

# Fractional Ecosystem Model and Its Solution by Homotopy Perturbation Method

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**Abstract** In the present paper we propose an algorithm based on Homotopy Perturbation Method to solve the fractional Ecosystem model and show the efficiency and accuracy of the algorithm. This Ecosystem model is solved with time fractional derivatives in the sense of Caputo. The nonlinear terms can be easily handled by the using He's Polynomials. The numerical solutions of the Ecosystem model reveal that only a few iterations are sufficient to obtained accurate approximate analytical solutions. The numerical results obtained are presented graphically. The four different cases are considered and proved that the method is extremely effective due to fractional approach and performance. Comparing the methodology (FHPM) with the some known technique (HPM) shows that the present approach is effective and powerful. The proposed scheme finds the solution with the help of Mathematica and without any restrictive assumptions. We implement it to four different problems.

**Keywords** Nonlinear System, Fractional Ecosystem Model, Homotopy Perturbation Method, Fractional Integral Operator

## 1. Introduction

Ecosystem models play an important role to know the dynamics of the system. Such models are formed by combining known ecological relations with data gathered from field observations ([1]-[8]). These models are then used to make predictions about the dynamics of the real system. However, the study of deficiencies in the model leads to the generation of hypotheses about the possible ecological relations that are not yet known or well understood. Models enable researchers to simulate the conclusions of large-scale experiments that might be too expensive to perform on a real ecosystem. These also enable the simulation of ecological processes over very long period of time (may be some process may take centuries of time in reality), However this can be done in a matter of minutes in a computer model. Ecosystem models have applications in a wide variety of disciplines, in particular, to natural resource management. An important aspect any model evaluation is the validation, that is, the process of confirming that the model behaviour corresponds to reality. The results of the model can be validated by comparison with field data, but this technique limits the validation to a particular set of field conditions. We obtain a sequence of equations in which the solution at any stage is close to the solution at nearby stages. In the present

paper we discuss the fractional Ecosystem model and solve it using Homotopy Perturbation Method.

We consider the following fractional ecosystem model given by fractional initial value problem of the type:

$$D^\alpha x(t) = f(x(t), y(t)) \quad D^\alpha y(t) = g(x(t), y(t))$$

with initial conditions  $x(0) = 1, y(0) = -1$ . Here  $f$  and  $g$  are arbitrary functions,  $D^\alpha$  denotes the fractional derivative in the sense of Caputo.

## 2. Basic Definitions

In this section, we give the related definitions and properties of the fractional calculus from [9].

**Definition 2.1.** A real function  $h(t), t > 0$  is said to be in the space  $C_\mu, \mu \in R$ , if there exists a real number  $p > \mu$ , such that  $h(t) = t^p h_1(t)$ , where  $h_1(t) \in C(0, \infty)$ , and it is said to be in the space  $C_\mu^n$  if and only if  $h^{(n)} \in C_\mu, n \in N$ .

**Definition 2.2** The Riemann–Liouville fractional integral operator  $(J^\alpha)$  of order  $\alpha \geq 0$ , of a function,  $h \in C_\mu, \mu \geq -1$ , is defined as

$$J^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau \quad (\alpha > 0),$$

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$J^0 h(t) = h(t)$ , where  $\Gamma(z)$  is the well-known Gamma function.

Some of the properties of the operator  $J^\alpha$ , which we need here, are as follows:

For  $h \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$  and  $\gamma \geq -1$ :

$$(1) J^\alpha J^\beta h(t) = J^{\alpha+\beta} h(t),$$

$$(2) J^\alpha J^\beta h(t) = J^\beta J^\alpha h(t),$$

$$(3) J^\alpha t^\gamma = \left( \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \right) t^{\alpha+\gamma}.$$

**Definition 2.3.** [10] The fractional derivative  $D^\alpha$  of  $h(t)$  in the Caputo's sense is defined as

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau) d\tau, \quad \text{for}$$

$$n-1 < \alpha \leq n, n \in \mathbb{N}, t > 0, h \in C_{-1}^n.$$

The following are the two basic properties of the Caputo's fractional derivative [10]:

(1) Let  $h \in C_{-1}^n, n \in \mathbb{N}$ . Then  $D^\alpha h, 0 \leq \alpha \leq n$  is well defined and  $D^\alpha h \in C_{-1}$ .

(2) Let  $n-1 < \alpha \leq n, n \in \mathbb{N}$  and  $h \in C_\mu^n, \mu \geq -1$ . This leads to

$$(J^\alpha D^\alpha) h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{t^k}{k!}.$$

