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Adaptive Model Reference Hierarchical Sliding Mode Control of Uncertain Underactuated Systems with Time Delay and Dead-Zone Input

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Abstract This paper focuses on a problem of adaptive model reference hierarchical sliding mode control for a class of uncertain underactuated systems with unknown dead-zone and time delay. An incremental hierarchical structure sliding-mode controller (IHSSMC) strategy based on the reference model is presented, which drives the system output to follow the reference model. Fuzzy logic systems are utilized to approximate the unknown nonlinear functions by some adaptive laws. An incremental hierarchical structure sliding-mode controller (IHSSMC) is developed by introducing the incremental hierarchical structure into sliding mode control (SMC) method. By choosing an appropriate Lyapunov–Krasovskii function, the proposed controller is designed to demonstrate that all the signals in the closed-loop system can not only guarantee uniformly ultimately bounded, but also achieve good tracking performance. Finally, some computer simulation results of a practical example are illustrated to verify the effectiveness of the proposed approach.

Keywords Underactuated system, Lyapunov–Krasovskii function, Incremental hierarchical structure, Sliding mode control, Model reference control, Time delay, Dead-zone input, Fuzzy logic systems

1. Introduction

In the past two decades, underactuated systems have received much attention and some attractive results have been presented in the literature. Systems are said to be underactuated when the control actuators are fewer than number of degrees of freedom. They have more advantages in case of the reduced weight, cost, energy consumption, and the system structure design than actuated ones. Related problems have been presented in many actual systems such as, the crane system [1], underwater vehicles [2], the four-link manipulator [3], the axisymmetric spacecraft [4], hypersonic vehicles [5], the linear quadratic regulator (LQR) [6], the pendubot [7], the inertia-wheel pendulum (IWP) [8-10], and so on.

Dead-zone input nonlinearity is a nonsmooth function that features certain insensitivity for small control inputs which is often encountered in a variety of practical systems such as the four-link PAAA (Passive-Active-Active-Active) manipulator, the single-link flexible joint manipulator, and so on. The work [11] investigated the position-posture control problem of a planar four-link underactuated manipulator. The problem of fuzzy adaptive control design and discretization for a class of nonlinear uncertain systems was studied in [12]. In [13], an observed-based adaptive fuzzy tracking controller has been presented for switched nonlinear systems with dead-zone.

Fuzzy logic systems (FLSs) with appropriate adaptive laws and projection algorithms are introduced to approximate the unknown nonlinear functions appearing in the structure of the underactuated system. A FLS consists of four parts: the knowledge base, the fuzzifier, the fuzzy inference engine working on fuzzy rules, and the defuzzifier. By using Lyapunov-Krasovskii stability theorem, some adaptive laws are presented for fuzzy model and time delay.

The sliding-mode control technique have some advantages such as insensitivity to system parameter variations, invariance to external disturbances, good transient performance, fast response, and so on. Basically, SMC laws comprise two parts: switching controller design and equivalent controller design. The switching control law is employed to lead the system's states to a given sliding surface and the equivalent control law guarantees the system's states to stay on the aforementioned sliding surface and converge to zero along the sliding surface. As a result of the underactuated characteristic of the controlled system, at least two sliding surfaces with specific relation are utilized for the controller design. The sliding-mode controller on the basis of the incremental hierarchical structure and

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aggregated hierarchical structure were used to deal with the under-actuated system. For the incremental hierarchical structure of a sliding-mode controller, one of the subsystems is defined as the first layer sliding surface. Next, the first-layer sliding surface and one of the left states are used to establish the second-layer sliding surface. The process continues till all the subsystems are included.

The key idea of model reference hierarchical sliding mode control is motivated by the concept of model reference sliding mode control. The model reference hierarchical sliding mode control is to combine or integrate model reference and aforementioned hierarchical sliding mode control in such a way that the advantages of both techniques can be realized. In this paper, incremental hierarchical structure sliding-mode controller (IHSSMC) scheme based on reference model is proposed, which tracks the reference model by itself.

