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Estimators for Finite Population Variance Using Mean and Variance of Auxiliary Variable

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Abstract For estimating finite population variance using information on single auxiliary variable in the form of mean and variance both, the Ratio-Product-Difference (RPD) type estimators are proposed. The generalized cases of these estimators leading to the classes of estimators are also proposed. The bias and mean square error (MSE) of the proposed estimators are found. Theoretical comparisons with the traditional estimator are supported by a numerical example. By this comparison it is shown that the proposed estimators are more efficient than the traditional one.

Keywords Auxiliary Variable, Taylor's Series Expansion, Bias, Mean Square Error (MSE) and Efficiency

1. Introduction

In sampling theory, auxiliary information is used widely at both the stages of selection and estimation. At the estimation stage, auxiliary information is used by formulating various types of estimators of different population parameters with a view of getting increased efficiency and are available in plenty in the literature.

Let U = (1, 2, . . . , N) be a finite population of N units with Y being the study variable taking the value Y_i for the unit *i* of U and X being the auxiliary variable taking the value X_i for the unit i of the population, i = 1, 2, . . . , N.

Let $\bar{Y}\left(=\frac{1}{N}\sum_{i=1}^{N}Y_i\right)$ be the population of mean of Y and $\bar{X}\left(=\frac{1}{N}\sum_{i=1}^{N}X_i\right)$ be the population mean of X. Also, let $S_Y^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^2$, $S_X^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(X_i-\bar{X}\right)^2$ and $\mu_{rs}=\frac{1}{N}\sum_{i=1}^{N}\left(Y_i-\bar{Y}\right)^r\left(X_i-\bar{X}\right)^s$

For a simple random sample of size n drawn from U with the sample observations $y_1, y_2, \ldots, y_n$ on y and $x_1, x_2, \ldots, x_n$ on x, let $\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$ and $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ be the sample means of y- values and x- values respectively.

2. The Suggested Estimators

We know that the finite population variance $\sigma_Y{}^2$ of the study variable is

$$\sigma_Y{}^2=\frac{1}{N}\sum_{i=1}^{N}Y_i{}^2-\bar{Y}^2=\theta-\bar{Y}^2, \text{ where } \theta=\frac{1}{N}\sum_{i=1}^{N}Y_i{}^2 \quad (2.1)$$

From (2.1) it is natural to get an estimator of $\sigma_Y{}^2$ if we replace $\theta=\frac{1}{N}\sum_{i=1}^{N}Y_i{}^2$ and $\bar{Y}^2$ by their some estimators. In particular if $\theta=\frac{1}{N}\sum_{i=1}^{N}Y_i{}^2$ is estimated by $\hat{\theta}$ and $\bar{Y}^2$ is estimated by $\bar{y}^2\left\{1+\frac{k_1\left(\bar{x}-\bar{X}\right)}{\bar{X}}\right\}\left\{1+\frac{k_2\left(s_x{}^2-S_x{}^2\right)}{S_x{}^2}\right\}$, we get the estimator of $\sigma_Y{}^2$ as follows

$$d_1=\hat{\theta}-\bar{y}^2\left\{1+\frac{k_1\left(\bar{x}-\bar{X}\right)}{\bar{X}}\right\}\left\{1+\frac{k_2\left(s_x{}^2-S_x{}^2\right)}{S_x{}^2}\right\} \quad (2.2)$$

and its generalized estimator as

$$d_{1g}=\hat{\theta}-\bar{y}^2 f(u,v) \quad (2.3)$$

where $u=\frac{\bar{x}}{\bar{X}}$ and $v=\frac{s_x{}^2}{S_x{}^2}$ and f (*u* , *v*) satisfying the validity conditions of Taylor's series expansion is a bounded function of (u , v) such that f(1, 1) = 1.

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Also if $\theta = \frac{1}{N}\sum_{i=1}^{N} Y_i^2$ is estimated by $\hat{\theta}\left\{1+\frac{k_3(\bar{x}-\bar{X})}{\bar{X}}\right\}$ and $\bar{Y}^2$ is estimated by $\bar{y}^2\left\{1+\frac{k_4\left(s_x^{\ 2}-S_x^{\ 2}\right)}{S_x^{\ 2}}\right\}$, we get another estimator of $\sigma_Y^{\ 2}$ as follows

$$d_2 = \hat{\theta}\left\{1+\frac{k_3(\bar{x}-\bar{X})}{\bar{X}}\right\} - \bar{y}^2\left\{1+\frac{k_4\left(s_x^{\ 2}-S_x^{\ 2}\right)}{S_x^{\ 2}}\right\} \quad (2.4)$$

and its generalized estimator as

$$d_{2g} = \hat{\theta} f(u) - \bar{y}^2 g(v) \quad (2.5)$$

where $u = \frac{\bar{x}}{\bar{X}}$ and $v = \frac{s_x^{\ 2}}{S_x^{\ 2}}$, f($u$) and g ($v$) both are bounded functions in u and v respectively such that $f(1)$ = 1 at the point u = 1 and $g(1)$ = 1 at the point v = 1 and both are satisfying the regularity conditions for the validity of Taylor's series expansion having first two derivatives with respect to u and v respectively to be bounded.

3. Bias and Mean Squared Error of Suggested Estimators

(a) Bias and Mean Square Error of Suggested Estimator d_1

Let $\bar{y} = \bar{Y} + e_0$, $\bar{x} = \bar{X} + e_1$, $s_x^{\ 2} = \mu_{02} + e_2$ and $\hat{\theta} = \theta + e_3$,

For simplicity, it is assumed that the population size N is large enough as compared to the sample size n so that finite population correction terms may be ignored. Now

$$E(e_0) = E(e_1) = E(e_2) = E(e_3) = 0 \quad (3.1)$$

$$\left.\begin{aligned}
E(e_0^{\ 2}) &= \frac{1}{n}S_Y^2 = \frac{\mu_{20}}{n}\\
E(e_1^{\ 2}) &= \frac{1}{n}S_X^2 = \frac{\mu_{02}}{n}\\
E(e_0e_1) &= \frac{1}{n}\rho S_Y S_X = \frac{\mu_{11}}{n}\\
E(e_2^{\ 2}) &= \frac{\mu_{02}^{\ 2}}{n}(\beta_2 - 1)\\
E(e_0e_2) &= \frac{\mu_{12}}{n}\\
E(e_1e_2) &= \frac{\mu_{03}}{n}
\end{aligned}\right\} \quad (3.2)$$

and

$$\left.\begin{aligned}
E(e_3^{\ 2}) &= \frac{1}{n}(\mu_{40} + 4\bar{Y}\mu_{30} + 4\bar{Y}^2\mu_{20} - \mu_{20}^{\ 2})\\
E(e_0e_3) &= \frac{1}{n}(\mu_{30} + 2\bar{Y}\mu_{20})\\
E(e_1e_3) &= \frac{1}{n}(\mu_{21} + 2\bar{Y}\mu_{11})\\
E(e_2e_3) &= \frac{1}{n}(\mu_{22} + 2\bar{Y}\mu_{12} - \mu_{02}\mu_{20})
\end{aligned}\right\} \quad (3.3)$$