### 3. Analysis of HPM

The Homotopy Perturbation Method (HPM), which provides analytically an approximate solution, is applied to various nonlinear problems ([11]-[14]). In this section, we introduce a reliable algorithm to handle the nonlinear ecosystems in a realistic and efficient way. The proposed algorithm will then be used to investigate the system given by:

$$\begin{aligned} D^\alpha y_1(t) &= \sum_{i=1}^n a_{1i}(t) y_i + g_1(t, y_1, y_2, \dots, y_n) \\ D^\alpha y_2(t) &= \sum_{i=1}^n a_{2i}(t) y_i + g_2(t, y_1, y_2, \dots, y_n) \\ D^\alpha y_3(t) &= \sum_{i=1}^n a_{3i}(t) y_i + g_3(t, y_1, y_2, \dots, y_n) \\ &\vdots \\ D^\alpha y_n(t) &= \sum_{i=1}^n a_{ni}(t) y_i + g_n(t, y_1, y_2, \dots, y_n) \end{aligned} \quad (1)$$

subject to the initial condition,

$$y_1(0) = c_1, \quad y_2(0) = c_2, \quad y_n(0) = c_n, \quad (2)$$

where  $g_i$  is a nonlinear function for each  $i = 1, 2, \dots, n$ . Using the homotopy perturbation method we can construct the following homotopy for  $i = 1, 2, \dots, n$ ,

$$D^\alpha y_i(t) - \sum_{j=1}^n a_{ij}(t) y_j = p g_i(t, y_1, y_2, \dots, y_n), \quad (3)$$

where  $p \in [0, 1]$  is the homotopy parameter, which takes values from zero to unity. In case  $p = 0$ , equation (3) becomes a linear equation given by:

$$D^\alpha y_i(t) = \sum_{j=1}^n a_{ij}(t) y_j, \quad (4)$$

and when  $p$  is 1, equation (3) turns out to be the original equation given as system (1). Assuming that the solution of the system (1) is a power series in  $p$  given by:

$$y_i = y_i^{(0)} + p y_i^{(1)} + p^2 y_i^{(2)} + p^3 y_i^{(3)} + p^4 y_i^{(4)} + \dots \quad (5)$$

Substituting equation (5) in equation (3), and equating the terms having identical powers of  $p$ , we obtain a series of linear equations in the form:

$$p^0 : D^\alpha y_i^{(0)}(t) = \sum_{j=1}^n a_{ij}(t) y_j^{(0)}, \quad y_i^{(0)}(0) = c_i,$$

$$\begin{aligned} p^1 : D^\alpha y_i^{(1)}(t) &= \sum_{j=1}^n a_{ij}(t) y_j^{(1)} + g_i^{(1)}(t, y^{(0)}), \quad y_i^{(1)}(0) = 0 \\ &= \sum_{j=1}^n a_{ij}(t) y_j^{(1)} + g_i^{(1)}(t, y^{(0)}), \quad y_i^{(1)}(0) = 0 \end{aligned}$$

$$\begin{aligned} p^2 : D^\alpha y_i^{(2)}(t) &= \sum_{j=1}^n a_{ij}(t) y_j^{(2)} + g_i^{(2)}(t, y^{(0)}, y^{(1)}), \quad y_i^{(2)}(0) = 0 \\ &= \sum_{j=1}^n a_{ij}(t) y_j^{(2)} + g_i^{(2)}(t, y^{(0)}, y^{(1)}), \quad y_i^{(2)}(0) = 0 \end{aligned}$$

$$\begin{aligned} p^3 : D^\alpha y_i^{(3)}(t) &= \sum_{j=1}^n a_{ij}(t) y_j^{(3)} + g_i^{(3)}(t, y^{(0)}, y^{(1)}, y^{(2)}), \quad y_i^{(3)}(0) = 0 \\ &= \sum_{j=1}^n a_{ij}(t) y_j^{(3)} + g_i^{(3)}(t, y^{(0)}, y^{(1)}, y^{(2)}), \quad y_i^{(3)}(0) = 0 \\ &\vdots \end{aligned}$$

where the function  $g_i^{(1)}, g_i^{(2)}, g_i^{(3)} \dots$  satisfy the following equation:

$$\begin{aligned} g_i &\left( t, y_1^{(0)} + p y_1^{(1)} + p^2 y_1^{(2)} + p^3 y_1^{(3)} + \dots, \right. \\ &\quad \left. y_n^{(0)} + p y_n^{(1)} + p^2 y_n^{(2)} + \dots \right) \\ &= g_i^{(1)}(t, y_1^{(0)}, y_2^{(0)}, \dots, y_n^{(0)}) \\ &\quad + p g_i^{(2)}(t, y_1^{(0)}, y_2^{(0)}, \dots, y_n^{(0)}, y_1^{(1)}, y_2^{(1)}, \dots, y_n^{(1)}) \\ &\quad + p^2 g_i^{(3)}(t, y_1^{(0)}, y_2^{(0)}, \dots, y_n^{(0)}, y_1^{(1)}, y_2^{(1)}, \dots, y_n^{(1)}, y_1^{(2)}, y_2^{(2)}, \dots, y_n^{(2)}) + \dots \end{aligned}$$

Now taking  $p = 1$  in the equation (5) it gives the solution of the system (1). It is obvious that the above linear equations are easy to solve, and the components  $y_i^{(k)}$ ,

$k \geq 0$ , of the homotopy perturbation solution can be completely determined as a series solution,

