship for M_o

The new parametric functional relationship for M_o is

$$M_o = \bar{Y} - k \frac{\mu_{30}}{s_y^2}, \quad (4.1)$$

where k is an unknown constant whose value is to be determined for the given population.

If k is optimised by minimising the MSE of conventional consistent estimator $\hat{M}_{01} = \bar{y} - k_1 \frac{m_{30}}{s_y^2}$, (up to the order n^{-1}), the above functional relationship reduced to

$$M_{01} = \bar{Y} - k_{1opt} \frac{\mu_{30}}{s_y^2}.$$

As in sampling theory, the nature of the population under study is not known. To see the authenticity of approximate parametric relation M_{01} , we have to consider the population of particular nature. The approximation is due to the fact that we are using the optimum value k_{1opt} of unknown constant k , not the exact value. When the population is univariate, one can check the relationship for M_{01} easily. But, if the population is (Y, X) -bivariate, then it is very difficult check this relationship because for this we need the bivariate moments and marginal distributions of X and Y . No doubt, this relationship can be checked by researchers in inference. So this problem is open for researchers involved in statistical Inference. But in sampling theory, for the known value of \bar{X} we can increase the precision of the estimators \hat{M}_{02} and \hat{M}_{03} . Following particular populations are considered to

check the authenticity of M_{01} :

(i) Population is Normal:

(a) Let $Y \sim N(\mu_y, \sigma_y^2)$, then we know $\beta_{1y} = \mu_{30} = \mu_{50} = 0$, $\lambda_{60} = 15$ and $\beta_{2y} = 3$. Using these values, we get,

$$M_{01} = \mu_y = \bar{Y} = M_o.$$

Because $\beta_y = \beta_{2y} - \beta_{1y}^2 - 3 = 0$ in numerator of k_{1opt} and also,

$$MSE(\hat{M}_{01}) = \frac{1}{n} S_y^2$$

which is variance of \bar{y} when fpc are ignored. So M_{01} is verified for univariate normal population.

(b) Let $(X, Y) \sim N(\mu_y, \mu_x, \sigma_y^2, \sigma_x^2, \rho)$, then we know $\lambda_{60} = 15$, $\lambda_{40} = 3$, $\lambda_{31} = 3\rho$, $\lambda_{22} = 1 + 2\rho^2$, $\lambda_{r,s} = 0$ if $r + s$ is odd. Also, $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$. Using these values, we get,

$$M_{01} = \mu_y = \bar{Y} = M_o.$$

The expressions for MSE's are

$$MSE(\hat{M}_{02}) = MSE(\hat{Y}_R) = \frac{1}{n} \bar{Y}^2 [C_y^2 + C_x^2 - 2C_{yx}]$$

and

$$MSE(\hat{M}_{03}) = MSE(\hat{Y}_P) = \frac{1}{n} \bar{Y}^2 [C_y^2 + C_x^2 + 2C_{yx}]$$

So, M_{01} is verified for bivariate normal population and upto terms of order n^{-1} , we have

$$MSE(\hat{M}_{02}) = MSE(\hat{Y}_R)$$

and

$$MSE(\hat{M}_{03}) = MSE(\hat{Y}_P)$$

So the relation $M_{01} = \bar{Y} - k_{1opt} \frac{\mu_{03}}{s_y^2}$ is true for univariate as well as for bivariate normal populations and in general true for all symmetrical populations.

(ii) Population is Poisson:

Let $Y \sim P(\lambda)$. We then have $\mu'_{10} = \mu_{20} = \mu_{30} = \lambda$, $\mu_{40} = 3\lambda^2 + \lambda$, $\beta_{1y} = \frac{1}{\sqrt{\lambda}}$, so Poisson distribution is always positively skewed, $\beta_{2y} = 3 + \frac{1}{\sqrt{\lambda}}$. Using these values, we get $\beta_y = \beta_{2y} - \beta_{1y}^2 - 3 = 0$, so $k_{1opt} = 0$ and $M_{01} = \lambda = M_o$ if λ is not integer and $MSE(\hat{M}_{01}) = \frac{1}{n} S_y^2$.

Hence this relation is also verified for positively skewed distributions.

