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On Posterior Analysis of Mixture of Two Components of Gumbel Type II Distribution

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Abstract This paper describes the Bayesian analysis of the parameters of mixture of two components of Gumbel type II distribution. A heterogeneous population has been modeled by means of two components mixture of the Gumbel type II distribution under type I censored data. The Bayes estimators of the said parameters have been derived under the assumption of non-informative priors on the basis of different loss functions. A censored mixture data is simulated by probabilistic mixing for the computational purpose. The comparisons among the estimators have been made in terms of corresponding posterior risks. The posterior predictive distributions and intervals have been derived and evaluated under each prior.

Keywords Bayes Estimators, Posterior Risks, Mixture Models, Loss Functions

1. Introduction

The Gumbel type II distribution is used to model the extreme events like extreme earthquake, rainfalls, temperature, floods etc. It has another side of applications which deals with life testing experiments. Chechile[1] obtained the posterior distribution assuming that the random sample is taken from the Gumbel distribution using the conjugate prior. Corsini et al.[2] discussed the maximum likelihood (ML) algorithms and Cramer-Rao (CR) bounds for the location and scale parameters of the Gumbel distribution. Mousa[3] obtained the Bayesian estimation for the two parameters of the Gumbel distribution based on record values. Koutsoyiannis and Baloutsos[4] described that the Gumbel distribution has been the prevailing model for quantifying risk associated with extreme rainfall. Rasmussen and Gautam[5] extended the probability weighted moments (PWM) to what is called the generalized method of probability weighted moments (GPWM) because there is no reason why the PWMs provide the most efficient estimators of Gumbel parameters and quantile especially in hydrology. Malinowska and Szynal[6] obtained the Bayes estimators for the two parameters of a Gumbel distribution based on kth lower record values. Nadarajah and Kotz[7] introduced the beta Gumbel (BG) distribution and provided closed-form expressions for the moments, the asymptotic distribution of the extreme order statistics and discussed the maximum likelihood estimation procedure. Miladinovic and Tsokos[8] modified the classical Gumbel probability distribution in order to study the failure times of a given system. Park et al.[9] gave a novel equation for the scale parameter of the Gumbel distribution. Heo and Salas[10] examined the log-Gumbel distribution regarding quantile estimation and confidence intervals of quantiles. Thompson et al.[11] introduced a distributional hypothesis test for left censored Gumbel observations based on the probability plot correlation coefficient (PPCC).

The mixture models have received great attention of the analysts in the recent era. These models include finite and infinite number of components that can analyze different datasets. A finite mixture of probability distribution is suitable to study a population categorized in number of subpopulations. A population of lifetimes of certain electrical elements can be classified into number of subpopulations based on causes of failures. The analysis of mixture models under Bayesian framework has developed a significant interest among the statisticians. The authors dealing with Bayesian analysis of mixture models include Saleem and Aslam[12], Saleem et al.[13], Majeed and Aslam[14] and Kazmi et al.[15]. These contributions to the mixture models are the great motivations for the resent study.

We considered two component mixture of Gumbel type II distribution. The population of certain items is assumed to be partitioned into two subpopulations. The randomly selected observations from the said population are considered to be a part of one of the above mentioned subpopulations. These subpopulations are assumed to follow the Gumbel type II distribution. Therefore, the two components mixture of Gumbel type II distributions has been proposed to model this population. The observations have been assumed to be right censored. The inverse transformation technique of

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simulation under a probabilistic mixing has been used to generate data and to evaluate the performance of different estimators.

2. The Model and Likelihood Function

A density function for mixture of two components densities with mixing weights (p,q) is:

$$f(x) = pf_1(x) + qf_2(x) \tag{1}$$

$$0 < p < 1 \;,\; q = 1 - p$$

The following Gumbel type II distribution is considered for both mixture densities:

$$f(x_i;\alpha_i,\beta_i) = \alpha_i\beta_i x_i^{-(\alpha_i+1)} e^{-\beta_i x_i^{-\alpha_i}} \tag{2}$$

$$x_i > 0,\; \alpha_i > 0,\; \beta_i > 0;\; i = 1,2$$

With the cumulative distribution function as:

$$F(x_i;\alpha_i,\beta_i) = 1 - e^{-\beta_i x_i^{-\alpha_i}} \tag{3}$$

The cumulative distribution function for the mixture model is:

$$F(x) = pF_1(x) + qF_2(x) \tag{4}$$

Suppose n items are put on a life testing experiment and r units failed until time T while, n – r units are still working. Now based on causes of failure, the failed items are assumed to come either from subpopulation 1 or from subpopulation 2. Therefore it can be observed that r_1 and r_2 failed items come from first and second subpopulation respectively. Where r = r_1 + r_2. The remaining n – r items are assumed to be censored observations. The likelihood function for above censored data can be obtained as:

$$L(\beta_1,\beta_2,p|\underline{x}) = \prod_{j=1}^{r_1}\{pf_1(x_{1j})\} \times \prod_{j=1}^{r_2}\{pf_2(x_{2j})\}[1-F(t)]^{n-r} \tag{5}$$

$$L(\beta_1,\beta_2,p|\underline{x}) \propto \beta_1^{r_1}\beta_2^{r_2}p^{r_1}q^{r_2}e^{-\beta_1\sum_{j=1}^{r_1}x_{1j}-\beta_2\sum_{j=1}^{r_2}x_{2j}} \times\left[pe^{-\beta_1 t^{-\alpha_1}} + qe^{-\beta_2 t^{-\alpha_2}}\right]^{n-r} \tag{6}$$

Expanding the last term by binomial expansion the likelihood function becomes:

$$L(\beta_1,\beta_2,p|\underline{x}) \propto \sum_{k=0}^{n-r}\binom{n-r}{k}\beta_1^{r_1}\beta_2^{r_2}p^{n-k-r_2}q^{r_2+k} \times e^{-\beta_1 A_{1k}}e^{-\beta_2 A_{2k}} \tag{7}$$

Where $A_{1k} = \sum_{j=1}^{r_1}x_{1j} + (n-r-k)t^{-\alpha_1}$

and $A_{2k} = \sum_{j=1}^{r_2}x_{2j} + kt^{-\alpha_2}$

3. The Posterior Analysis under the Assumption of Uniform Prior

One of the most widely used non-informative priors, proposed by Laplace[16], is a uniform prior. It has been applied to many problems, and often the results are entirely satisfactory. This prior has been used for the posterior estimation.