Finally, we approximate the solution  $y_i(t) = \sum_{k=0}^{\infty} p^k y_i^{(k)}(t)$  by the truncated series

$$\phi_i(t) = \sum_{k=0}^{N-1} p^k y_i^{(k)}(t). \quad (6)$$

It is also useful, for the system (1) to construct the homotopy, for  $i = 1, 2, \dots, n$

$$D^\alpha y_i(t) - p \sum_{j=1}^n a_{ij}(t) y_j = p g_i(t, y_1, y_2, \dots, y_n) \quad (7)$$

where  $p \in [0, 1]$ . In this case, the term  $\sum_{j=1}^n a_{ij}(t) y_j^{(0)}$  is combined with the component  $y_i^{(1)}$  and the term  $\sum_{j=1}^n a_{ij}(t) y_j^{(1)}$  is combined with the component  $y_i^{(2)}$  and so on. This variation reduces the number of terms in each component and also minimizes the size of calculations. By substituting equation (5) in equation (7) we obtain the following series of linear equations:

$$\begin{aligned} p^0 : D^\alpha y_i^{(0)}(t) &= 0, \quad y_i^{(0)}(0) = c_i, \\ p^1 : D^\alpha y_i^{(1)}(t) &= \sum_{j=1}^n a_{ij}(t) y_j^{(0)} + g_i^{(1)}(t, y^{(0)}), \quad y_i^{(1)}(0) = 0 \\ p^2 : D^\alpha y_i^{(2)}(t) &= \sum_{j=1}^n a_{ij}(t) y_j^{(1)} + g_i^{(2)}(t, y^{(0)}, y^{(1)}), \quad y_i^{(2)}(0) = 0 \\ p^3 : D^\alpha y_i^{(3)}(t) &= \sum_{j=1}^n a_{ij}(t) y_j^{(2)} + g_i^{(3)}(t, y^{(0)}, y^{(1)}, y^{(2)}), \quad y_i^{(3)}(0) = 0 \end{aligned}$$

$$p^4 : D^\alpha y_i^{(4)}(t) =$$

$$\sum_{j=1}^n a_{ij}(t) y_j^{(3)} + g_i^{(4)}(t, y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}), \quad y_i^{(4)}(0) = 0$$

$\vdots$

## 4. Numerical Example

In this section we show the application of the algorithm based on HPM given above to solve four different linear and non linear fractional Ecosystem model.

**Example 4.1.** Consider the linear system [15]

$$D^\alpha x(t) = y(t),$$

$$D^\alpha y(t) = 2x(t) - y(t), \quad (8)$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = -1, \quad (9)$$

We construct the Homotopy

$$D^\alpha x(t) = p y(t)$$

$$D^\alpha y(t) = p(2x(t) - y(t))$$

To solve the system given in Eq. (8) we construct the Homotopy as given in Eq. (7) putting Eq. (5) then initial condition (9) in Eq. (7) and then comparing the like powers of  $p$  we get,

$$p^0 : D^\alpha x^{(0)} = 0, \quad p^0 : D^\alpha y^{(0)} = 0, \quad (10)$$

$$p^1 : D^\alpha x^{(1)} = y^{(0)}, \quad p^1 : D^\alpha y^{(1)} = 2x^{(0)} - y^{(0)}, \quad (11)$$

$$p^2 : D^\alpha x^{(2)} = y^{(1)}, \quad p^2 : D^\alpha y^{(2)} = 2x^{(1)} - y^{(1)}, \quad (12)$$

$$p^3 : D^\alpha x^{(3)} = y^{(2)}, \quad p^3 : D^\alpha y^{(3)} = 2x^{(2)} - y^{(2)}, \quad (13)$$

$$p^4 : D^\alpha x^{(4)} = y^{(3)}, \quad p^4 : D^\alpha y^{(4)} = 2x^{(3)} - y^{(3)}, \quad (14)$$

$$p^5 : D^\alpha x^{(5)} = y^{(4)}, \quad p^5 : D^\alpha y^{(5)} = 2x^{(4)} - y^{(4)}, \quad (15)$$

Using the operator  $J_t^\alpha$ , the inverse operator of  $D^\alpha$ , on both side of linear equations from (10)–(15) with initial conditions (9) we get

$$x^{(0)} = x^{(0)}(0) = 1, \quad x^{(1)} = J_t^\alpha [y^{(0)}] = (-1) \frac{t^\alpha}{(1+\alpha)},$$

$$x^{(2)} = J_t^\alpha [y^{(1)}] = (3) \frac{t^{2\alpha}}{(1+2\alpha)}, \quad x^{(3)} = J_t^\alpha [y^{(2)}] = (-5) \frac{t^{3\alpha}}{(1+3\alpha)},$$

$$x^{(4)} = J_t^\alpha [y^{(3)}] = (11) \frac{t^{4\alpha}}{(1+4\alpha)}, \quad x^{(5)} = J_t^\alpha [y^{(4)}] = (-21) \frac{t^{5\alpha}}{(1+5\alpha)},$$

$$\begin{aligned}
 y^{(0)} &= y^{(0)}(0) = -1, \quad y^{(1)} = J_t^\alpha [2x^{(0)} - y^{(0)}] = (3) \frac{t^\alpha}{\Gamma(1+\alpha)}, \\
 y^{(2)} &= J_t^\alpha [2x^{(1)} - y^{(1)}] = (-5) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad y^{(3)} = J_t^\alpha [2x^{(2)} - y^{(2)}] = (11) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\
 y^{(4)} &= J_t^\alpha [2x^{(3)} - y^{(3)}] = (-21) \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}, \quad y^{(5)} = J_t^\alpha [2x^{(4)} - y^{(4)}] = (43) \frac{t^{5\alpha}}{\Gamma(1+5\alpha)},
 \end{aligned}$$