Let us consider the proposed estimator d_1 defined in (2.2)

$$d_1 = \hat{\theta} - \bar{y}^2\left\{1+\frac{k_1(\bar{x}-\bar{X})}{\bar{X}}\right\}\left\{1+\frac{k_2\left(s_x^{\ 2}-S_x^{\ 2}\right)}{S_x^{\ 2}}\right\}$$

$$= (\theta + e_3) - (\bar{Y} + e_0)^2\left(1+\frac{k_1e_1}{\bar{X}}\right)\left(1+\frac{k_2e_2}{\mu_{02}}\right)$$

$$= (\theta + e_3) - (\bar{Y}^2 + e_0^{\ 2} + 2\bar{Y}e_0)\left(1+\frac{k_2e_2}{\mu_{02}}+\frac{k_1e_1}{\bar{X}}+\frac{k_1k_2e_1e_2}{\bar{X}\mu_{02}}\right)$$

$$= (\theta + e_3) - \bar{Y}^2 - \frac{k_2\bar{Y}^2e_2}{\mu_{02}} - \frac{k_1\bar{Y}^2e_1}{\bar{X}} - \frac{k_1k_2\bar{Y}^2e_1e_2}{\bar{X}\mu_{02}} - e_0^{\ 2} - 2\bar{Y}e_0 - \frac{2k_2\bar{Y}e_0e_2}{\mu_{02}} - \frac{2k_1\bar{Y}e_0e_1}{\bar{X}}$$

$$= \left(\frac{1}{N}\sum_{i=1}^{N} Y_i^2 - \bar{Y}^2\right) + e_3 - 2\bar{Y}e_0 - \frac{k_1\bar{Y}^2e_1}{\bar{X}} - \frac{k_2\bar{Y}^2e_2}{\mu_{02}} - e_0^{\ 2} - \frac{2k_1\bar{Y}e_0e_1}{\bar{X}} - \frac{2k_2\bar{Y}e_0e_2}{\mu_{02}} - \frac{k_1k_2\bar{Y}^2e_1e_2}{\bar{X}\mu_{02}}$$

or

$$(d_1-\sigma_Y^2)=e_3-2\bar{Y}e_0-\frac{k_1\bar{Y}^2e_1}{\bar{X}}-\frac{k_2\bar{Y}^2e_2}{\mu_{02}}-e_0{}^2-\frac{2k_1\bar{Y}e_0e_1}{\bar{X}}-\frac{2k_2\bar{Y}e_0e_2}{\mu_{02}}-\frac{k_1k_2\bar{Y}^2e_1e_2}{\bar{X}\mu_{02}} \tag{3.4}$$

Taking expectation on both sides of (3.4), the bias in $d_1\left(=E(d_1)-\sigma_Y^2\right)$ to the order $\left(\frac{1}{n}\right)$ is given by

$$\text{Bias}(d_1)=E(d_1)-\sigma_Y^2=E(e_3)-2\bar{Y}E(e_0)-\frac{k_1\bar{Y}^2}{\bar{X}}E(e_1)-\frac{k_2\bar{Y}^2}{\mu_{02}}E(e_2)-E(e_0{}^2)$$
$$-\frac{2k_1\bar{Y}}{\bar{X}}E(e_0e_1)-\frac{2k_2\bar{Y}}{\mu_{02}}E(e_0e_2)-\frac{k_1k_2\bar{Y}^2}{\bar{X}\mu_{02}}E(e_1e_2)$$

Using values of the expectations given from (3.1) to (3.3), we have

$$\text{Bias}(d_1)=-\frac{\mu_{20}}{n}-\frac{2k_1\bar{Y}}{\bar{X}}\frac{\mu_{11}}{n}-\frac{2k_2\bar{Y}}{\mu_{02}}\frac{\mu_{12}}{n}-\frac{k_1k_2\bar{Y}^2}{\bar{X}\mu_{02}}\frac{\mu_{03}}{n} \tag{3.5}$$

Now squaring (3.4) on both sides and then taking expectation, the mean square error of $d_1(=E(d_1)-\sigma_Y^2)^2$ to the first degree of approximation is given by

$$E(d_1-\sigma_Y{}^2)^2=E\left(e_3-2\bar{Y}e_0-\frac{k_1\bar{Y}^2e_1}{\bar{X}}-\frac{k_2\bar{Y}^2e_2}{\mu_{02}}-e_0{}^2-\frac{2k_1\bar{Y}e_0e_1}{\bar{X}}-\frac{2k_2\bar{Y}e_0e_2}{\mu_{02}}-\frac{k_1k_2\bar{Y}^2e_1e_2}{\bar{X}\mu_{02}}\right)^2$$
$$=E(e_3{}^2)+4\bar{Y}^2E(e_0{}^2)+\frac{k_1{}^2\bar{Y}^4}{\bar{X}^2}E(e_1{}^2)+\frac{k_2{}^2\bar{Y}^4}{\mu_{02}{}^2}E(e_2{}^2)-4\bar{Y}E(e_0e_3)-\frac{2k_1\bar{Y}^2}{\bar{X}}E(e_1e_3)$$
$$-\frac{2k_2\bar{Y}^2}{\mu_{02}}E(e_2e_3)+\frac{4k_1\bar{Y}^3}{\bar{X}}E(e_0e_1)+\frac{4k_2\bar{Y}^3}{\mu_{02}}E(e_0e_2)+\frac{2k_1k_2\bar{Y}^4}{\bar{X}\mu_{02}}E(e_1e_2)$$
$$=\frac{1}{n}(\mu_{40}+4\bar{Y}\mu_{30}+4\bar{Y}^2\mu_{20}-\mu_{20}{}^2)+4\bar{Y}^2\frac{\mu_{20}}{n}-4\bar{Y}\frac{1}{n}(\mu_{30}+2\bar{Y}\mu_{20})+\frac{k_1{}^2\bar{Y}^4}{\bar{X}^2}\frac{\mu_{02}}{n}$$
$$+\frac{k_2{}^2\bar{Y}^4}{\mu_{02}{}^2}\frac{\mu_{02}{}^2}{n}(\beta_2-1)-\frac{2k_1\bar{Y}^2}{\bar{X}}\frac{1}{n}\left(\mu_{21}+2\bar{Y}\mu_{11}\right)-\frac{2k_2\bar{Y}^2}{\mu_{02}}\frac{1}{n}(\mu_{22}+2\bar{Y}\mu_{12}-\mu_{02}\mu_{20})$$
$$+\frac{4k_1\bar{Y}^3}{\bar{X}}\frac{\mu_{11}}{n}+\frac{4k_2\bar{Y}^3}{\mu_{02}}\frac{\mu_{12}}{n}+\frac{2k_1k_2\bar{Y}^4}{\bar{X}\mu_{02}}\frac{\mu_{03}}{n}$$