Let $\beta_1 \in \text{Uniform}\ \forall\ \beta_1 \in (0,\infty)$, $\beta_2 \in \text{Uniform}\ \forall\ \beta_2 \in (0,\infty)$ and $p \sim U(0,1)$. Assuming independence, these priors result into a joint prior that is proportional to a constant. That joint prior has been used to derive the joint posterior distribution of β_1, β_2 and p. The marginal distribution for each parameter can be obtained by integrating the joint posterior distribution with respect to nuisance parameters. The joint posterior distribution is:

$$p(\beta_1,\beta_2,p|\underline{x}) = \frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta_1^{r_1}\beta_2^{r_2}p^{n-k-r_2}q^{r_2+k} \times e^{-\beta_1 A_{1k}}e^{-\beta_2 A_{2k}} \tag{8}$$

Where

$$C_1 = \sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}},$$

$$\theta_{1k} = n-k-r_2+1 \text{ and } \theta_{2k} = r_2+k+1$$

Using the posterior distribution, discussed in (8), the Bayes estimators and posterior risks under different loss functions have been derived and presented in the following.

Bayes estimators and associated risks under uniform prior using squared error loss function (SELF) are:

$$\beta_{1,SELF} = \frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+2)}{(A_{1k})^{r_1+2}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}$$

$$\rho(\beta_{1,SELF}) = \frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+3)}{(A_{1k})^{r_1+3}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}} - \left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+2)}{(A_{1k})^{r_1+2}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^2$$

$$\beta_{2,SELF} = \frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+2)}{(A_{2k})^{r_2+2}}$$

$$\rho(\beta_{2,SELF}) = \frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+3)}{(A_{2k})^{r_2+3}} - \left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+2)}{(A_{2k})^{r_2+2}}\right]^2$$

$$p_{SELF}=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+1,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}$$

$$\rho(p_{SELF})=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+2,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}-\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+1,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^2$$

Bayes estimators and risk under uniform prior using quadratic loss function (QLF) are:

$$\beta_{1,QLF}=\frac{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1-1)}{(A_{1k})^{r_1-1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}}$$

$$\rho(\beta_{1,QLF})=1-\frac{\frac{1}{C_1}\left[\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^2}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1-1)}{(A_{1k})^{r_1-1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}}$$

$$\beta_{2,QLF}=\frac{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2-1)}{(A_{2k})^{r_2-1}}}$$

$$\rho(\beta_{2,QLF})=1-\frac{\frac{1}{C_1}\left[\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}\right]^2}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2-1)}{(A_{2k})^{r_2-1}}}$$

$$p_{QLF}=\frac{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-1,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-2,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}}$$

$$\rho(p_{QLF})=1-\frac{\frac{1}{C_1}\left[\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-1,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^2}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-2,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}}$$

Bayes estimators and risk under uniform prior using weighted loss function (WLF) are:

$$\beta_{1,WLF}=\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^{-1}$$

$$\rho(\beta_{1,WLF})=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+2)}{(A_{1k})^{r_1+2}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}-\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^{-1}$$

$$\beta_{2,WLF}=\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}\right]^{-1}$$

$$\rho(\beta_{2,WLF})=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+2)}{(A_{2k})^{r_2+2}}-\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}\right]^{-1}$$

$$p_{WLF}=\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-1,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^{-1}$$

$$\rho(p_{WLF})=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+1,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}-\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-1,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^{-1}$$

Bayes estimators and risk under uniform prior using precautionary loss function (PLF) are:

$$\beta_{1,PLF}=\left[\frac{1}{C_1}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+3)}{(A_{1k})^{r_1+3}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^{\frac{1}{2}}$$

$$\rho(\beta_{1,PLF})=2\left[\frac{1}{C_1}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+3)}{(A_{1k})^{r_1+3}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^{\frac{1}{2}}-\frac{2}{C_1}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+2)}{(A_{1k})^{r_1+2}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}$$

$$\beta_{2,PLF}=\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+3)}{(A_{2k})^{r_2+3}}\right]^{\frac{1}{2}}$$

$$\rho(\beta_{2,PLF})=2\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+3)}{(A_{2k})^{r_2+3}}\right]^{\frac{1}{2}}-\frac{2}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+2)}{(A_{2k})^{r_2+2}}$$

$$p_{PLF}=\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+2,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^{\frac{1}{2}}$$

$$\rho(p_{PLF})=2\left[\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+2,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^{\frac{1}{2}}$$
$$-\frac{2}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+1,\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}$$

4. The Posterior Analysis under the Assumption of Jeffreys Prior

Another non-informative prior has been suggested by Jeffreys[17] which is frequently used in situations where one does not have much information about the parameters. This is defined as the distribution of the parameters proportional to the square root of the determinants of the Fisher information matrix i.e.

$$p(\underline{\beta})\propto\left\{\left|I(\underline{\beta})\right|\right\}^{\frac{1}{2}}$$

Where $\underline{\beta}=(\beta_1,\beta_2)$ is the vector of parameters and $I(\underline{\beta})$ is (2×2) Fisher information matrix, defined as;

$$I(\underline{\beta})=-\mathrm{E}\left[\frac{\partial^2\log f_i(x_i|\beta_i)}{\partial\beta_i^2}\right]\quad i=1,2$$

Where $f_i(x_i|\beta_i)$ have been defined in (2) and $p\sim U(0,1)$. Assuming independence, the joint prior is obtained as:

$$h(\beta_1,\beta_2,p)\propto\frac{1}{\beta_1\beta_2}. \tag{9}$$

The joint posterior distribution using the above prior is:

$$p(\beta_1,\beta_2,p|\underline{x})=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta_1^{r_1-1}\beta_2^{r_2-1}p^{n-k-r_2}q^{r_2+k}\times e^{-\beta_1A_{1k}}e^{-\beta_2A_{2k}} \tag{10}$$

Where $C_2=\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}$

The Bayes estimators and associated posterior risks have been derived under different loss functions using the posterior distribution (10).

Bayes estimators and associated risks under Jeffreys prior using squared error loss function (SELF) are:

$$\beta_{1,SELF}=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}$$

$$\rho(\beta_{1,SELF})=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+2)}{(A_{1k})^{r_1+2}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}-\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1+1)}{(A_{1k})^{r_1+1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}\right]^2$$

$$\beta_{2,SELF}=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}$$

$$\rho(\beta_{2,SELF})=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2+2)}{(A_{2k})^{r_2+2}}-\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\right]^2$$

$$p_{SELF}=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+1,\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}$$

$$\rho(p_{SELF})=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+2,\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}-\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}+1,\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}\right]^2$$