The solution of (8) is given by:

$$\begin{aligned}
 x(t) &= \sum_{i=0}^{\infty} x^{(i)}(t) = 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + 3 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - 5 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + 11 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} - 21 \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} + \dots \\
 y(t) &= \sum_{i=0}^{\infty} y^{(i)}(t) = -1 + 3 \frac{t^\alpha}{\Gamma(1+\alpha)} - 5 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + 11 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - 21 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + 43 \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} + \dots
 \end{aligned}$$

the graph of which is given below:

From Fig. 1 and Fig. 2 we see that the solution obtained by the proposed algorithm for  $\alpha = 1$  is same as the solution obtained by [16]. Also figures 1 and 2 show the solution for other different values of  $\alpha$ . Fig 1 and 2 show the solutions in red by HPM of the original system and by blue thick dots by HPM of the fractional ecosystem model.

**Example 4.2.** Consider the predator – prey system [15].

$$D^\alpha x(t) = x(t) - x(t)y(t),$$

$$D^\alpha y(t) = -y(t) + x(t)y(t), \quad (16)$$

subject with the initial conditions

$$x(0) = 1, \quad y(0) = 0.5, \quad (17)$$

We construct the Homotopy

$$D^\alpha x(t) = p(x(t) - x(t)y(t)),$$

$$D^\alpha y = p(-y(t) + x(t)y(t)),$$

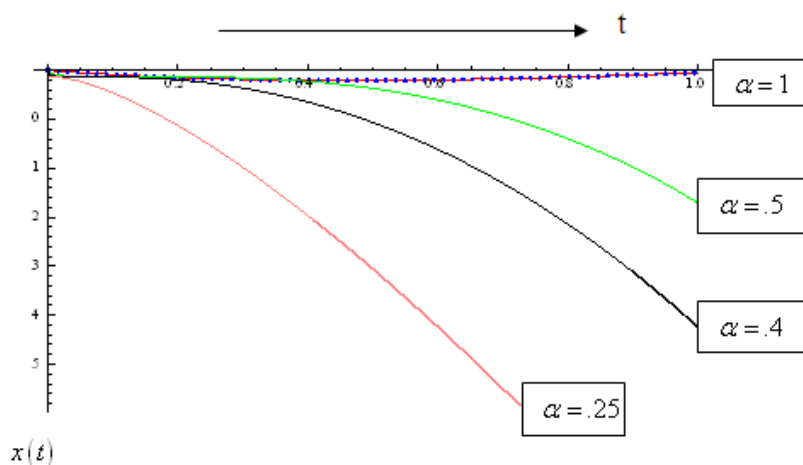


Figure 1. Plot of  $x(t)$  vs  $t$

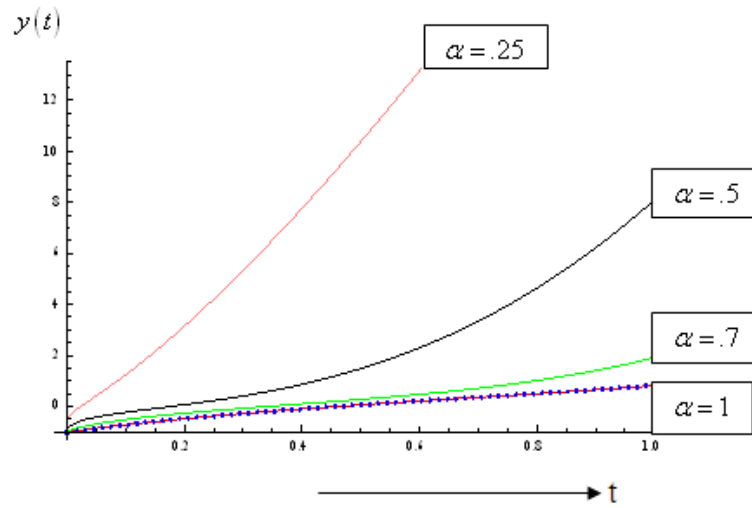


Figure 2. Plot of  $y(t)$  vs  $t$

To solve the system given in Eq. (16) we construct the Homotopy as given in Eq. (7) putting Eq. (5) then initial condition (17) in Eq. (7) and then comparing the like powers of  $p$  we get,