or

$$\text{MSE}(d_1)=\frac{1}{n}(\mu_{40}-\mu_{20}{}^2)+\frac{k_1{}^2\bar{Y}^4}{\bar{X}^2}\frac{\mu_{02}}{n}+\frac{k_2{}^2\bar{Y}^4}{n}(\beta_2-1)-\frac{2k_1\bar{Y}^2}{\bar{X}}\frac{\mu_{21}}{n}-\frac{2k_2\bar{Y}^2}{\mu_{02}}\frac{\mu_{22}}{n}$$
$$+2\bar{Y}^2k_2\frac{\mu_{20}}{n}+\frac{2k_1k_2\bar{Y}^4}{\bar{X}\mu_{02}}\frac{\mu_{03}}{n} \tag{3.6}$$

For minimizing (3.6) in two unknowns k_1 and k_2, the two normal equations after differentiating (3.6) partially with respect to k_1 and k_2 are

$$\frac{2\bar{Y}^4\mu_{02}}{n\bar{X}^2}k_1-\frac{2\bar{Y}^2\mu_{21}}{n\bar{X}}+\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}k_2=0 \tag{3.7}$$

$$\frac{2\bar{Y}^4(\beta_2-1)}{n}k_2+\left(\frac{2\bar{Y}^2\mu_{20}}{n}-\frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right)+\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}k_1=0 \tag{3.8}$$

Solving (3.7) and (3.8) for k_1 and k_2, we get the minimizing optimum values to be

$$k_1{*} = -\frac{\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right)\left(\frac{-2\bar{Y}^2\mu_{21}}{n\bar{X}}\right)-\left(\frac{2\bar{Y}^2\mu_{20}}{n}-\frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right)\left(\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}\right)}{\left(\frac{2\bar{Y}^2\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right)-\left(\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}\right)^2} \tag{3.9}$$

$$k_2{*} = -\frac{\left(\frac{2\bar{Y}^4\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^2\mu_{20}}{n}-\frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right)-\left(\frac{-2\bar{Y}^2\mu_{21}}{n\bar{X}}\right)\left(\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}\right)}{\left(\frac{2\bar{Y}^4\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right)-\left(\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}\right)^2} \tag{3.10}$$

which when substituted in (3.6) gives the minimum value of mean square error of the estimator d_1 as

$$\text{MSE}(d_1)\text{min} = \frac{1}{n}\left(\mu_{40}-\mu_{20}{}^2\right)-\frac{\left\{\frac{2\mu_{21}\mu_{03}}{n\mu_{02}}\left(\mu_{20}-\frac{\mu_{22}}{\mu_{02}}\right)-\frac{(\beta_2-1)\mu_{21}{}^2}{n}-\frac{\mu_{02}}{n}\left(\mu_{20}-\frac{\mu_{22}}{\mu_{02}}\right)^2\right\}}{\left\{(\beta_2-1)\mu_{02}-\frac{\mu_{03}{}^2}{\mu_{02}{}^2}\right\}} \tag{3.11}$$

(b) Bias and Mean Square Error of Suggested Estimator d_{1g}

For f_1 and f_2 being the first order partial derivatives of $f(u,v)$ with respect to u and v respectively at the point (1,1), that is

$$\left(\frac{\partial}{\partial u}f(u,v)\right)_{(1,1)} = f_1 \text{ and } \left(\frac{\partial}{\partial v}f(u,v)\right)_{(1,1)} = f_2,$$

expanding $f(u,v)$ in (2.3) in third order Taylor's series about point (1, 1), we have

$$d_{1g} = \hat{\theta}-\bar{y}^2\Big[f(1,1)+(u-1)f_1+(v-1)f_2+\frac{1}{2!}\{(u-1)^2 f_{11}+(v-1)^2 f_{22}+2(u-1)(v-1)f_{12}\}$$
$$+\frac{1}{3!}\left\{(u-1)\frac{\partial}{\partial u}+(v-1)\frac{\partial}{\partial v}\right\}^3 f(u{*},v{*})\Big]$$

where f_1 and f_2 are already defined; f_{11}, f_{22} and f_{12} are the second order partial derivatives given by

$$f_{11} = \left(\frac{\partial^2}{\partial u^2}f(u,v)\right)_{(1,1)},\ f_{22} = \left(\frac{\partial^2}{\partial v^2}f(u,v)\right)_{(1,1)},\ f_{12} = \left(\frac{\partial^2}{\partial u\partial v}f(u,v)\right)_{(1,1)}$$

and

$$u{*} = 1 + h(u-1),\ v{*} = 1 + h(v-1),\ 0 < h < 1.$$

or

$$d_{1g} = (\theta+e_3)-(\bar{Y}+e_0)^2\Big[1+\frac{e_1}{\bar{X}}f_1+\frac{e_2}{\mu_{02}}f_2+\frac{1}{2!}\left\{\frac{e_1{}^2}{\bar{X}^2}f_{11}+\frac{e_2{}^2}{\mu_{02}{}^2}f_{22}+2\frac{e_1e_2}{\bar{X}\mu_{02}}f_{12}\right\}$$
$$+\frac{1}{3!}\left\{\frac{e_1}{\bar{X}}\frac{\partial}{\partial u}+\frac{e_2}{\mu_{02}}\frac{\partial}{\partial v}\right\}^3 f(u{*},v{*})\Big]$$

$$= (\theta+e_3)-\bar{Y}^2-\frac{\bar{Y}^2e_1}{\bar{X}}f_1-\frac{\bar{Y}^2e_2}{\mu_{02}}f_2-\frac{\bar{Y}^2}{2}\left(\frac{e_1{}^2}{\bar{X}^2}f_{11}+\frac{e_2{}^2}{\mu_{02}{}^2}f_{22}+2\frac{e_1e_2}{\bar{X}\mu_{02}}f_{12}\right)$$
$$-e_0{}^2-2\bar{Y}e_0-\frac{2\bar{Y}e_0e_1}{\bar{X}}f_1-\frac{2\bar{Y}e_0e_2}{\mu_{02}}f_2$$

$$=\left(\frac{1}{N}\sum_{i=1}^{N}Y_i^2-\bar{Y}^2\right)+e_3-2\bar{Y}e_0-\frac{\bar{Y}^2e_1}{\bar{X}}f_1-\frac{\bar{Y}^2e_2}{\mu_{02}}f_2-e_0^2-\frac{\bar{Y}^2e_1^2}{2\bar{X}^2}f_{11}-\frac{\bar{Y}^2e_2^2}{2\mu_{02}}f_{22}$$