Bayes estimators and risk under Jeffreys prior using quadratic loss function (QLF) are:

$$\beta_{1,QLF}=\frac{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1-1)}{(A_{1k})^{r_1-1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1-2)}{(A_{1k})^{r_1-2}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}}$$

$$\rho(\beta_{1,QLF})=1-\frac{\frac{1}{C_2}\left[\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1-1)}{(A_{1k})^{r_1-1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}\right]^2}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1-2)}{(A_{1k})^{r_1-2}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}}$$

$$\beta_{2,QLF}=\frac{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2-1)}{(A_{2k})^{r_2-1}}}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2-2)}{(A_{2k})^{r_2-2}}}$$

$$\rho(\beta_{2,QLF})=1-\frac{\frac{1}{C_2}\left[\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2-1)}{(A_{2k})^{r_2-1}}\right]^2}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k},\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2-2)}{(A_{2k})^{r_2-2}}}$$

$$p_{QLF}=\frac{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-1,\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-2,\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}}$$

$$\rho(p_{QLF})=1-\frac{\frac{1}{C_2}\left[\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-1,\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}\right]^2}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\theta_{1k}-2,\theta_{2k})\frac{\Gamma(r_1)}{(A_{1k})^{r_1}}\frac{\Gamma(r_2)}{(A_{2k})^{r_2}}}$$

Bayes estimators and risk under Jeffreys prior using weighted loss function (WLF) are:

$$\beta_{1,WLF}=\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1-1\right)}{\left(A_{1k}\right)^{r_1-1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}\right]^{-1}$$

$$\rho\left(\beta_{1,WLF}\right)=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1+1\right)}{\left(A_{1k}\right)^{r_1+1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}-\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1-1\right)}{\left(A_{1k}\right)^{r_1-1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}\right]^{-1}$$

$$\beta_{2,WLF}=\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2-1\right)}{\left(A_{2k}\right)^{r_2-1}}\right]^{-1}$$

$$\rho\left(\beta_{2,WLF}\right)=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2+1\right)}{\left(A_{2k}\right)^{r_2+1}}-\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2-1\right)}{\left(A_{2k}\right)^{r_2-1}}\right]^{-1}$$

$$p_{WLF}=\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k}-1,\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}\right]^{-1}$$

$$\rho\left(p_{WLF}\right)=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k}+1,\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}-\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k}-1,\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}\right]^{-1}$$

Bayes estimators and risk under Jeffreys prior using precautionary loss function (PLF) are:

$$\beta_{1,PLF}=\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1+2\right)}{\left(A_{1k}\right)^{r_1+2}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}\right]^{\frac{1}{2}}$$

$$\rho\left(\beta_{1,PLF}\right)=2\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1+2\right)}{\left(A_{1k}\right)^{r_1+2}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}\right]^{\frac{1}{2}}-\frac{2}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1+1\right)}{\left(A_{1k}\right)^{r_1+1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}$$

$$\beta_{2,PLF}=\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2+2\right)}{\left(A_{2k}\right)^{r_2+2}}\right]^{\frac{1}{2}}$$

$$\rho\left(\beta_{2,PLF}\right)=2\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2+2\right)}{\left(A_{2k}\right)^{r_2+2}}\right]^{\frac{1}{2}}-\frac{2}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k},\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2+1\right)}{\left(A_{2k}\right)^{r_2+1}}$$

$$p_{PLF}=\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k}+2,\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}\right]^{\frac{1}{2}}$$

$$\rho\left(p_{PLF}\right)=2\left[\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k}+2,\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}\right]^{\frac{1}{2}}-\frac{2}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}B\left(\theta_{1k}+1,\theta_{2k}\right)\frac{\Gamma\left(r_1\right)}{\left(A_{1k}\right)^{r_1}}\frac{\Gamma\left(r_2\right)}{\left(A_{2k}\right)^{r_2}}$$

6. Posterior Predictive Distributions and Intervals

The posterior predictive distributions under uniform and Jeffreys priors are:

$$p\left(y|\underline{x}\right)=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}\left[\frac{B\left(\theta_{1k}+1,\theta_{2k}\right)\Gamma\left(r_1+2\right)\Gamma\left(r_2+1\right)\alpha_1 y^{-(\alpha_1+1)}}{\left(A_{1k}+y^{-\alpha_1}\right)^{r_1+2}\left(A_{2k}\right)^{r_2+1}}+\frac{B\left(\theta_{1k},\theta_{2k}+1\right)\Gamma\left(r_1+1\right)\Gamma\left(r_2+2\right)\alpha_2 y^{-(\alpha_2+1)}}{\left(A_{1k}\right)^{r_1+1}\left(A_{2k}+y^{-\alpha_2}\right)^{r_2+2}}\right]$$

$$p\left(y|\underline{x}\right)=\frac{1}{C_2}\sum_{k=0}^{n-r}\binom{n-r}{k}\left[\frac{B\left(\theta_{1k}+1,\theta_{2k}\right)\Gamma\left(r_1+1\right)\Gamma\left(r_2\right)\alpha_1 y^{-(\alpha_1+1)}}{\left(A_{1k}+y^{-\alpha_1}\right)^{r_1+1}\left(A_{2k}\right)^{r_2}}+\frac{B\left(\theta_{1k},\theta_{2k}+1\right)\Gamma\left(r_1\right)\Gamma\left(r_2+1\right)\alpha_2 y^{-(\alpha_2+1)}}{\left(A_{1k}\right)^{r_1}\left(A_{2k}+y^{-\alpha_2}\right)^{r_2+1}}\right]$$

The posterior predictive intervals under uniform prior can be obtained by solving the following two equations respectively.

$$\frac{k}{2}=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}\left[\frac{B\left(\theta_{1k}+1,\theta_{2k}\right)\Gamma\left(r_1+1\right)\Gamma\left(r_2+1\right)}{\left(A_{2k}\right)^{r_2+1}}\left\{\frac{1}{\left(A_{1k}\right)^{r_1+1}}-\frac{1}{\left(A_{1k}+L^{-\alpha_1}\right)^{r_1+1}}\right\}+\frac{B\left(\theta_{1k},\theta_{2k}+1\right)\Gamma\left(r_1+1\right)\Gamma\left(r_2+1\right)}{\left(A_{1k}\right)^{r_1+1}}\left\{\frac{1}{\left(A_{2k}\right)^{r_2+1}}-\frac{1}{\left(A_{2k}+L^{-\alpha_2}\right)^{r_2+1}}\right\}\right]$$

$$1-\frac{k}{2}=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}\left[\frac{B(\theta_{1k}+1,\theta_{2k})\Gamma(r_1+1)\Gamma(r_2+1)}{(A_{2k})^{r_2+1}}\left\{\frac{1}{(A_{1k})^{r_1+1}}-\frac{1}{\left(A_{1k}+U^{-\alpha_1}\right)^{r_1+1}}\right\}+\frac{B(\theta_{1k},\theta_{2k}+1)\Gamma(r_1+1)\Gamma(r_2+1)}{(A_{1k})^{r_1+1}}\left\{\frac{1}{(A_{2k})^{r_2+1}}-\frac{1}{\left(A_{2k}+U^{-\alpha_2}\right)^{r_2+1}}\right\}\right]$$