$$p^0 : D^\alpha x^{(0)} = 0, \quad p^0 : D^\alpha y^{(0)} = 0, \quad (18)$$

$$p^1 : D^\alpha x^{(1)} = x^{(0)} - x^{(0)}y^{(0)}, \quad (19)$$

$$p^1 : D^\alpha y^{(1)} = x^{(0)}y^{(0)} - y^{(0)}, \quad (20)$$

$$p^2 : D^\alpha x^{(2)} = x^{(1)} - x^{(0)}y^{(1)} - x^{(1)}y^{(0)}, \quad (21)$$

$$p^2 : D^\alpha y^{(2)} = y^{(0)}x^{(1)} - y^{(1)} + x^{(0)}y^{(1)},$$

$$p^3 : D^\alpha x^{(3)} = x^{(2)} - x^{(0)}y^{(2)} - x^{(1)}y^{(1)} - x^{(2)}y^{(0)},$$

$$p^3 : D^\alpha y^{(3)} = y^{(0)}x^{(2)} - y^{(2)} + x^{(0)}y^{(2)} - x^{(1)}y^{(1)},$$

$$\vdots$$

Now using the operator  $J_t^\alpha$ , the inverse operator of  $D^\alpha$ , on both side of linear equations from (18)–(21) with initial condition (17) we get,

$$x^{(0)} = x^{(0)}(0) = 1, \quad x^{(1)} = J_t^\alpha [x^{(0)} - x^{(0)}y^{(0)}] = (0.5) \frac{t^\alpha}{(1+\alpha)},$$

$$x^{(2)} = J_t^\alpha [x^{(1)} - x^{(0)}y^{(1)} - x^{(1)}y^{(0)}] = (0.25) \frac{t^{2\alpha}}{(1+2\alpha)},$$

$$x^{(3)} = J_t^\alpha [x^{(2)} - x^{(0)}y^{(2)} - x^{(1)}y^{(1)} - x^{(2)}y^{(0)}] = (0.125) \frac{t^{3\alpha}}{(1+3\alpha)},$$

$$y^{(0)} = y^{(0)}(0) = 0.5, \quad y^{(1)} = J_t^\alpha [x^{(0)}y^{(0)} - y^{(0)}] = 0,$$

$$y^{(2)} = J_t^\alpha [y^{(0)}x^{(1)} - y^{(1)} + x^{(0)}y^{(1)}] = (.25) \frac{t^{2\alpha}}{(1+2\alpha)},$$

$$y^{(3)} = J_t^\alpha [y^{(0)}x^{(2)} - y^{(2)} + y^{(2)}x^{(0)} + y^{(1)}x^{(1)}] = (.125) \frac{t^{3\alpha}}{(1+3\alpha)},$$

hence the solution of (16) is given by

$$x(t) = \sum_{i=0}^{\infty} x^{(i)}(t) = 1 + (0.5) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + (0.25) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + (.125) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots$$

$$y(t) = \sum_{i=0}^{\infty} y^{(i)}(t) = (0.5) + (0.25) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + (.125) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots$$

This is the accurate and exact solution of the system (16), [7], using Homotopy Perturbation Method.