In this paper, an adaptive model reference hierarchical sliding mode control method is developed for a class of uncertain underactuated systems with time delay and dead-zone input. A fuzzy logic system with adaptive laws is introduced to approximate the nonlinear functions appearing in the structure of the underactuated system. Choosing an appropriate Lyapunov-Krasovskii function, it is theoretically verified that all the signals in the closed-loop system are uniformly ultimately bounded under our designed adaptive model reference control method. By introducing a sliding-mode controller with incremental hierarchical structure, the errors between the system outputs and model reference outputs are driven onto the sliding surface and kept on the surface afterward.

The rest of this paper is organized as follows. In Section II, the problem statement and basic preliminaries together with the definition of fuzzy logic systems (FLSs) are given. In Section III, the controller design based on the model reference hierarchical sliding mode control method is proposed to cope with the control problem of a class of uncertain underactuated nonlinear time-delay systems with dead-zone input. The stability analysis of the whole system is also verified by Lyapunov-Krasovskii functional. Simulation results are provided in Section IV to demonstrate the advantages and effectiveness of the proposed approaches. Finally, the concluding remarks are gathered in Section V.

2. Problem Statement and Preliminaries

2.1. System Description

Consider a class of single-input-multi-output uncertain underactuated nonlinear systems with unknown dead-zone input and time delay as follows:

$$\begin{cases} \dot{x}_1 = x_2 + d_1(\mathbf{x},t) \\ \dot{x}_2 = f_1(\mathbf{x}(t-\tau)) + b_1(\mathbf{x})D(u(t)) + d_2(\mathbf{x},t) \\ \dot{x}_3 = x_4 + d_3(\mathbf{x},t) \\ \dot{x}_4 = f_2(\mathbf{x}(t-\tau)) + b_2(\mathbf{x})D(u(t)) + d_4(\mathbf{x},t) \\ \qquad\vdots \\ \dot{x}_{2n-1} = x_{2n} + d_{2n-1}(\mathbf{x},t) \\ \dot{x}_{2n} = f_n(\mathbf{x}(t-\tau)) + b_n(\mathbf{x})D(u(t)) + d_{2n}(\mathbf{x},t) \end{cases}$$

$$\mathbf{y}(t) = \left[x_1, x_3, \cdots, x_{2n-1}\right]^{\mathrm{T}} \tag{1}$$

where $\mathbf{x} = [x_1, x_{2,} \cdots, x_{2n}]^{\mathrm{T}} \in R^{2n}$ is the system state vector which is assumed to be available for measurement, $u \in R$ and $\mathbf{y}(t) \in R^n$ are input and output of the system output, respectively. τ is the value of time delay. $f_1(\mathbf{x}(t-\tau)),\ f_2(\mathbf{x}(t-\tau)),\ \cdots, f_n(\mathbf{x}(t-\tau))$, and $b_1(\mathbf{x}),\ b_2(\mathbf{x}),\ \cdots, b_n(\mathbf{x})$ are unknown real continuous nonlinear functions, $d_1(\mathbf{x},t),\ d_2(\mathbf{x},t),\ \cdots, d_n(\mathbf{x},t)$ are unknown external bound disturbances. $D(u(t)): R \to R$ is the nonlinear input function containing a dead-zone. Without loss of generality, we assume that $b_i(\mathbf{x}) > 0$ for i=1,2,...,n, and the following assumptions are made for the controller design:

Assumption 1: The time delay τ is a fixed and known constant.