Where ‘k’ is level of significance

The posterior predictive intervals under uniform prior can be obtained by solving the following two equations respectively.

$$\frac{k}{2}=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}\left[\frac{B(\theta_{1k}+1,\theta_{2k})\Gamma(r_1)\Gamma(r_2)}{(A_{2k})^{r_2}}\left\{\frac{1}{(A_{1k})^{r_1}}-\frac{1}{\left(A_{1k}+L^{-\alpha_1}\right)^{r_1}}\right\}+\frac{B(\theta_{1k},\theta_{2k}+1)\Gamma(r_1)\Gamma(r_2)}{(A_{1k})^{r_1}}\left\{\frac{1}{(A_{2k})^{r_2}}-\frac{1}{\left(A_{2k}+L^{-\alpha_2}\right)^{r_2}}\right\}\right]$$

$$1-\frac{k}{2}=\frac{1}{C_1}\sum_{k=0}^{n-r}\binom{n-r}{k}\left[\frac{B(\theta_{1k}+1,\theta_{2k})\Gamma(r_1)\Gamma(r_2)}{(A_{2k})^{r_2}}\left\{\frac{1}{(A_{1k})^{r_1}}-\frac{1}{\left(A_{1k}+U^{-\alpha_1}\right)^{r_1}}\right\}+\frac{B(\theta_{1k},\theta_{2k}+1)\Gamma(r_1)\Gamma(r_2)}{(A_{1k})^{r_1+1}}\left\{\frac{1}{(A_{2k})^{r_2}}-\frac{1}{\left(A_{2k}+U^{-\alpha_2}\right)^{r_2}}\right\}\right]$$

7. Simulation Study

A simulation study has been conducted to assess and compare the performance of Bayes estimators and to analyse the impact of sample size, mixing weight and magnitude of parametric values on the Bayes estimators. Samples of sizes n = 100, 200, 300, 400 and 500 have been generated by inverse transformation method from two components mixture of Gumbel type II distribution. The parametric values used are: $(\beta_1,\beta_2)\in\{(3,6),(6,9),(10,12)\}$ and $p\in(0.30,0.45)$. The probabilistic mixing has been used to generate the mixture data. For each observation a random number u has been generated from $U(0,1)$. If $u<p$ the observation has been randomly taken from first subpopulation and if $u>p$ then the observation have been taken from the second subpopulation.

The observations above a fixed censoring time T have been assumed to be right censored. Under each combination of parametric values, the choice of censoring time has been made so that the censoring rate in the respective sample has been 20%. As one sample cannot completely describe the behaviour and properties of the Bayes estimators, the results have been replicated 1000 times and the average of results has been presented in the tables below.

Table 1. Bayes estimates and posterior risks under uniform prior using β1 = 3, β2 = 6 and p = 0.30

n	$\beta_1=3$			
	SELF	QLF	WLF	PLF
100	3.28620 0.05344	3.22048 0.00270	3.25301 0.01195	3.30310 0.01216
200	3.26174 0.02592	3.22912 0.00135	3.24535 0.00590	3.27001 0.00595
300	3.19922 0.01237	3.18322 0.00068	3.19120 0.00289	3.20324 0.00290
400	3.10967 0.00777	3.09931 0.00045	3.10448 0.00187	3.11228 0.00187
500	3.03695 0.00555	3.02936 0.00034	3.03315 0.00137	3.03886 0.00137
n	$\beta_2=6$			
	SELF	QLF	WLF	PLF
100	6.56364 0.06136	6.43237 0.00310	6.49734 0.01372	6.59739 0.01397
200	6.51478 0.02976	6.44963 0.00155	6.48204 0.00677	6.53130 0.00683
300	6.38990 0.01421	6.35795 0.00078	6.37389 0.00331	6.39794 0.00333
400	6.21105 0.00893	6.19035 0.00052	6.20068 0.00215	6.21625 0.00215
500	6.06581 0.00638	6.05064 0.00039	6.05821 0.00157	6.06961 0.00157
n	p = 0.30			
	SELF	QLF	WLF	PLF
100	0.33300 0.00297	0.32634 0.00015	0.32964 0.00066	0.33471 0.00068
200	0.33052 0.00144	0.32722 0.00008	0.32886 0.00033	0.33136 0.00033
300	0.32419 0.00069	0.32257 0.00004	0.32337 0.00016	0.32460 0.00016
400	0.31511 0.00043	0.31406 0.00003	0.31459 0.00010	0.31538 0.00010
500	0.30774 0.00031	0.30698 0.00002	0.30736 0.00008	0.30794 0.00008

Table 2. Bayes estimates and posterior risks under uniform prior using β1 = 3, β2 = 6 and p = 0.45

n	$\beta_1 = 3$			
	SELF	QLF	WLF	PLF
100	3.26977	3.20438	3.23674	3.28658
	0.05237	0.00265	0.01171	0.01192
200	3.24543	3.21298	3.22912	3.25366
	0.02540	0.00132	0.00578	0.00583
300	3.18322	3.16730	3.17524	3.18723
	0.01213	0.00066	0.00283	0.00284
400	3.09412	3.08381	3.08896	3.09671
	0.00762	0.00044	0.00183	0.00184
500	3.02177	3.01421	3.01799	3.02366
	0.00544	0.00033	0.00134	0.00134
n	$\beta_2 = 6$			
	SELF	QLF	WLF	PLF
100	6.69491	6.56101	6.62729	6.72933
	0.06258	0.00316	0.01399	0.01425
200	6.64508	6.57863	6.61168	6.66192
	0.03036	0.00158	0.00691	0.00697
300	6.51770	6.48511	6.50136	6.52590
	0.01449	0.00079	0.00338	0.00340
400	6.33527	6.31415	6.32470	6.34058
	0.00911	0.00053	0.00219	0.00220
500	6.18712	6.17165	6.17938	6.19100
	0.00651	0.00040	0.00160	0.00161
n	$p = 0.45$			
	SELF	QLF	WLF	PLF
100	0.49284	0.48299	0.48786	0.49538
	0.00413	0.00021	0.00092	0.00094
200	0.48917	0.48428	0.48672	0.49041
	0.00200	0.00010	0.00046	0.00046
300	0.47980	0.47740	0.47859	0.48040
	0.00096	0.00005	0.00022	0.00022
400	0.46637	0.46481	0.46559	0.46676
	0.00060	0.00003	0.00014	0.00014
500	0.45546	0.45432	0.45489	0.45575
	0.00043	0.00003	0.00011	0.00011