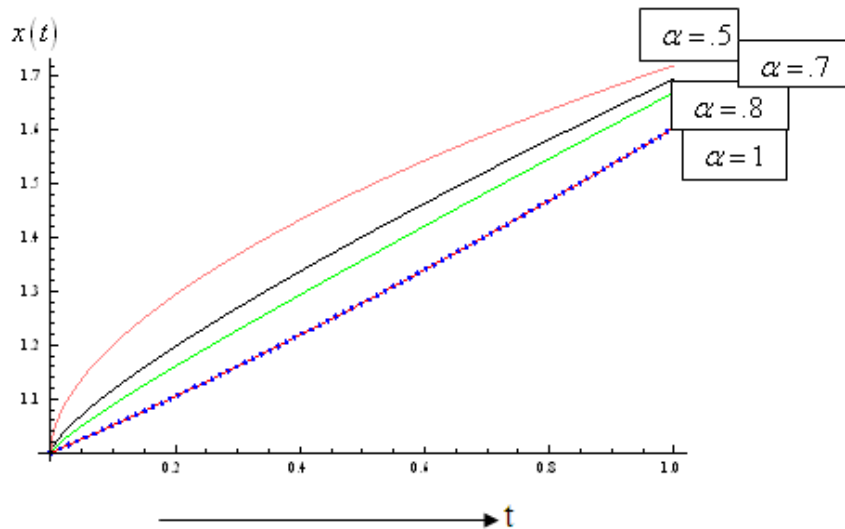


Figure 3. Plot of  $x(t)$  vs  $t$

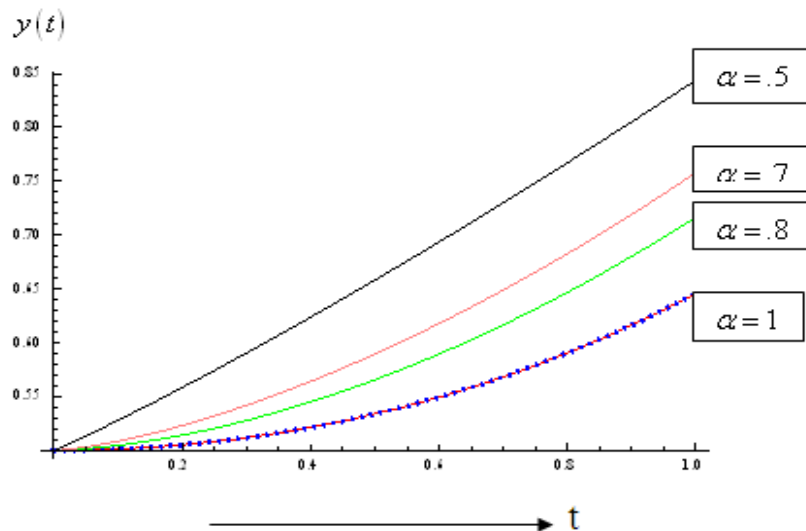


Figure 4. Plot of  $y(t)$  vs  $t$

From the Fig. 3 and Fig. 4 we see that the solution obtained by the proposed algorithm for  $\alpha = 1$  is same as the solution obtained by [16]. Also fig. 3 and 4 show the solution for other different values of  $\alpha$ .

Figure 3 and 4 show the solutions of HPM to the ecosystem model by red colour and that of the fractional ecosystem model by blue dots.

**Example 4.3.** Consider the predator – prey system

$$D^\alpha x(t) = ax(t) - bx(t)y(t) - cx(t)z(t), \quad (22)$$

$$D^\alpha y(t) = -dy(t) + ex(t)y(t) - fy(t)z(t),$$

$$D^\alpha z(t) = -gz(t) + hx(t)z(t) - iy(t)z(t),$$

where  $a, b, c, d, e, f, g, h$  and  $i$  are constants, subject to the initial conditions

$$x(0) = c_1, \quad y(0) = c_2, \quad z(0) = c_3, \quad (23)$$

We construct the Homotopy

$$D^\alpha x(t) = p(ax(t) - bx(t)y(t) - cx(t)z(t)),$$

$$D^\alpha y = p(-dy(t) + ex(t)y(t) - fy(t)z(t)),$$

$$D^\alpha z(t) = p(-gz(t) + hx(t)z(t) - iy(t)z(t)),$$

To solve the system given in Eq. (22) we construct the Homotopy as given in Eq. (7) putting Eq. (5) then initial condition (23) in Eq. (7) and then comparing the like powers of  $p$  we get,

$$p^0 : D^\alpha x^{(0)} = 0, \quad p^0 : D^\alpha y^{(0)} = 0, \quad p^0 : D^\alpha z^{(0)} = 0, \quad (24)$$

$$p^1 : D^\alpha x^{(1)} = ax^{(0)} - bx^{(0)}y^{(0)} - cx^{(0)}z^{(0)},$$

$$p^1 : D^\alpha y^{(1)} = -dy^{(0)} + ex^{(0)}y^{(0)} - fy^{(0)}z^{(0)}, \quad (25)$$

$$p^1 : D^\alpha z^{(1)} = -gz^{(0)} + hx^{(0)}z^{(0)} - iy^{(0)}z^{(0)},$$

$$p^2 : D^\alpha x^{(2)} = ax^{(1)} - b(x^{(0)}y^{(1)} + x^{(1)}y^{(0)}) - c(x^{(0)}z^{(1)} + x^{(1)}z^{(0)}),$$

$$p^2 : D^\alpha y^{(2)} = -dy^{(1)} + e(x^{(0)}y^{(1)} + x^{(1)}y^{(0)}) - f(y^{(0)}z^{(1)} + y^{(1)}z^{(0)}), \quad (26)$$

$$p^2 : D^\alpha z^{(2)} = -gz^{(1)} + h(x^{(0)}z^{(1)} + x^{(1)}z^{(0)}) + i(y^{(0)}z^{(1)} + y^{(1)}z^{(0)}),$$

$\vdots$

Using the operator  $J_t^\alpha$  the inverse operator of  $D^\alpha$  on both side of linear equations from (24)–(26) with initial condition (23) we get,

$$x^{(0)} = x^{(0)}(0) = c_1,$$

$$x^{(1)} = J_t^\alpha [ax^{(0)} - bx^{(0)}y^{(0)} - cx^{(0)}z^{(0)}] = c_1(a - bc_2 - cc_3) \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$\begin{aligned} x^{(2)} &= J_t^\alpha [ax^{(1)} - b(x^{(0)}y^{(1)} + x^{(1)}y^{(0)}) - c(x^{(0)}z^{(1)} + x^{(1)}z^{(0)})] \\ &= \left( ac_1(a - bc_2 - cc_3) - bc_1c_2(-d + ec_1 - fc_3 + a - bc_2 - cc_3) \right. \\ &\quad \left. - cc_1c_2(-g + hc_1 + ic_2 + a - bc_2 - cc_3) \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \end{aligned}$$