$$-\frac{2\bar{Y}e_0e_1}{\bar{X}}f_1-\frac{2\bar{Y}e_0e_2}{\mu_{02}}f_2-\frac{\bar{Y}^2e_1e_2}{\bar{X}\mu_{02}}f_{12}$$

$$d_{1g}-\sigma_Y^2=e_3-2\bar{Y}e_0-\frac{\bar{Y}^2e_1}{\bar{X}}f_1-\frac{\bar{Y}^2e_2}{\mu_{02}}f_2-e_0^2-\frac{\bar{Y}^2e_1^2}{2\bar{X}^2}f_{11}-\frac{\bar{Y}^2e_2^2}{2\mu_{02}^2}f_{22}$$

$$-\frac{2\bar{Y}e_0e_1}{\bar{X}}f_1-\frac{2\bar{Y}e_0e_2}{\mu_{02}}f_2-\frac{\bar{Y}^2e_1e_2}{\bar{X}\mu_{02}}f_{12} \tag{3.12}$$

Taking expectation on both sides of (3.12) and using values of the expectations given from (3.1) to (3.3), the bias in d_{1g} $\left(=E(d_{1g})-\sigma_Y^2\right)$ to the order $\left(\frac{1}{n}\right)$ is given by

Bias (d_{1g}) $=E(d_{1g})-\sigma_Y^2$

$$=-\frac{\mu_{20}}{n}-\frac{\bar{Y}^2}{2\bar{X}^2}f_{11}\frac{\mu_{02}}{n}-\frac{\bar{Y}^2}{2\mu_{02}^2}f_{22}\frac{\mu_{02}^2}{n}(\beta_2-1)-\frac{2\bar{Y}}{\bar{X}}f_1\frac{\mu_{11}}{n}-\frac{2\bar{Y}}{\mu_{02}}f_2\frac{\mu_{12}}{n}-\frac{\bar{Y}^2}{\bar{X}\mu_{02}}f_{12}\frac{\mu_{03}}{n}$$

$$\text{Bias}(d_{1g})=-\frac{\mu_{20}}{n}-\frac{2\bar{Y}}{\bar{X}}\frac{\mu_{11}}{n}f_1-\frac{2\bar{Y}}{\mu_{02}}\frac{\mu_{12}}{n}f_2-\frac{\bar{Y}^2}{2\bar{X}^2}\frac{\mu_{02}}{n}f_{11}-\frac{\bar{Y}^2}{2n}f_{22}(\beta_2-1)-\frac{\bar{Y}^2}{\bar{X}\mu_{02}}\frac{\mu_{03}}{n}f_{12} \tag{3.13}$$

Now squaring (3.12) on both sides and then taking expectation, the mean square error of d_{1g} to the first degree of approximation is given by

$$E(d_{1g}-\sigma_Y^2)^2=E\left(e_3-2\bar{Y}e_0-\frac{\bar{Y}^2e_1}{\bar{X}}f_1-\frac{\bar{Y}^2e_2}{\mu_{02}}f_2\right)^2$$

$$=E(e_3^2)+4\bar{Y}^2E(e_0^2)+\frac{\bar{Y}^4}{\bar{X}^2}f_1^2E(e_1^2)+\frac{\bar{Y}^4}{\mu_{02}^2}f_2^2E(e_2^2)-4\bar{Y}E(e_0e_3)-\frac{2\bar{Y}^2}{\bar{X}}f_1E(e_1e_3)$$

$$-\frac{2\bar{Y}^2}{\mu_{02}}f_2E(e_2e_3)+\frac{4\bar{Y}^3}{\bar{X}}f_1E(e_0e_1)+\frac{4\bar{Y}^3}{\mu_{02}}f_2E(e_0e_2)+\frac{2\bar{Y}^4}{\bar{X}\mu_{02}}f_1f_2E(e_1e_2)$$

$$=\frac{1}{n}\left(\mu_{40}+4\bar{Y}\mu_{30}+4\bar{Y}^2\mu_{20}-\mu_{20}^2\right)+4\bar{Y}^2\frac{\mu_{20}}{n}-4\bar{Y}\frac{1}{n}\left(\mu_{30}+2\bar{Y}\mu_{20}\right)+\frac{\bar{Y}^4}{\bar{X}^2}f_1^2\frac{\mu_{02}}{n}$$

$$+\frac{\bar{Y}^4}{\mu_{02}^2}f_2^2\frac{\mu_{02}^2(\beta_2-1)}{n}-\frac{2\bar{Y}^2}{\bar{X}}f_1\frac{1}{n}(\mu_{21}+2\bar{Y}\mu_{11})-\frac{2\bar{Y}^2}{\mu_{02}}f_2\frac{1}{n}\left(\mu_{22}+2\bar{Y}\mu_{12}-\mu_{02}\mu_{20}\right)$$

$$+\frac{4\bar{Y}^3}{\bar{X}}f_1\frac{\mu_{11}}{n}+\frac{4\bar{Y}^3}{\mu_{02}}f_2\frac{\mu_{12}}{n}+\frac{2\bar{Y}^4}{\bar{X}\mu_{02}}f_1f_2\frac{\mu_{03}}{n}$$

$$\text{MSE}(d_{1g})=\frac{1}{n}\left(\mu_{40}-\mu_{20}^2\right)+\frac{\bar{Y}^4}{\bar{X}^2}f_1^2\frac{\mu_{02}}{n}+\frac{\bar{Y}^4(\beta_2-1)}{n}f_2^2-\frac{2\bar{Y}^2}{\bar{X}}f_1\frac{\mu_{21}}{n}-\frac{2\bar{Y}^2}{\mu_{02}}f_2\frac{\mu_{22}}{n}$$

$$+2\bar{Y}^2f_2\frac{\mu_{20}}{n}+\frac{2\bar{Y}^4}{\bar{X}\mu_{02}}f_1f_2\frac{\mu_{03}}{n} \tag{3.14}$$

For minimizing (3.14) in two unknowns f_1 and f_2, the two normal equations after differentiating (3.14) partially with respect to f_1 and f_2 are

$$\frac{2\bar{Y}^4\mu_{02}}{n\bar{X}^2} f_1 - \frac{2\bar{Y}^2\mu_{21}}{n\bar{X}} + \frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}} f_2 = 0 \tag{3.15}$$

$$\frac{2\bar{Y}^4(\beta_2-1)}{n} f_2 + \left(\frac{2\bar{Y}^2\mu_{20}}{n} - \frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right) + \frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}} f_1 = 0 \tag{3.16}$$