(iii) Population is χ^2 :

Let $Y \sim \chi_n^2$, then we have $\mu'_{10} = n$, $\mu_{20} = 2n$, $\mu_{30} = 8n$, $\mu_{40} = 48 + 12n^2$, $\beta_{2y} = \frac{12}{n} + 3$, $\lambda_{50} = \frac{8(5n+12)}{n\sqrt{2n}}$, $\lambda_{60} = 5 \frac{(12n^2 + 253n + 384)}{4n^2}$, $\beta_{1y}^2 = \frac{8}{n}$ (positively skewed).

(iv) Particular cases:

(a) When $n = 16$ then mean $\bar{Y} = 16$, $M_o = 14$, and $M_{01} = 15.9188$.

(b) When $n = 10$ then mean $\bar{Y} = 10$, $M_o = 8$, and

$$M_{o1} = 9.9166.$$

Hence we conclude that the new functional relationship for M_o is always true for symmetric populations whether univariate or bivariate. In general, it is approximately true for all populations, whereas for the skewed distribution, it is approximately true. The approximation is due to the fact that we use the optimum value of k not the exact one.

5. Numerical Illustrations

To illustrate the result numerically, we have made computations for 10 populations taken from literature by

using Microsoft Excel 2010.

The source of the populations, the nature of the variables, the values of \bar{Y} , k_{1opt} , μ_{20} , β_{1y} and ρ_{xy} are listed in Table 1.

In Table 2, we have given the true value of population Mode (M_o), the values of the population mode obtained by using the new parametric functional relationship of M_o and values of population mode M_{oE} obtained by using the empirical relationship ' $Mode = 3 \text{ Median} - 2 \text{ mean}$ ' for different populations.

The efficiencies of proposed estimators are given in Table 3.

Table 1. Description of populations

Sr. No.	Source	y	x	\bar{Y}	k_{1opt}	μ_{20}	β_{1y}	ρ_{xy}
1	Murthy (1967) P.91 (1-35)	Cultivated area (acres)	Holding size (acres)	2.3650	-0.2839	1.5818	0.9119	0.3685
2	Chakravarti et al.(1967) P-207	Weight (kg) of female	Height (cm) of female	28.5313	-0.4080	1.8109	0.1099	0.2306
3	Murthy (1967) P.398	No. of absentees	No. of workers	9.6512	0.0670	42.1341	1.5575	0.6608
4	Murthy (1967) P.399	Area under wheat in 1964	Cultivated area in 1961	199.441	-0.1442	21900.8936	1.1295	0.9043
5	Chakravarti et al.(1967) P-183	Length (cm) measured by 1 st person	Length (cm) measured by 2 nd person	4.9737	-.02875	0.1346	-0.0546	0.9317
6	Chakravarti et al.(1967) P-207	Weight (kg) of male	Height (cm) of male	29.2625	-0.0841	6.5836	0.3670	0.7709
7	Chakravarti et al.(1967) P-185 (1-35)	Weight (lb) of Kayastha males	Stature (cm) of Kayastha males	82.2000	-0.4975	191.7029	0.0439	0.8578
8	Chakravarti et al.(1967) P-185 (1-76)	Weight (lb) of Kayastha males	Stature (cm) of Kayastha males	89.4211	0.0503	278.4806	0.6068	0.4361
9	Maddala p-316	Consumption per capital of Lamb	Deflated prices of Lamb	4.5188	-0.1696	0.2103	-0.6578	-0.7517
10	http://content.hccfl.edu	highway fuel efficiency of vehicles (in miles)	weight of vehicles (in 1000 lbs.)	30.6154	-0.5950	15.6213	0.0549	-0.8978

Table 2. Values of population mode (M_o), values obtained from new parametric relationship M_{o1} and empirical relationship M_{oE}

Pop. No.	M_o	M_{o1}	M_{oE}
1	2.7000	2.6906	1.1950
2	27.8000	28.5916	28.4375
3	9.0000	8.9739	4.6977
4	278.0000	223.5494	28.6176
5	4.8500	4.9679	4.9026
6	30.6000	29.3416	29.0750
7	70.0000	82.5022	81.6000
8	88.0000	88.9114	85.1579
9	5.1000	4.4676	4.7625
10	30.0000	30.7444	28.7692