Assumption 2: $0 < \left| f_i(\mathbf{x}(t-\tau)) \right| \le F_i < \infty$, $0 < b_i(\mathbf{x}) \le B_i \le \quad < \infty$, $0 < \left| d_i(\mathbf{x}) \right| \le E_i < \infty$, for $\mathbf{x} \in \Gamma$ $i = 1, 2, \cdots, n$, where F_i, B_i, and E_i are known constants, and Γ is a set given as follows:

$$\Gamma = \left\{ \mathbf{x} \middle| \left\| \mathbf{x} - \mathbf{x_0} \right\|_{p,\omega} \le \Delta \right\}.$$

Here $\omega = \{\omega_1, \omega_2, ..., \omega_{2n}\}$ is a set of weight, and Δ is a positive constant which denotes all state variables' boundary. $\mathbf{x}_0 \in R^{2\mathrm{n}}$ is a fixed point, and $\|\mathbf{x}\|_{p,\omega}$ is a weighted p-norm, which is defined as

$$\|\mathbf{x}\|_{p,\omega} = \left[\sum_{i=1}^{2n} \left(\frac{x_i}{\omega_i} \right)^p \right]^{1/p}$$

If $p = \infty$,

$$\|\mathbf{x}\|_{p,\omega} = \max\left(\frac{|x_1|}{\omega_1}, \frac{|x_2|}{\omega_2}, \cdots, \frac{|x_{2n}|}{\omega_{2n}} \right)$$

If $p = 2$ $\omega_i = 1$ for $i = 1,2\ldots,2n$, $\|\mathbf{x}\|_{p,\omega}$ will denote Euclidean norm $\|\mathbf{x}\|$.

The non-symmetric dead-zone with input $u(t)$ and output as shown in the above Fig. 1. is described by

$$D(u(t))=\begin{cases} m_r(u(t)-E_a) & \text{for } u(t)\ge E_a \\ 0 & \text{for } -E_b\le u(t)\le E_a \\ m_l(u(t)+E_b) & \text{for } u(t)\ge -E_b \end{cases} \quad (2)$$

where, E_a, E_b, and m_r, m_l are parameters and slopes of the dead-zone, respectively.

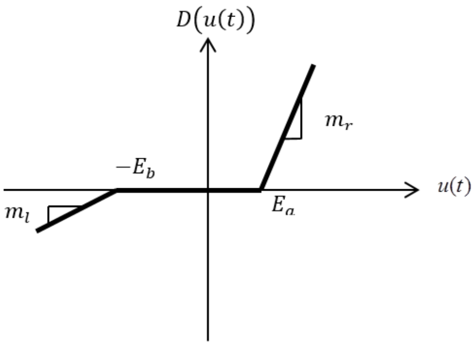

Figure 1. Dead-zone model

In order to investigate the key features of the dead-zone in the control problems, the following assumptions should be made:

Assumption 3： The dead-zone output $D(u(t))$ is not available to obtain.

Assumption 4： The coefficients E_a, E_b ,and m_r, m_l are unknown.

Assumption 5： There exist known constants $m_{\min}$, $m_{\max}$, $E_{a\min}$, $E_{a\max}$, $E_{b\min}$, $E_{b\max}$ such that the unknown dead-zone parameters m_r, m_l, E_a, E_b are bounded, i.e. $0<E_a \in[E_{a\ \min},E_{a\ \max}]$, $0<E_b \in[E_{b\ \min},E_{b\ \max}]$, $0<m_r,m_l\in[m_{min},m_{max}]$

Based on the above assumptions the expression (2) can be represented as

$$D(u(t))=m[u(t)-d(u(t))] \quad \text{for } m>0 \quad (3)$$

where $d(u(t))$ can be calculated form (2) and (3) as

$$d(u(t))=\begin{cases} E_a & \text{for} u(t)\ge E_a \\ u(t) & \text{for } -E_b\le u(t)\le E_a \\ -E_b & \text{for} u(t)\ge -E_b \end{cases} \quad (4)$$