Solving (3.15) and (3.16) for f_1 and f_2, we get the minimizing optimum values to be

$$f_1{}^* = -\frac{\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right)\left(\frac{-2\bar{Y}^2\mu_{21}}{n\bar{X}}\right) - \left(\frac{2\bar{Y}^2\mu_{20}}{n} - \frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right)\left(\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}\right)}{\left(\frac{2\bar{Y}^2\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right) - \left(\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}\right)^2} \tag{3.17}$$

$$f_2{}^* = -\frac{\left(\frac{2\bar{Y}^4\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^2\mu_{20}}{n} - \frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right) - \left(\frac{-2\bar{Y}^2\mu_{21}}{n\bar{X}}\right)\left(\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}\right)}{\left(\frac{2\bar{Y}^4\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right) - \left(\frac{2\bar{Y}^4\mu_{03}}{n\bar{X}\mu_{02}}\right)^2} \tag{3.18}$$

which when substituted in (3.14) gives the minimum value of mean square error of the estimator d_{1g} as

$$\text{MSE}(d_{1g})\text{min} = \frac{1}{n}\left(\mu_{40} - \mu_{20}{}^2\right) - \frac{\left\{\frac{2\mu_{21}\mu_{03}}{n\mu_{02}}\left(\mu_{20} - \frac{\mu_{22}}{\mu_{02}}\right) - \frac{(\beta_2-1)\mu_{21}{}^2}{n} - \frac{\mu_{02}}{n}\left(\mu_{20} - \frac{\mu_{22}}{\mu_{02}}\right)^2\right\}}{\left\{(\beta_2-1)\mu_{02} - \frac{\mu_{03}{}^2}{\mu_{02}{}^2}\right\}} \tag{3.19}$$

(c) Bias and Mean Square Error of Suggested Estimator d_2

Let us consider the proposed estimator d_2 defined in (2.4)

$$d_2 = \hat{\theta}\left\{1 + \frac{k_3(\bar{x}-\bar{X})}{\bar{X}}\right\} - \bar{y}^2\left\{1 + \frac{k_4(s_x{}^2 - S_x{}^2)}{S_x{}^2}\right\}$$

$$= \hat{\theta}\left\{1 + \frac{k_3(\bar{X}+e_1-\bar{X})}{\bar{X}}\right\} - \bar{y}^2\left\{1 + \frac{k_4(S_x{}^2 + e_2 - S_x{}^2)}{S_x{}^2}\right\}$$

$$= (\theta + e_3)\left\{1 + \frac{k_3 e_1}{\bar{X}}\right\} - (\bar{Y}^2 + e_0{}^2 + 2\bar{Y}e_0)\left\{1 + \frac{k_4 e_2}{\mu_{02}}\right\}$$

$$= \theta + \frac{\theta\, k_3 e_1}{\bar{X}} + e_3 + \frac{k_3 e_1 e_3}{\bar{X}} - \bar{Y}^2 - \frac{\bar{Y}^2 k_4 e_2}{\mu_{02}} - e_0{}^2 - 2\bar{Y}e_0 - \frac{2\bar{Y}k_4 e_0 e_2}{\mu_{02}}$$

$$= \left(\frac{1}{N}\sum_{i=1}^{N} Y_i^2 - \bar{Y}^2\right) + e_3 - 2\bar{Y}e_0 + \frac{\theta\, k_3 e_1}{\bar{X}} - \frac{\bar{Y}^2 k_4 e_2}{\mu_{02}} - e_0{}^2 + \frac{k_3 e_1 e_3}{\bar{X}} - \frac{2\bar{Y}k_4 e_0 e_2}{\mu_{02}}$$

or

$$d_2 - \sigma_Y^2 = e_3 - 2\bar{Y}e_0 + \frac{\theta\, k_3 e_1}{\bar{X}} - \frac{\bar{Y}^2 k_4 e_2}{\mu_{02}} - e_0{}^2 + \frac{k_3 e_1 e_3}{\bar{X}} - \frac{2\bar{Y}k_4 e_0 e_2}{\mu_{02}} \tag{3.20}$$

Taking expectation on both sides of (3.20), the bias in $d_2\ (= E(d_2) - \sigma_Y^2)$ up to terms of order $\left(\frac{1}{n}\right)$ is given by

Bias (d_2) = $E(e_3)-2\bar{Y}E(e_0)+\frac{\theta\ k_3E(e_1)}{\bar{X}}-\frac{\bar{Y}^2k_4E(e_2)}{\mu_{02}}-E(e_0{}^2)+\frac{k_3E(e_1e_3)}{\bar{X}}-\frac{2\bar{Y}k_4E(e_0e_2)}{\mu_{02}}$

Using values of the expectations given from (3.1) to (3.3), we have

$$\text{Bias } (d_2)=E(d_2)-\sigma_Y^2=\frac{k_3}{\bar{X}}\frac{1}{n}\left(\mu_{21}+2\bar{Y}\mu_{11}\right)-\frac{2\bar{Y}k_4}{\mu_{02}}\frac{\mu_{12}}{n}-\frac{\mu_{20}}{n} \tag{3.21}$$

Now squaring (3.20) on both sides and then taking expectation, the mean square error to the first degree of approximation is

$$E(d_2-\sigma_Y^2)^2 = E\left(e_3-2\bar{Y}e_0+\frac{\theta\ k_3e_1}{\bar{X}}-\frac{\bar{Y}^2k_4e_2}{\mu_{02}}\right)^2$$

$$=E\left(e_3{}^2\right)+4\bar{Y}^2E\left(e_0{}^2\right)+\frac{\theta^2k_3{}^2}{\bar{X}^2}E\left(e_1{}^2\right)+\frac{\bar{Y}^4k_4{}^2}{\mu_{02}{}^2}E\left(e_2{}^2\right)-4\bar{Y}E\left(e_0e_3\right)+\frac{2\theta\ k_3}{\bar{X}}E\left(e_1e_3\right)$$

$$-\frac{2\bar{Y}^2k_4}{\mu_{02}}E\left(e_2e_3\right)-\frac{4\bar{Y}\theta\ k_3}{\bar{X}}E\left(e_0e_1\right)+\frac{4\bar{Y}^3k_4}{\mu_{02}}E\left(e_0e_2\right)-\frac{2\theta\ \bar{Y}^2k_3k_4}{\bar{X}\mu_{02}}E\left(e_1e_2\right)$$

$$=\frac{1}{n}(\mu_{40}+4\bar{Y}\mu_{30}+4\bar{Y}^2\mu_{20}-\mu_{20}{}^2)+4\bar{Y}^2\frac{\mu_{20}}{n}-4\bar{Y}\frac{1}{n}(\mu_{30}+2\bar{Y}\mu_{20})+\frac{\theta^2}{\bar{X}^2}k_3{}^2\frac{\mu_{02}}{n}$$