Table 3. Bayes estimates and posterior risks under Jeffreys prior using β1 = 3, β2 = 6 and p = 0.30

n	$\beta_1 = 3$			
	SELF	QLF	WLF	PLF
100	3.26429	3.19901	3.23132	3.28108
	0.05245	0.00265	0.01173	0.01194
200	3.23999	3.20759	3.22371	3.24821
	0.02544	0.00133	0.00579	0.00584
300	3.17789	3.16200	3.16992	3.18189
	0.01215	0.00066	0.00283	0.00285
400	3.08894	3.07864	3.08378	3.09153
	0.00763	0.00044	0.00183	0.00184
500	3.01671	3.00916	3.01293	3.01860
	0.00545	0.00033	0.00134	0.00135
n	$\beta_2 = 6$			
	SELF	QLF	WLF	PLF
100	6.55488	6.42378	6.48867	6.58858
	0.06037	0.00305	0.01350	0.01374
200	6.50608	6.44102	6.47339	6.52258
	0.02928	0.00153	0.00667	0.00672
300	6.38137	6.34946	6.36538	6.38940
	0.01398	0.00076	0.00326	0.00328
400	6.20276	6.18208	6.19240	6.20795
	0.00878	0.00051	0.00211	0.00212
500	6.05771	6.04256	6.05013	6.06151
	0.00627	0.00038	0.00155	0.00155
n	$p = 0.30$			
	SELF	QLF	WLF	PLF
100	0.33081	0.32419	0.32747	0.33251
	0.00287	0.00015	0.00064	0.00065
200	0.32835	0.32506	0.32670	0.32918
	0.00139	0.00007	0.00032	0.00032
300	0.32205	0.32044	0.32125	0.32246
	0.00066	0.00004	0.00016	0.00016
400	0.31304	0.31200	0.31252	0.31330
	0.00042	0.00002	0.00010	0.00010
500	0.30572	0.30496	0.30534	0.30591
	0.00030	0.00002	0.00007	0.00007

Table 4. Bayes estimates and posterior risks under Jeffreys prior using β1 = 3, β2 = 6 and p = 0.45

n	$\beta_1 = 3$			
	SELF	QLF	WLF	PLF
100	3.24797 0.05140	3.18301 0.00260	3.21516 0.01149	3.26467 0.01170
200	3.22379 0.02493	3.19156 0.00130	3.20759 0.00568	3.23197 0.00573
300	3.16200 0.01190	3.14619 0.00065	3.15407 0.00278	3.16598 0.00279
400	3.07350 0.00748	3.06325 0.00043	3.06836 0.00180	3.07607 0.00180
500	3.00162 0.00534	2.99412 0.00032	2.99787 0.00132	3.00351 0.00132
n	$\beta_2 = 6$			
	SELF	QLF	WLF	PLF
100	6.68597 0.06157	6.55225 0.00311	6.61844 0.01377	6.72035 0.01402
200	6.63620 0.02987	6.56984 0.00156	6.60286 0.00680	6.65303 0.00686
300	6.50900 0.01426	6.47645 0.00078	6.49268 0.00333	6.51719 0.00334
400	6.32681 0.00896	6.30572 0.00052	6.31625 0.00215	6.33211 0.00216
500	6.17886 0.00640	6.16341 0.00039	6.17113 0.00158	6.18274 0.00158
n	p = 0.45			
	SELF	QLF	WLF	PLF
100	0.48960 0.00399	0.47981 0.00020	0.48465 0.00089	0.49212 0.00091
200	0.48596 0.00194	0.48110 0.00010	0.48351 0.00044	0.48719 0.00044
300	0.47664 0.00092	0.47426 0.00005	0.47545 0.00022	0.47724 0.00022
400	0.46330 0.00058	0.46176 0.00003	0.46253 0.00014	0.46369 0.00014
500	0.45247 0.00041	0.45133 0.00003	0.45190 0.00010	0.45275 0.00010

Table 5. Bayes estimates and posterior risks under uniform prior using β_1 = 6, β_2 = 9 and p = 0.45

n	$\beta_1 = 6$			
	SELF	QLF	WLF	PLF
100	6.21328 0.06556	6.15114 0.01495	6.18205 0.03122	6.22903 0.03150
200	6.11691 0.03153	6.08633 0.00747	6.10158 0.01533	6.12461 0.01540
300	6.06531 0.02062	6.04509 0.00498	6.05519 0.01013	6.07039 0.01016
400	6.07220 0.01548	6.05702 0.00374	6.06460 0.00760	6.07600 0.00762
500	5.97946 0.01200	5.96750 0.00299	5.97347 0.00599	5.98246 0.00600
n	$\beta_2 = 9$			
	SELF	QLF	WLF	PLF
100	9.73303 0.21487	9.63570 0.01119	9.68412 0.04891	9.75770 0.04934
200	9.54646 0.10257	9.49873 0.00560	9.52253 0.02393	9.55847 0.02403
300	9.27926 0.06445	9.24833 0.00373	9.26377 0.01549	9.28703 0.01554
400	9.06227 0.04604	9.03961 0.00280	9.05092 0.01134	9.06795 0.01137
500	9.00161 0.03631	8.98361 0.00224	8.99260 0.00901	9.00613 0.00903
n	p = 0.45			
	SELF	QLF	WLF	PLF
100	0.46979 0.00496	0.46509 0.00113	0.46743 0.00236	0.47098 0.00238
200	0.46250 0.00238	0.46019 0.00057	0.46134 0.00116	0.46309 0.00116
300	0.45860 0.00156	0.45707 0.00038	0.45784 0.00077	0.45899 0.00077
400	0.45912 0.00117	0.45797 0.00028	0.45855 0.00057	0.45941 0.00058
500	0.45211 0.00091	0.45121 0.00023	0.45166 0.00045	0.45234 0.00045

Table 6. Bayes estimates and posterior risks under Jeffreys prior using $\beta_1 = 6$, $\beta_2 = 9$ and $p = 0.45$

n	$\beta_1=6$			
	SELF	QLF	WLF	PLF
100	6.18205 0.06458	6.15114 0.01487	6.15114 0.03091	6.19765 0.03118
200	6.10158 0.03130	6.08633 0.00745	6.08633 0.01525	6.10924 0.01532
300	6.05519 0.02051	6.04509 0.00497	6.04509 0.01009	6.06025 0.01012
400	6.06460 0.01542	6.05702 0.00373	6.05702 0.00758	6.06839 0.00760
500	5.97347 0.01196	5.96750 0.00299	5.96750 0.00597	5.97647 0.00598
n	$\beta_2=9$			
	SELF	QLF	WLF	PLF
100	9.68412 0.21164	9.63570 0.01113	9.63570 0.04842	9.70855 0.04885
200	9.52253 0.10180	9.49873 0.00558	9.49873 0.02381	9.53449 0.02391
300	9.26377 0.06412	9.24833 0.00372	9.24833 0.01544	9.27151 0.01548
400	9.05092 0.04587	9.03961 0.00279	9.03961 0.01131	9.05659 0.01134
500	8.99260 0.03621	8.98361 0.00224	8.98361 0.00899	8.99711 0.00901
n	$p=0.45$			
	SELF	QLF	WLF	PLF
100	0.46743 0.00488	0.46509 0.00112	0.46509 0.00234	0.46861 0.00236
200	0.46134 0.00237	0.46019 0.00056	0.46019 0.00115	0.46192 0.00116
300	0.45784 0.00155	0.45707 0.00038	0.45707 0.00076	0.45822 0.00077
400	0.45855 0.00117	0.45797 0.00028	0.45797 0.00057	0.45883 0.00057
500	0.45166 0.00090	0.45121 0.00023	0.45121 0.00045	0.45188 0.00045