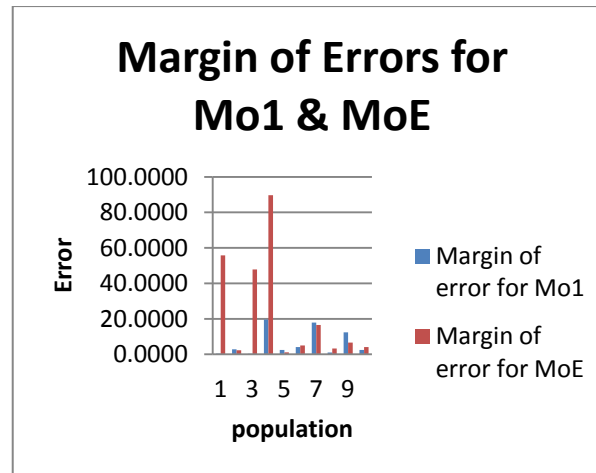
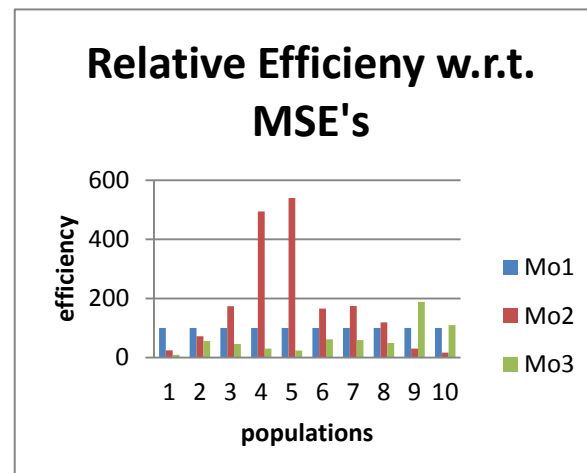


Table 3. MSE's of \bar{y} , \hat{M}_{o1} , \hat{M}_{o2} , \hat{M}_{o3} , \hat{Y}_R and \hat{Y}_P up to terms of n^{-1}

Pop. No.	\bar{y}	$1/n \times \text{MSE of}$				
		\hat{M}_{o1}	\hat{M}_{o2}	\hat{M}_{o3}	\hat{Y}_R	\hat{Y}_P
1	1.5818	1.2908	5.2902	14.1623	-	-
2	1.8109	1.2534	1.7332	2.2410	-	-
3	42.1341	38.9929	22.4016	84.6724	23.745	-
4	21900.9	20163.19	4081.67	66594.1	4286.4	-
5	0.1346	0.1085	0.0201	0.4480	0.0201	-
6	6.5836	6.4593	3.8957	10.4341	3.9590	-
7	191.702	102.2873	58.5715	172.635	105.52	-
8	278.480	263.7385	220.986	532.868	237.22	-
9	0.2103	0.1926	0.6273	0.1022	-	0.102
10	15.6213	6.7198	39.3146	6.1189	-	6.764



From Table 2, we observe that the values obtained from the new parametric functional relationship for mode i.e. $M_{o1} = \bar{Y} - k_{1opt} \frac{\mu_{o3}}{s_y^2}$, are close to the true value of the population mode (M_o), as compared to the values of population mode obtained by using the empirical relationship for it. The small amount of errors ($M_o - M_{o1}$)

is due to the fact that we are using the optimum value of k not the exact value. So, empirically, this relation is verified.

From Table 3, at the end, we observe that in the sampling theory, for the bivariate population, the known value of \bar{x} , increases the accuracy of the estimators of population mode (M_o). We note that the MSE's of all \hat{M}_{o1} , \hat{M}_{o2} and \hat{M}_{o3} are less than $V(\bar{y})$, at the same time MSE's of \hat{M}_{o2} and \hat{M}_{o3} are

less than $V(\hat{V}_R)$ and $V(\hat{V}_P)$ respectively. When the conditions (3.7) and (3.8) are satisfied then \hat{M}_{o2} and \hat{M}_{o3} are more efficient than \hat{M}_{o1} respectively, otherwise \hat{M}_{o1} is more efficient than the others.

6. Conclusions

The empirical study shows that the new parametric relationship for population mode is more beneficial for estimating population mode than the empirical relationship between mean, median and mode. Also, the values of MSEs of the proposed estimators for mode are less than the values of MSEs of mean per unit estimator of population mean. It means that the proposed estimators of population mode are more stable than the mean per unit estimator of population mean. Further, the conditions have been found out where ratio-type and product-type estimators of population mode are more efficient than \hat{M}_{o1} . Also, the new parametric relationship for population mode provides a new way to estimate population mode for further study.

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