$$+\frac{\bar{Y}^4k_4{}^2}{\mu_{02}{}^2}\frac{\mu_{02}{}^2}{n}(\beta_2-1)+\frac{2\theta\ k_3}{\bar{X}}\frac{1}{n}\left(\mu_{21}+2\bar{Y}\mu_{11}\right)-\frac{2\bar{Y}^2k_4}{\mu_{02}}\frac{1}{n}(\mu_{22}+2\bar{Y}\mu_{12}-\mu_{02}\mu_{20})$$

$$-\frac{4\bar{Y}\theta\ k_3}{\bar{X}}\frac{\mu_{11}}{n}+\frac{4\bar{Y}^3k_4}{\mu_{02}}\frac{\mu_{12}}{n}-\frac{2\theta\ \bar{Y}^2k_3k_4}{\bar{X}\mu_{02}}\frac{\mu_{03}}{n}$$

or

$$\text{MSE}(d_2)=\frac{1}{n}\left(\mu_{40}-\mu_{20}{}^2\right)+\frac{\theta^2k_3{}^2}{\bar{X}^2}\frac{\mu_{02}}{n}+\frac{\bar{Y}^4k_4{}^2}{n}\left(\beta_2-1\right)+\frac{2\theta k_3}{\bar{X}}\frac{\mu_{21}}{n}-\frac{2\bar{Y}^2k_4}{\mu_{02}}\frac{\mu_{22}}{n}$$

$$+2\,\bar{Y}^2k_4\frac{\mu_{20}}{n}-\frac{2\theta\ \bar{Y}^2k_3k_4}{\bar{X}\mu_{02}}\frac{\mu_{03}}{n} \tag{3.22}$$

For minimizing (3.22) in two unknowns k_3 and k_4, the two normal equations after differentiating (3.22) partially with respect to k_3 and k_4 are

$$\frac{2\theta^2}{n\bar{X}^2}\mu_{02}k_3+\frac{2\theta}{n\bar{X}}\mu_{21}-\frac{2\theta\bar{Y}^2\mu_{03}k_4}{n\bar{X}\mu_{02}}=0 \tag{3.23}$$

$$\frac{2\bar{Y}^4}{n}\left(\beta_2-1\right)k_4+\left(\frac{2\bar{Y}^2\mu_{20}}{n}-\frac{2\bar{Y}^2}{\mu_{02}}\frac{\mu_{22}}{n}\right)-\frac{2\theta\bar{Y}^2\mu_{03}}{n\bar{X}\mu_{02}}k_3=0 \tag{3.24}$$

Solving (3.23) and (3.24) for k_3 and k_4, we get the minimizing optimum values to be

$$k_3{}^*=-\frac{\left(\frac{2\bar{Y}^4\left(\beta_2-1\right)}{n}\right)\left(\frac{2\theta\mu_{21}}{n\bar{X}}\right)-\left(\frac{2\bar{Y}^2\mu_{20}}{n}-\frac{2\bar{Y}^2}{\mu_{02}}\frac{\mu_{22}}{n}\right)\left(-\frac{2\theta\bar{Y}^2\mu_{03}}{n\bar{X}\mu_{02}}\right)}{\left(\frac{2\theta^2\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^4\left(\beta_2-1\right)}{n}\right)-\left(\frac{-2\theta\bar{Y}^2\mu_{03}}{n\bar{X}\mu_{02}}\right)^2} \tag{3.25}$$

and

$$k_4{*}=-\frac{\left(\frac{2\theta^2\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^2\mu_{20}}{n}-\frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right)-\left(\frac{2\theta\mu_{21}}{n\bar{X}}\right)\left(\frac{-2\theta\bar{Y}^2\mu_{03}}{n\bar{X}\mu_{02}}\right)}{\left(\frac{2\theta^2\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right)-\left(\frac{-2\theta\bar{Y}^2\mu_{03}}{n\bar{X}\mu_{02}}\right)^2} \tag{3.26}$$

which when substituted in (3.22) gives the minimum value of mean square error of the estimator d_2 to be

$$\text{MSE}(d_2)_{\min}=\frac{1}{n}\left(\mu_{40}-\mu_{20}^2\right)-\frac{\left\{\left(\frac{2\mu_{21}\mu_{03}}{n\mu_{02}}\right)\left(\mu_{20}-\frac{\mu_{22}}{\mu_{02}}\right)+\frac{\mu_{02}}{n}\left(\mu_{20}-\frac{\mu_{22}}{\mu_{02}}\right)^2+\frac{(\beta_2-1)}{n}\mu_{21}^2\right\}}{\left\{\mu_{02}(\beta_2-1)-\left(\frac{\mu_{03}}{\mu_{02}}\right)^2\right\}} \tag{3.27}$$

(d) Bias and Mean Square Error of Suggested Estimator d_{2g}

For $f'(1)$, $f''(1)$ and $f'''(1)$ to be first, second and third order derivatives of $f(u)$ at the point u = 1, u* = 1 + h_1 (u-1), 0 < h_1 < 1 , also $g'(1)$, $g''(1)$ and $g'''(1)$ to be first, second and third order derivatives of $g(v)$ at the point v = 1 and v* = 1 + h_2 (v-1), 0 < h_2 < 1, expanding $f(u)$ and $g(v)$ in d_{2g} in third order Taylor's series, we have

$$\begin{aligned}
d_{2g}&=\hat{\theta}\left\{f(1)+(u-1)f'(1)+\frac{(u-1)^2}{2!}f''(1)+\frac{(u-1)^3}{3!}f'''(u^*)\right\}\\
&\quad-\bar{y}^2\left\{g(1)+(v-1)g'(1)+\frac{(v-1)^2}{2!}g''(1)+\frac{(v-1)^3}{3!}g'''(v^*)\right\}\\
&=\hat{\theta}\left\{1+\frac{e_1}{\bar{X}}f'(1)+\frac{1}{2!}\frac{e_1^2}{\bar{X}^2}f''(1)+\frac{1}{3!}\frac{e_1^3}{\bar{X}^3}f'''(u^*)\right\}-\bar{y}^2\left\{1+\frac{e_2}{\mu_{02}}g'(1)+\frac{1}{2!}\frac{e_2^2}{\mu_{02}^2}g''(1)+\frac{1}{3!}\frac{e_2^3}{\mu_{02}^3}g'''(v^*)\right\}\\
&=(\theta+e_3)\left\{1+\frac{e_1}{\bar{X}}f'(1)+\frac{1}{2}\frac{e_1^2}{\bar{X}^2}f''(1)\right\}-\left(\bar{Y}^2+e_0^2+2\bar{Y}e_0\right)\cdot\left\{1+\frac{e_2}{\mu_{02}}g'(1)+\frac{1}{2}\frac{e_2^2}{\mu_{02}^2}g''(1)\right\}\\
&=\theta+\frac{\theta e_1}{\bar{X}}f'(1)+\frac{\theta}{2}\frac{e_1^2}{\bar{X}^2}f''(1)+e_3+\frac{e_1e_3}{\bar{X}}f'(1)-\bar{Y}^2-\frac{\bar{Y}^2e_2}{\mu_{02}}g'(1)-\frac{\bar{Y}^2}{2}\frac{e_2^2}{\mu_{02}^2}g''(1)-e_0^2-2\bar{Y}e_0-\frac{2\bar{Y}e_0e_2}{\mu_{02}}g'(1)\\
&=\left(\frac{1}{N}\sum_{i=1}^{N}Y_i^2-\bar{Y}^2\right)+e_3-2\bar{Y}e_0+\frac{\theta e_1}{\bar{X}}f'(1)-\frac{\bar{Y}^2e_2}{\mu_{02}}g'(1)-e_0^2+\frac{\theta e_1^2}{2\bar{X}^2}f''(1)\\
&\quad-\frac{\bar{Y}^2e_2^2}{2\mu_{02}^2}g''(1)+\frac{e_1e_3}{\bar{X}}f'(1)-\frac{2\bar{Y}e_0e_2}{\mu_{02}}g'(1)
\end{aligned}$$