Table 7. Bayes estimates and posterior risks under uniform prior using $\beta_1 = 6$, $\beta_2 = 9$ and $p = 0.30$

n	$\beta_1=6$			
	SELF	QLF	WLF	PLF
100	6.34008 0.06690	6.27668 0.01525	6.30822 0.03186	6.35615 0.03214
200	6.24175 0.03218	6.21054 0.00763	6.22610 0.01564	6.24960 0.01571
300	6.18909 0.02104	6.16846 0.00508	6.17876 0.01033	6.19428 0.01036
400	6.19612 0.01579	6.18063 0.00381	6.18836 0.00775	6.20000 0.00777
500	6.10149 0.01224	6.08929 0.00305	6.09538 0.00611	6.10455 0.00612
n	$\beta_2=9$			
	SELF	QLF	WLF	PLF
100	9.69824 0.21410	9.60126 0.01115	9.64950 0.04873	9.72282 0.04917
200	9.51233 0.10221	9.46477 0.00558	9.48849 0.02384	9.52431 0.02395
300	9.24609 0.06422	9.21527 0.00372	9.23066 0.01544	9.25383 0.01548
400	9.02987 0.04588	9.00730 0.00279	9.01857 0.01130	9.03554 0.01133
500	8.96944 0.03619	8.95150 0.00223	8.96046 0.00898	8.97393 0.00899
n	$p=0.30$			
	SELF	QLF	WLF	PLF
100	0.31201 0.00329	0.30890 0.00075	0.31045 0.00157	0.31281 0.00158
200	0.30718 0.00158	0.30564 0.00038	0.30641 0.00077	0.30756 0.00077
300	0.30459 0.00104	0.30357 0.00025	0.30408 0.00051	0.30484 0.00051
400	0.30493 0.00078	0.30417 0.00019	0.30455 0.00038	0.30512 0.00038
500	0.30027 0.00060	0.29967 0.00015	0.29997 0.00030	0.30042 0.00030

Table 8. Bayes estimates and posterior risks under Jeffreys prior using $\beta_1 = 6$, $\beta_2 = 9$ and $p = 0.30$

n	$\beta_1 = 6$			
	SELF	QLF	WLF	PLF
100	6.30822	6.27668	6.27668	6.32413
	0.06589	0.01517	0.03154	0.03182
200	6.22610	6.21054	6.21054	6.23392
	0.03193	0.00761	0.01557	0.01563
300	6.17876	6.16846	6.16846	6.18393
	0.02093	0.00507	0.01030	0.01033
400	6.18836	6.18063	6.18063	6.19224
	0.01573	0.00381	0.00774	0.00775
500	6.09538	6.08929	6.08929	6.09844
	0.01221	0.00305	0.00610	0.00611
n	$\beta_2 = 9$			
	SELF	QLF	WLF	PLF
100	9.64950	9.60126	9.60126	9.67384
	0.21088	0.01109	0.04825	0.04867
200	9.48849	9.46477	9.46477	9.50041
	0.10144	0.00556	0.02372	0.02383
300	9.23066	9.21527	9.21527	9.23837
	0.06389	0.00371	0.01538	0.01543
400	9.01857	9.00730	9.00730	9.02422
	0.04571	0.00278	0.01127	0.01130
500	8.96046	8.95150	8.95150	8.96495
	0.03608	0.00223	0.00896	0.00898
n	$p = 0.30$			
	SELF	QLF	WLF	PLF
100	0.30817	0.30663	0.30663	0.30895
	0.00322	0.00074	0.00154	0.00155
200	0.30416	0.30340	0.30340	0.30454
	0.00156	0.00037	0.00076	0.00076
300	0.30185	0.30134	0.30134	0.30210
	0.00102	0.00025	0.00050	0.00050
400	0.30232	0.30194	0.30194	0.30251
	0.00077	0.00019	0.00038	0.00038
500	0.29777	0.29748	0.29748	0.29792
	0.00060	0.00015	0.00030	0.00030

Table 9. Bayes estimates and posterior risks under uniform prior using $\beta_1 = 10$, $\beta_2 = 12$ and $p = 0.45$

n	$\beta_1 = 10$			
	SELF	QLF	WLF	PLF
100	10.39357	10.28963	10.34134	10.41992
	0.10967	0.02500	0.05223	0.05269
200	10.23237	10.18121	10.20673	10.24525
	0.05275	0.01250	0.02565	0.02576
300	10.14606	10.11224	10.12912	10.15455
	0.03449	0.00833	0.01694	0.01699
400	10.15757	10.13218	10.14486	10.16394
	0.02589	0.00625	0.01271	0.01274
500	10.03245	10.01238	10.02241	10.03748
	0.02007	0.00500	0.01001	0.01003
n	$\beta_2 = 12$			
	SELF	QLF	WLF	PLF
100	13.04696	12.91649	12.98140	13.08003
	0.28803	0.01500	0.06556	0.06614
200	12.79686	12.73288	12.76479	12.81297
	0.13750	0.00750	0.03207	0.03221
300	12.43869	12.39723	12.41792	12.44910
	0.08639	0.00500	0.02077	0.02083
400	12.14781	12.11744	12.13261	12.15543
	0.06172	0.00375	0.01520	0.01524
500	12.06651	12.04237	12.05443	12.07255
	0.04868	0.00300	0.01208	0.01210
n	$p = 0.45$			
	SELF	QLF	WLF	PLF
100	0.47395	0.46921	0.47157	0.47515
	0.00500	0.00114	0.00238	0.00240
200	0.46660	0.46426	0.46543	0.46718
	0.00241	0.00057	0.00117	0.00117
300	0.46266	0.46112	0.46189	0.46305
	0.00157	0.00038	0.00077	0.00077
400	0.46319	0.46203	0.46261	0.46348
	0.00118	0.00029	0.00058	0.00058
500	0.45611	0.45520	0.45565	0.45634
	0.00092	0.00023	0.00046	0.00046