From **Assumption 5**, we can conclude that $d(u(t))$ is bounded, and satisfies:

$$|d(u(t))|\le\varepsilon \quad (5)$$

where ε is an upper bound, which can be chosen as

$$\varepsilon=\max\{E_{a\max},E_{b\max}\} \quad (6)$$

The stable model reference of system is given by

$$\dot{\mathbf{x}}_m(t)=\mathbf{A}_m\mathbf{x}_m(t)+\mathbf{B}_m r(t) \quad (7)$$

where $\dot{\mathbf{x}}_m(t)=[x_{m1}(t),x_{m2}(t),\cdots,x_{m2n}(t)]^{\mathrm{T}}\in R^{2n}$ is the state vector of reference model, $r(t)\in R$ is the bounded reference input. $\mathbf{A}_m\in R^{2n\times 2n}$ is Hurwitz. Let the model reference output vector as $\mathbf{y}_m(t)=[x_{m1},x_{m3},\cdots,x_{m2n-1}]^{\mathrm{T}}$. Because the underactuated system is divided into several different subsystems and for the state variables, there is not obvious differential relationship between these subsystems. As the result, $\mathbf{A}_m$ comprises n subsystems with the controllable canonical form for the plant.

Thus,

$$\mathbf{A}_m=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -a_{m1} & -a_{m2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -a_{m3} & -a_{m4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -a_{m2n-1} & -a_{m2n} \end{bmatrix}\in R^{2n\times 2n}, \quad (8)$$

where a_{mi} for $i=1, 2, \cdots, 2n$ are positive real constants to be chosen.

$$\mathbf{B}_m=[0\ 1\ 0\ 1\ \cdots\ 0\ 1]^T\in R^{2n\times 1} \quad (9)$$

is the known and real constant matrix.

Define the dynamic of the tracking error:

$$\mathbf{e}(t)=\mathbf{x}(t)-\mathbf{x}_m(t) \quad (10)$$

Control objective： Design a controller for (1) such that the system output $\mathbf{y}(t)$ would track the reference model output vector $\mathbf{y}_m(t)$. Define the vector of the output tracking error as

$$\mathbf{y}(t)-\mathbf{y}_m(t)=[e_1,e_3,\cdots,e_{2n-1}]^{\mathrm{T}}\in R^n \quad (11)$$

where the model reference output vector is defined as

$$\mathbf{y}_m(t)=[x_{m1},x_{m3},\cdots,x_{m2n-1}]^{\mathrm{T}}. \quad (12)$$

2.2. Description of Fuzzy Logic Systems

The fuzzy logic system performs a mapping from $U\subset\Re^n$ to $V\subset\Re$. Let $U=U_1\times\cdots\times U_n$ where $U_i\subset\Re$, $i=1,2,\cdots,n$. The fuzzy rule base consists of a collection of fuzzy IF-THEN rules:

$$R^{(l)}:\ \text{IF } x_1 \text{ is } F_1^l, \text{ and } x_2 \text{ is } F_2^l, \text{ and } \cdots \text{ and, } x_n \text{ is } F_n^l \quad \text{THEN } y \text{ is } G^l, \quad \text{for } l=1,2,\cdots,M. \quad (13)$$

in which $\mathbf{x}=[x_1,x_2,\cdots,x_n]^T\in U$ and $y\in V\subset\Re$ are the input and output of the fuzzy logic system, F_i^l and G^l are fuzzy sets in U_i and V, respectively. The fuzzifier maps a crisp point $\mathbf{x}=[x_1,x_2,\cdots,x_n]^T$ into a fuzzy set in U. The fuzzy inference engine performs a mapping from fuzzy sets in U to fuzzy sets in V, based upon the fuzzy IF-THEN rules in the fuzzy rule base and the compositional rule of inference. The defuzzifier maps a fuzzy set in V to a crisp point in V.

The fuzzy systems with center-average defuzzifier, product inference and singleton fuzzifier are of the following form:

$$y = \boldsymbol{\theta}^T \boldsymbol{\xi}(\mathbf{x}) \tag{14}$$

where $\boldsymbol{\theta}^T = \left[\theta^1, \ldots, \theta^M\right]$ with each variable θ^l as the point at which the fuzzy membership function of G^l achieves the maximum value and $\boldsymbol{\xi}(\mathbf{x}) = \left[\xi^1(\mathbf{x}), \ldots, \xi^M(\mathbf{x})\right]^T$ with each variable $\xi^l(\mathbf{x})$ as the fuzzy basis function defined as

$$\xi^l(\mathbf{x}) = \frac{\prod_{i=1}^n \mu_{F_i^l}(x_i)}{\sum_{l=1}^M \left(\prod_{i=1}^n \mu_{F_i^l}(x_i)\right)} \tag{15}$$

where $\mu_{F_i^l}(x_i)$ is the membership function of the fuzzy set.