or

$$\begin{aligned}
d_{2g}-\sigma_Y^2&=e_3-2\bar{Y}e_0+\frac{\theta e_1}{\bar{X}}f'(1)-\frac{\bar{Y}^2e_2}{\mu_{02}}g'(1)-e_0^2+\frac{\theta e_1^2}{2\bar{X}^2}f''(1)\\
&\quad-\frac{\bar{Y}^2e_2^2}{2\mu_{02}^2}g''(1)+\frac{e_1e_3}{\bar{X}}f'(1)-\frac{2\bar{Y}e_0e_2}{\mu_{02}}g'(1)
\end{aligned} \tag{3.28}$$

Taking expectation on both sides of (3.28) and using values of the expectations given from (3.1) to (3.3), the bias in d_{2g} $(=E(d_{2g})-\sigma_Y^2)$ to the order $\left(\frac{1}{n}\right)$ is given by

$$\text{Bias}(d_{2g}) = -\frac{\mu_{20}}{n} + \frac{\theta^2}{2\bar{X}^2} f''(1)\frac{\mu_{02}}{n} - \frac{\bar{Y}^2}{2\mu_{02}^2} g''(1)\frac{\mu_{04}}{n} + \frac{1}{\bar{X}} f'(1)\frac{1}{n}\left(\mu_{21} + 2\bar{Y}\mu_{11}\right) - \frac{2\bar{Y}}{\mu_{02}} g'(1)\frac{\mu_{12}}{n}$$

$$= \frac{\theta^2}{2\bar{X}^2} f''(1)\frac{\mu_{02}}{n} - \frac{\bar{Y}^2}{2\mu_{02}^2} g''(1)\frac{\mu_{04}}{n} + \frac{1}{\bar{X}} f'(1)\frac{1}{n}\left(\mu_{21} + 2\bar{Y}\mu_{11}\right) - \frac{2\bar{Y}}{\mu_{02}} g'(1)\frac{\mu_{12}}{n} - \frac{\mu_{20}}{n} \quad (3.29)$$

Now squaring (3.28) on both sides and then taking expectation, the mean square error of d_{2g} to the first degree of approximation is given by

$$E(d_{2g} - \sigma_Y^2)^2 = E\left(e_3 - 2\bar{Y}e_0 + \frac{\theta e_1}{\bar{X}} f'(1) - \frac{\bar{Y}^2 e_2}{\mu_{02}} g'(1)\right)^2$$

$$= E\left(e_3^2\right) + 4\bar{Y}^2 E\left(e_0^2\right) + \frac{\theta^2}{\bar{X}^2}\{f'(1)\}^2 E\left(e_1^2\right) + \frac{\bar{Y}^4}{\mu_{02}^2}\{g'(1)\}^2 E\left(e_2^2\right) - 4\bar{Y}E(e_0e_3) + \frac{2\theta}{\bar{X}} f'(1)E(e_1e_3)$$

$$- \frac{2\bar{Y}^2}{\mu_{02}} g'(1)E(e_2e_3) - \frac{4\bar{Y}\theta}{\bar{X}} f'(1)E\left(e_0e_1\right) + \frac{4\bar{Y}^3}{\mu_{02}} g'(1)E(e_0e_2) - \frac{2\bar{Y}^2\theta}{\bar{X}\mu_{02}} f'(1)g'(1)E(e_1e_2)$$

Using values of the expectations given from (3.1) to (3.3), we have

$$\text{MSE}(d_{2g}) = E\left(d_{2g} - \sigma_Y^2\right)^2$$

$$= \frac{1}{n}\left(\mu_{40} + 4\bar{Y}\mu_{30} + 4\bar{Y}^2\mu_{20} - \mu_{20}^2\right) + 4\bar{Y}^2\frac{\mu_{20}}{n}$$

$$- 4\bar{Y}\frac{1}{n}\left(\mu_{30} + 2\bar{Y}\mu_{20}\right) + \frac{\theta^2}{\bar{X}^2}\{f'(1)\}^2\frac{\mu_{02}}{n} + \frac{\bar{Y}^4}{\mu_{02}^2}\{g'(1)\}^2\frac{\mu_{02}^2}{n}(\beta_2 - 1) + \frac{2\theta}{\bar{X}} f'(1)\frac{1}{n}(\mu_{21} + 2\bar{Y}\mu_{11})$$

$$- \frac{2\bar{Y}^2}{\mu_{02}} g'(1)\frac{1}{n}\left(\mu_{22} + 2\bar{Y}\mu_{12} - \mu_{02}\mu_{20}\right) - \frac{4\bar{Y}\theta}{\bar{X}} f'(1)\frac{\mu_{11}}{n} + \frac{4\bar{Y}^3}{\mu_{02}} g'(1)\frac{\mu_{12}}{n} - \frac{2\bar{Y}^2\theta}{\bar{X}\mu_{02}} f'(1)g'(1)\frac{\mu_{03}}{n}$$

or

$$\text{MSE}(d_{2g}) = E\left(d_{2g} - \sigma_Y^2\right)^2$$

$$= \frac{1}{n}\left(\mu_{40} - \mu_{20}^2\right) + \frac{\theta^2}{\bar{X}^2}\{f'(1)\}^2\frac{\mu_{02}}{n} + \frac{\bar{Y}^4(\beta_2 - 1)}{n}\{g'(1)\}^2 + \frac{2\theta}{\bar{X}} f'(1)\frac{\mu_{21}}{n} - \frac{2\bar{Y}^2}{\mu_{02}}\frac{\mu_{22}}{n} g'(1)$$

$$+ 2\bar{Y}^2\frac{\mu_{20}}{n} g'(1) - \frac{2\bar{Y}^2\theta}{\bar{X}\mu_{02}}\frac{\mu_{03}}{n} f'(1)g'(1) \quad (3.30)$$