Table 10. Bayes estimates and posterior risks under Jeffreys prior using $\beta_1 = 10$, $\beta_2 = 12$ and $p = 0.45$

n	$\beta_1 = 10$			
	SELF	QLF	WLF	PLF
100	10.34134	10.28963	10.28963	10.36742
	0.10802	0.02488	0.05171	0.05216
200	10.20673	10.18121	10.18121	10.21954
	0.05235	0.01247	0.02552	0.02563
300	10.12912	10.11224	10.11224	10.13758
	0.03431	0.00832	0.01688	0.01693
400	10.14486	10.13218	10.13218	10.15121
	0.02579	0.00624	0.01268	0.01271
500	10.02241	10.01238	10.01238	10.02743
	0.02001	0.00500	0.00999	0.01001
n	$\beta_2 = 12$			
	SELF	QLF	WLF	PLF
100	12.98140	12.91649	12.91649	13.01414
	0.28370	0.01493	0.06491	0.06548
200	12.76479	12.73288	12.73288	12.78082
	0.13647	0.00748	0.03191	0.03205
300	12.41792	12.39723	12.39723	12.42830
	0.08596	0.00499	0.02070	0.02076
400	12.13261	12.11744	12.11744	12.14021
	0.06149	0.00375	0.01517	0.01520
500	12.05443	12.04237	12.04237	12.06046
	0.04853	0.00300	0.01205	0.01208
n	$p = 0.45$			
	SELF	QLF	WLF	PLF
100	0.47157	0.46921	0.46921	0.47275
	0.00493	0.00113	0.00236	0.00238
200	0.46543	0.46426	0.46426	0.46601
	0.00239	0.00057	0.00116	0.00117
300	0.46189	0.46112	0.46112	0.46227
	0.00156	0.00038	0.00077	0.00077
400	0.46261	0.46203	0.46203	0.46290
	0.00118	0.00028	0.00058	0.00058
500	0.45565	0.45520	0.45520	0.45588
	0.00091	0.00023	0.00046	0.00046

Table 11. Bayes estimates and posterior risks under uniform prior using $\beta_1 = 10$, $\beta_2 = 12$ and $p = 0.30$

n	$\beta_1 = 10$			
	SELF	QLF	WLF	PLF
100	10.64302	10.53659	10.58953	10.66999
	0.11230	0.02560	0.05348	0.05396
200	10.47795	10.42556	10.45169	10.49114
	0.05401	0.01280	0.02626	0.02638
300	10.38956	10.35493	10.37222	10.39826
	0.03531	0.00853	0.01734	0.01740
400	10.40135	10.37535	10.38833	10.40788
	0.02651	0.00640	0.01302	0.01305
500	10.24250	10.22202	10.23225	10.24764
	0.02055	0.00512	0.01025	0.01027
n	$\beta_2 = 12$			
	SELF	QLF	WLF	PLF
100	12.99912	12.86913	12.93380	13.03207
	0.28697	0.01495	0.06532	0.06590
200	12.74994	12.68619	12.71799	12.76599
	0.13699	0.00747	0.03195	0.03210
300	12.39308	12.35177	12.37239	12.40345
	0.08607	0.00498	0.02069	0.02075
400	12.10327	12.07301	12.08812	12.11086
	0.06149	0.00374	0.01515	0.01518
500	12.02226	11.99822	12.01023	12.02829
	0.04850	0.00299	0.01203	0.01206
n	$p = 0.30$			
	SELF	QLF	WLF	PLF
100	0.31596	0.31280	0.31438	0.31677
	0.00333	0.00076	0.00159	0.00160
200	0.31106	0.30951	0.31028	0.31146
	0.00160	0.00038	0.00078	0.00078
300	0.30844	0.30741	0.30793	0.30870
	0.00105	0.00025	0.00051	0.00052
400	0.30879	0.30802	0.30840	0.30898
	0.00079	0.00019	0.00039	0.00039
500	0.30407	0.30347	0.30377	0.30423
	0.00061	0.00015	0.00030	0.00030

Table 12. Bayes estimates and posterior risks under Jeffreys prior using $\beta_1 = 10$, $\beta_2 = 12$ and p = 0.30

n	$\beta_1 = 10$			
	SELF	QLF	WLF	PLF
100	10.58953 0.11062	10.53659 0.02547	10.53659 0.05295	10.61624 0.05342
200	10.45169 0.05361	10.42556 0.01277	10.42556 0.02613	10.46481 0.02624
300	10.37222 0.03514	10.35493 0.00852	10.35493 0.01729	10.38088 0.01734
400	10.38833 0.02641	10.37535 0.00639	10.37535 0.01299	10.39484 0.01301
500	10.23225 0.02049	10.22202 0.00511	10.22202 0.01023	10.23737 0.01025
n	$\beta_2 = 12$			
	SELF	QLF	WLF	PLF
100	12.93380 0.28266	12.86913 0.01487	12.86913 0.06467	12.96642 0.06524
200	12.71799 0.13597	12.68619 0.00745	12.68619 0.03179	12.73395 0.03193
300	12.37239 0.08564	12.35177 0.00497	12.35177 0.02062	12.38273 0.02068
400	12.08812 0.06126	12.07301 0.00373	12.07301 0.01511	12.09569 0.01514
500	12.01023 0.04836	11.99822 0.00299	11.99822 0.01201	12.01624 0.01203
n	p = 0.30			
	SELF	QLF	WLF	PLF
100	0.31024 0.00324	0.30869 0.00075	0.30869 0.00155	0.31102 0.00156
200	0.30620 0.00157	0.30544 0.00037	0.30544 0.00077	0.30659 0.00077
300	0.30387 0.00103	0.30337 0.00025	0.30337 0.00051	0.30413 0.00051
400	0.30435 0.00077	0.30397 0.00019	0.30397 0.00038	0.30454 0.00038
500	0.29977 0.00060	0.29947 0.00015	0.29947 0.00030	0.29992 0.00030

The simulation study indicates that by increasing the sample size the estimated values each parameter converges to the true value and the magnitude of corresponding posterior risks decreases. However, the posterior risks seem to be quite large for relatively larger values of the parameters. It can also be observed that all the parameters are over estimated with few exceptions. The extend of over estimation is more intensive for larger values of the parameters. This indicates that the posterior distributions are positively skewed. In comparison of priors, the performance of estimates under the Jeffreys prior seems better than those under uniform prior for each loss function and mixing weight. While, in case of loss functions, the estimates under quadratic loss function (QLF) are associated with the minimum risks. It is also interesting to note that the amount of risks under precautionary loss function (PLF) and weighted loss function (WLF) are converging to each other with increase in sample size. The posterior risks under QLF are approximately half of the risks under WLF and PLF. The increase in the value of mixing proportion p imposes positive impact on the performance of estimators of β_1 and it negatively affects the performance of the estimators of β_2 and vice versa. This is because the increasing value of p tend to increase the value of r_1 (the number of observations selected from the Gumbel type II distribution having parameter β_1), which will result in lesser posterior risks. The increasing values of the main parameters ($\beta_1$and β_2) are having a negative effect on the behavior of the estimators of mixing proportion p. The expressions for complete samples can simply be obtained by increasing the test termination time to unity. The risks associated with estimates under complete samples are expected to reduce as no information will be lost then.