$$y^{(0)} = y^{(0)}(0) = c_2,$$

$$y^{(1)} = J_t^\alpha [-dy^{(0)} + ex^{(0)}y^{(0)} - fy^{(0)}z^{(0)}] = c_2(-d - ec_1 - fc_3) \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$\begin{aligned} y^{(2)} &= J_t^\alpha [-dy^{(1)} + e(x^{(0)}y^{(1)} + x^{(1)}y^{(0)}) - f(y^{(0)}z^{(1)} + y^{(1)}z^{(0)})] \\ &= \left( -dc_2((-d + ec_1 - fc_3) + ec_1c_2(-d + ec_1 - fc_3 + a - bc_2 - cc_3)) \right. \\ &\quad \left. - fc_2c_3(-g + hc_1 + ic_2 - d + ec_1 - fc_3) \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \end{aligned}$$

$$\begin{aligned}
z^{(0)} &= z^{(0)}(0) = c_3, \\
z^{(1)} &= J_t^\alpha \left[ -gz^{(0)} + hx^{(0)}z^{(0)} - iy^{(0)}z^{(0)} \right] = c_3(-g + hc_1 - ic_2) \frac{t^\alpha}{\Gamma(1+\alpha)}, \\
z^{(2)} &= J_t^\alpha \left[ -gz^{(1)} + h(x^{(0)}z^{(1)} + x^{(1)}z^{(0)}) + i(y^{(0)}z^{(1)} + y^{(1)}z^{(0)}) \right] \\
&= \left( -gc_3((-g + hc_1 - ic_2) + hc_1c_3(-g + hc_1 + ic_2 + a - bc_2 - cc_3)) \right. \\
&\quad \left. + ic_2c_3(-g - d + hc_1 + ic_2 + ec_1 - fc_3) \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}
\end{aligned}$$

Hence the solution is given as

$$\begin{aligned}
x(t) &= \sum_{i=0}^{\infty} x^{(i)}(t) \\
&= c_1 + c_1(a - bc_2 - cc_3) \frac{t^\alpha}{\Gamma(1+\alpha)} + \left( ac_1(a - bc_2 - cc_3) - bc_1c_2(-d + ec_1 - fc_3 + a - bc_2 - cc_3) \right. \\
&\quad \left. - cc_1c_2(-g + hc_1 + ic_2 + a - bc_2 - cc_3) \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
y(t) &= \sum_{i=0}^{\infty} y^{(i)}(t) \\
&= c_2 + c_2(-d - ec_1 - fc_3) \frac{t^\alpha}{\Gamma(1+\alpha)} \\
&\quad + \left( -dc_2((-d + ec_1 - fc_3) + ec_1c_2(-d + ec_1 - fc_3 + a - bc_2 - cc_3)) \right. \\
&\quad \left. - fc_2c_3(-g + hc_1 + ic_2 - d + ec_1 - fc_3) \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
z(t) &= \sum_{i=0}^{\infty} z^{(i)}(t) \\
&= c_3 + c_3(-g + hc_1 - ic_2) \frac{t^\alpha}{\Gamma(1+\alpha)} \\
&\quad + \left( -gc_3((-g + hc_1 - ic_2) + hc_1c_3(-g + hc_1 + ic_2 + a - bc_2 - cc_3)) + ic_2c_3(-g - d + hc_1 + ic_2 + ec_1 - fc_3) \right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}
\end{aligned}$$

which is the accurate /exact solution of the system (22) using Homotopy Perturbation Method .

**Example 4.4.** Consider the system [17]

$$\begin{aligned}
D^\alpha x(t) &= a_0x(t) - b_0x^2(t) - \frac{v_0x(t)y(t)}{d_0 + x(t)}, \\
D^\alpha y(t) &= -a_1y(t) + \frac{v_1x(t)y(t)}{d_1 + x(t)} - \frac{v_2y(t)z(t)}{d_2 + y(t)}, \\
D^\alpha z(t) &= a_2z(t) - \frac{v_3z^2(t)}{d_3 + y(t)},
\end{aligned} \tag{27}$$

Where  $a_0, b_0, d_0, v_0, a_1, v_1, d_1, a_2, v_2, d_2, v_3$ , and  $d_3$  are model parameters assuming only positive values , with the initial conditions

$$x(0) = c_1, \quad y(0) = c_2, \quad z(0) = c_3, \tag{28}$$

multiplying the first equation of (27) by the factor  $(d_0 + x(t))$  , the second equation of (27) by the factor  $((d_1 + x(t)), (d_2 + y(t)))$  and the third equation (27) by the factor  $(d_3 + y(t))$  To solve the system given in Eq.