For minimizing (3.30) in two unknowns $f'(1)$ and $g'(1)$, the two normal equations after differentiating (3.30) partially with respect to $f'(1)$ and $g'(1)$ are

$$\frac{2\theta^2\mu_{02}}{n\bar{X}^2} f'(1) + \frac{2\theta\,\mu_{21}}{n\bar{X}} - \frac{2\bar{Y}^2\theta\mu_{03}}{n\bar{X}\mu_{02}} g'(1) = 0 \quad (3.31)$$

and

$$\frac{2\bar{Y}^4\left(\beta_2 - 1\right)}{n} g'(1) + \left(\frac{2\bar{Y}^2\mu_{20}}{n} - \frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right) - \frac{2\bar{Y}^2\theta\mu_{03}}{n\bar{X}\mu_{02}} f'(1) = 0 \quad (3.32)$$

Solving (3.31) and (3.32) for $f'(1)$ and $g'(1)$, we get the minimizing optimum values to be

$$f'(1)^* = -\frac{\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right)\left(\frac{2\theta\mu_{21}}{n\bar{X}}\right) - \left(2\bar{Y}^2\frac{\mu_{20}}{n} - \frac{2\bar{Y}^2}{\mu_{02}}\frac{\mu_{22}}{n}\right)\left(-\frac{2\bar{Y}^2\theta\mu_{03}}{n\bar{X}\mu_{02}}\right)}{\left(\frac{2\theta^2\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right) - \left(-\frac{2\bar{Y}^2\theta\ \mu_{03}}{n\bar{X}\mu_{02}}\right)^2} \tag{3.33}$$

$$g'(1)^* = -\frac{\left(\frac{2\theta^2\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^2\mu_{20}}{n} - \frac{2\bar{Y}^2\mu_{22}}{n\mu_{02}}\right) - \left(\frac{2\theta\ \mu_{21}}{n\bar{X}}\right)\left(-\frac{2\bar{Y}^2\theta\ \mu_{03}}{n\bar{X}\mu_{02}}\right)}{\left(\frac{2\theta^2\mu_{02}}{n\bar{X}^2}\right)\left(\frac{2\bar{Y}^4(\beta_2-1)}{n}\right) - \left(-\frac{2\bar{Y}^2\theta\ \mu_{03}}{n\bar{X}\mu_{02}}\right)^2} \tag{3.34}$$

which when substituted in (3.30) gives the minimum value of mean square error as

$$\text{MSE}(d_{2g})_{\min} = \frac{1}{n}\left(\mu_{40} - \mu_{20}{}^2\right) - \frac{\left\{\left(\frac{2\mu_{21}\mu_{03}}{n\mu_{02}}\right)\left(\mu_{20} - \frac{\mu_{22}}{\mu_{02}}\right) + \frac{\mu_{02}}{n}\left(\mu_{20} - \frac{\mu_{22}}{\mu_{02}}\right)^2 + \frac{(\beta_2-1)}{n}\mu_{21}{}^2\right\}}{\left\{\mu_{02}(\beta_2-1) - \left(\frac{\mu_{03}}{\mu_{02}}\right)^2\right\}} \tag{3.35}$$

4. Efficiency Comparison with the Traditional Estimator

As we know that the mean square error of usual conventional unbiased estimator $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$ of population variance $\sigma_Y{}^2$ is $\frac{1}{n}\left(\mu_{40} - \mu_{20}{}^2\right)$ and

MSE(d_1)min = MSE(d_{1g})min

$$= \frac{1}{n}\left(\mu_{40} - \mu_{20}{}^2\right) - \frac{\left\{\frac{2\mu_{21}\mu_{03}}{n\mu_{02}}\left(\mu_{20} - \frac{\mu_{22}}{\mu_{02}}\right) - \frac{(\beta_2-1)\mu_{21}{}^2}{n} - \frac{\mu_{02}}{n}\left(\mu_{20} - \frac{\mu_{22}}{\mu_{02}}\right)^2\right\}}{\left\{(\beta_2-1)\mu_{02} - \frac{\mu_{03}{}^2}{\mu_{02}{}^2}\right\}} \tag{4.1}$$

and

MSE (d_2)$_{\min}$ = MSE(d_{2g})$_{\min}$

$$= \frac{1}{n}\left(\mu_{40} - \mu_{20}{}^2\right) - \frac{\left\{\left(\frac{2\mu_{21}\mu_{03}}{n\mu_{02}}\right)\left(\mu_{20} - \frac{\mu_{22}}{\mu_{02}}\right) + \frac{\mu_{02}}{n}\left(\mu_{20} - \frac{\mu_{22}}{\mu_{02}}\right)^2 + \frac{(\beta_2-1)}{n}\mu_{21}{}^2\right\}}{\left\{\mu_{02}(\beta_2-1) - \left(\frac{\mu_{03}}{\mu_{02}}\right)^2\right\}} \tag{4.2}$$

Comparative studies regarding their efficiency over the usual conventional unbiased estimator are carried out with the help of a numerical illustration.

Considering the data given in Cochran (1977) dealing with Paralytic Polio Cases 'Placebo' (Y) group, Paralytic Polio Cases in not inoculated group (X), computations of required values of μ_{rs} have been done and comparisons are made for a simple random sample of size n. For the data considered, we have

n = 34, $\bar{Y}$ = 2.58, $\bar{X}$ = 8370.6

μ_{20} = 9.8894, μ_{02} = 7.1865882 × 10^7

μ_{30} = 47.015235, μ_{03} = 1.4510955 × 10^{12}

μ_{40} = 421.96088, μ_{04} = 4.5961952 × 10^{16}

μ_{21} = 93.464705 × 10^3 μ_{12} = 3.443287 × 10^8

μ_{11} = 19.435294 × 10^3 μ_{22} = 3.0156658 × 10^9

Using above values, we have

Mean Square Error of usual conventional unbiased estimator = 9.534136695 and

$$\text{MSE}(d_1) = \text{MSE}(d_{1g}) = \text{MSE}(d_2)$$

$$= \text{MSE}(d_{2g}) = 5.512540843,$$

Hence the percent relative efficiency (PRE) of the proposed estimators d_1, d_{1g}, d_2, and d_{2g} over the usual conventional estimator are given by

$$\text{PRE}(d_1) = \text{PRE}(d_{1g}) = \text{PRE}(d_2)$$

$$= \text{PRE}(d_{2g}) = 172.9535792,$$

showing that the proposed estimators d_1, d_{1g}, d_2, and d_{2g} are more efficient with highly significant percent relative efficiency over the usual conventional unbiased estimator of the population variance.

5. Conclusions

We have derived new sampling estimators of population variance using auxiliary information in the form of mean and variance both, the bias and mean square error equations are obtained. Using these equations, MSE of proposed estimators are compared with the traditional estimator in theory and shown that the proposed estimators have smaller MSE than the traditional one.

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