Table 13. 95% posterior predictive intervals for $\beta_1 = 3, \beta_2 = 6$, and p = 0.30

n	Uniform		
	Lower Limit	Upper Limit	Difference
50	2.95364	16.40910	13.45546
100	2.93165	16.28695	13.35530
200	2.87545	15.97475	13.09929
300	2.79497	15.52763	12.73266
400	2.72961	15.16452	12.43490
500	2.71134	15.06302	12.35168
n	Jeffreys		
	Lower Limit	Upper Limit	Difference
50	2.65827	15.58864	12.93037
100	2.63849	15.47261	12.83412
200	2.58791	15.17601	12.58810
300	2.51548	14.75125	12.23577
400	2.45665	14.40629	11.94964
500	2.44021	14.30987	11.86966

From tables 13-16 it can be assessed that posterior predictions tend to be more accurate in larger samples. Increasing values of actual and weight parameter imposes negative impact on the performance of the predictions. It is interesting to note that the posterior predictive intervals are shorter in case of Jeffreys prior. This simply indicates the supremacy of Jeffreys prior over uniform prior. Anyhow, the results under the posterior predictive intervals are in accordance with the corresponding point estimation.

Table 14. 95% posterior predictive intervals for $\beta_1 = 3$, $\beta_2 = 6$, and $p = 0.45$

n	Uniform		
	Lower Limit	Upper Limit	Difference
50	3.01271	16.73728	13.72457
100	2.99028	16.61269	13.62241
200	2.93296	16.29424	13.36128
300	2.85087	15.83818	12.98731
400	2.78421	15.46781	12.68360
500	2.76557	15.36428	12.59871
n	Jeffreys		
	Lower Limit	Upper Limit	Difference
50	2.71144	15.90042	13.18898
100	2.69126	15.78206	13.09080
200	2.63967	15.47953	12.83986
300	2.56579	15.04627	12.48049
400	2.50578	14.69442	12.18863
500	2.48901	14.59607	12.10705

Table 15. 95% posterior predictive intervals for $\beta_1 = 6$, $\beta_2 = 9$, and $p = 0.30$

n	Uniform		
	Lower Limit	Upper Limit	Difference
50	4.43046	22.97274	18.54228
100	4.39748	22.80173	18.40426
200	4.31318	22.36465	18.05147
300	4.19246	21.73868	17.54622
400	4.09442	21.23032	17.13590
500	4.06702	21.08823	17.02121
n	Jeffreys		
	Lower Limit	Upper Limit	Difference
50	3.98741	21.82410	17.83669
100	3.95773	21.66165	17.70392
200	3.88186	21.24642	17.36455
300	3.77321	20.65175	16.87853
400	3.68498	20.16881	16.48383
500	3.66031	20.03382	16.37350

Table 16. 95% posterior predictive intervals for $\beta_1 = 3, \beta_2 = 6$, and $p = 0.30$

n	Uniform		
	Lower Limit	Upper Limit	Difference
50	4.87350	26.64838	21.77488
100	4.83723	26.45001	21.61279
200	4.74450	25.94299	21.19849
300	4.61171	25.21687	20.60516
400	4.50386	24.62717	20.12331
500	4.47372	24.46235	19.98863
n	Jeffreys		
	Lower Limit	Upper Limit	Difference
50	4.38615	25.31596	20.92981
100	4.35350	25.12751	20.77401
200	4.27005	24.64584	20.37579
300	4.15054	23.95602	19.80549
400	4.05348	23.39582	19.34234
500	4.02635	23.23923	19.21288

8. Analysis under Real Life Data

The real life data (following Gumbel distribution) regarding monthly wind speed in Cameron Highland from year 2004-2006 presented by Zaharimi et al.[18] is used to illustrate the applicability of the results obtained in previous sections.

Table 17. Bayes estimates and risks for real life data

Prior	β_1			
	SELF	QLF	WLF	PLF
Uniform	3.18955 0.30934	3.00000 0.02984	3.09422 0.09533	3.23672 0.09435
Jeffreys	3.09422 0.30098	2.33867 0.03109	3.00000 0.09422	3.14152 0.09460
	β_2			
	SELF	QLF	WLF	PLF
Uniform	3.58011 0.34722	3.36735 0.03350	3.47311 0.10700	3.63306 0.10590
Jeffreys	3.47311 0.33784	2.62504 0.03490	3.36735 0.10576	3.52620 0.10619
	$p = 0.30$			
	SELF	QLF	WLF	PLF
Uniform	0.32546 0.03157	0.30612 0.00305	0.31574 0.00973	0.33028 0.00963
Jeffreys	0.31574 0.03071	0.23864 0.00317	0.30612 0.00961	0.32056 0.00965

Table 18. Bayes estimates and risks for real life data

Prior	β_1			
	SELF	QLF	WLF	PLF
Uniform	3.28205 0.31831	3.08700 0.03071	3.18396 0.09809	3.33059 0.09708
Jeffreys	3.18396 0.30971	2.40650 0.03199	3.08700 0.09696	3.23263 0.09735
	β_2			
	SELF	QLF	WLF	PLF
Uniform	3.30346 0.32039	3.10714 0.03091	3.20473 0.09873	3.35232 0.09772
Jeffreys	3.20473 0.31173	2.42220 0.03220	3.10714 0.09759	3.25372 0.09798
	p = 0.45			
	SELF	QLF	WLF	PLF
Uniform	0.47192 0.04577	0.44388 0.00442	0.45782 0.01410	0.47890 0.01396
Jeffreys	0.45782 0.04453	0.34603 0.00460	0.44388 0.01394	0.46482 0.01400

The real life data replicated the patterns observed in the simulation study. The minimum amount of risks has been observed for the estimates under the assumption of Jeffreys prior using quadratic loss function.

9. Conclusions

The purpose of the article is to find out the appropriate combination of prior distribution and loss function to estimate the parameters of two-component mixture of Gumbel type II distribution. The parameters have been estimated under the assumption of two non-informative priors and four loss functions (symmetric and asymmetric). The posterior predictive intervals have also been evaluated. From the findings of the study it can be concluded that in order to estimate the said parameters, the use of Jeffreys prior and quadratic loss function can be preferred.

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