(27) we construct the Homotopy as given in Eq. (7) putting Eq. (5) then initial condition (28) in Eq. (7) and then comparing the like powers of  $P$  we get,

$$p^0 : D^\alpha x^{(0)} = 0, \quad p^0 : D^\alpha y^{(0)} = 0, \quad p^0 : D^\alpha z^{(0)} = 0, \quad (29)$$

$$\begin{aligned} p^1 : (d_0 + x^{(0)}) D^\alpha x^{(1)} &= a_0 d_0 x^{(0)} + a_0 (x^{(0)})^{(2)} - b_0 d_0 (x^{(0)})^{(2)} - b_0 (x^{(0)})^{(3)} - v_0 x^{(0)} y^{(0)}, \\ p^1 : (d_1 d_2 + d_1 y^{(0)} + d_2 x^{(0)} + x^{(0)} y^{(0)}) D^\alpha y^{(1)} &= \\ &-a_1 d_1 d_2 y^{(0)} - a_1 d_1 (y^{(0)})^{(2)} - a_1 d_2 x^{(0)} y^{(0)} - a_1 x^{(0)} (y^{(0)})^{(2)} \\ &+ v_1 d_2 x^{(0)} y^{(0)} + v_1 x^{(0)} (y^{(0)})^{(2)} - v_2 d_1 y^{(0)} z^{(0)} - v_2 x^{(0)} y^{(0)} z^{(0)} \\ p^1 : (d_3 + y^{(0)}) D^\alpha z^{(1)} &= -a_2 d_3 z^{(0)} + a_2 y^{(0)} z^{(0)} - v_3 (z^{(0)})^{(2)}, \end{aligned} \quad (30)$$

using the operator  $J_t^\alpha$  the inverse operator of  $D^\alpha$  on both side of linear equations from (29)–(30) with initial condition (28) we obtained,

$$\begin{aligned} x^{(0)} &= x^{(0)}(0) = c_1, \\ x^{(1)} &= J_t^\alpha \left[ \frac{a_0 d_0 x^{(0)} + a_0 (x^{(0)})^{(2)} - b_0 d_0 (x^{(0)})^{(2)} - b_0 (x^{(0)})^{(3)} - v_0 x^{(0)} y^{(0)}}{(d_0 + x^{(0)})} \right] \\ &= \left( a_0 c_1 - b_0 (c_1)^{(2)} - \frac{v_0 c_1 c_3}{d_0 + c_1} \right) \frac{t^\alpha}{(1+\alpha)} \\ y^{(0)} &= y^{(0)}(0) = c_2, \\ y^{(1)} &= J_t^\alpha \left[ \frac{-a_1 d_1 d_2 y^{(0)} - a_1 d_1 (y^{(0)})^{(2)} - a_1 d_2 x^{(0)} y^{(0)} - a_1 x^{(0)} (y^{(0)})^{(2)} + v_1 d_2 x^{(0)} y^{(0)} + v_1 x^{(0)} (y^{(0)})^{(2)} - v_2 d_1 y^{(0)} z^{(0)} - v_2 x^{(0)} y^{(0)} z^{(0)}}{(d_1 d_2 + d_1 y^{(0)} + d_2 x^{(0)} + x^{(0)} y^{(0)})} \right] \\ &= \left( -a_1 c_2 + \frac{v_1 c_1 c_2}{d_1 + c_1} - \frac{v_2 c_2 c_3}{d_2 + c_2} \right) \frac{t^\alpha}{(1+\alpha)} \\ z^{(0)} &= z^{(0)}(0) = c_3, \\ z^{(1)} &= J_t^\alpha \left[ \frac{-a_2 d_3 z^{(0)} + a_2 y^{(0)} z^{(0)} - v_3 (z^{(0)})^{(2)}}{(d_3 + y^{(0)})} \right] \\ &= \left( a_2 c_3 - \frac{v_3 (c_3)^{(2)}}{d_3 + c_2} \right) \frac{t^\alpha}{(1+\alpha)} \end{aligned}$$

Hence the solution is given by

$$\begin{aligned}
x(t) &= \sum_{i=0}^{\infty} x^{(i)}(t) = c_1 + \left( a_0 c_1 - b_0 (c_1)^{(2)} - \frac{v_0 c_1 c_3}{d_0 + c_1} \right) \frac{t^\alpha}{\Gamma(1+\alpha)} + \dots \\
y(t) &= \sum_{i=0}^{\infty} y^{(i)}(t) = c_2 + \left( -a_1 c_2 + \frac{v_1 c_1 c_2}{d_1 + c_1} - \frac{v_2 c_2 c_3}{d_2 + c_2} \right) \frac{t^\alpha}{\Gamma(1+\alpha)} + \dots \\
z(t) &= \sum_{i=0}^{\infty} z^{(i)}(t) = c_3 + \left( a_2 c_3 - \frac{v_3 (c_3)^{(2)}}{d_3 + c_2} \right) \frac{t^\alpha}{\Gamma(1+\alpha)} + \dots
\end{aligned}$$

This is closed form of solution of the system (27) using HPM.

## 4. Conclusions

In this paper we have introduced an algorithm based Homotopy Perturbation Method to solve the fractional ecosystem model. The method has been applied to get exact and accurate solution for linear and nonlinear systems of fractional Ecosystem model, the solution obtained matches with the solution of the original ecosystem for  $\alpha=1$  and both the variables (populations) increases with time while value of  $\alpha$  decreases and the solution converges to the original solution of the linear and nonlinear ecosystem model as the fractional parameter  $\alpha$  tends to 1 as found in two examples. This method can also be applied to similar linear and nonlinear models in other systems. Further it is also verified that the FHPM is more efficient than HPM.

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