3. Controller Design and Stability Analysis

From (7), the model reference of system can be rewritten as follows:

$$\begin{cases} \dot{x}_{m1} = x_{m2} \\ \dot{x}_{m2} = -a_{m1}x_{m1} - a_{m2}x_{m2} + r(t) \\ \dot{x}_{m3} = x_{m4} \\ \dot{x}_{m4} = -a_{m3}x_{m3} - a_{m4}x_{m4} + r(t) \\ \vdots \\ \dot{x}_{m2n-1} = x_{m2n} \\ \dot{x}_{m2n} = x_{m2n-1} - a_{m2n}x_{m2n} + r(t) \end{cases} \tag{16}$$

Then the dynamic equation of tracking error can be described by

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_m(t) \tag{17}$$

Substituting (1) and (16) into (17) yields

$$\begin{cases} \dot{e}_1 = e_2 + d_1(\mathbf{x},t) \\ \dot{e}_2 = \left[f_1(\mathbf{x}(t-\tau)) + b_1(\mathbf{x})D(u(t)) + d_2(\mathbf{x},t)\right] - \dot{x}_{m2} \\ \dot{e}_3 = e_4 + d_3(\mathbf{x},t) \\ \dot{e}_4 = \left[f_2(\mathbf{x}(t-\tau)) + b_2(\mathbf{x})D(u(t)) + d_4(\mathbf{x},t)\right] - \dot{x}_{m4} \\ \vdots \\ \dot{e}_{2n-1} = e_{2n} + d_{2n-1}(\mathbf{x},t) \\ \dot{e}_{2n} = \left[f_n(\mathbf{x}(t-\tau)) + b_n(\mathbf{x})u + d_{2n}(t)\right] - \dot{x}_{m2n} \end{cases} \tag{18}$$

Then the change of output tracking error vector can be expressed as

$$\dot{\mathbf{y}}(t) - \dot{\mathbf{y}}_m(t) = \left[\dot{e}_1, \dot{e}_3, \cdots, \dot{e}_{2n-1}\right]^{\mathrm{T}} \tag{19}$$

Let the suitable sliding surfaces be defined as follows:

$$s_i = c_i e_{2i-1} + \dot{e}_{2i-1} \quad \text{for } i = 1, 2, \ldots, n \tag{20}$$

where c_i are positive constants.

Differentiating s_i with respect to time, we have

$$\begin{aligned} s_i &= c_i e_{2i-1} + \dot{e}_{2i-1} \\ \dot{s}_i &= c_i \dot{e}_{2i-1} + f_i(\mathbf{x}(t-\tau)) \\ &\quad + b_i(\mathbf{x})m[u(t) - d(u(t))] - \dot{x}_{m2i} + Q_i(\mathbf{x},t) \end{aligned} \tag{21}$$

where $Q_i(\mathbf{x},t) = d_{2i}(\mathbf{x},t) + \dot{d}_{2i-1}(\mathbf{x},t)$ for $i = 1, 2, \ldots, n$

Assumption 6:

$0 < \left|Q_i(\mathbf{x},t)\right| \le \rho_i(\mathbf{x}) < \infty$ for $i = 1, 2, \ldots, n$

where $\rho_i(\mathbf{x})$ are unknown positive smooth continuous functions.

According to the equivalent control method, the equivalent control law of the systems can be obtained as:

$$u_{eqi} = (mb_i(\mathbf{x}))^{-1}\left[-c_i \dot{e}_{2i-1} + \dot{x}_{m2i}\right] \quad \text{for } i = 1, 2, \ldots, n \tag{22}$$

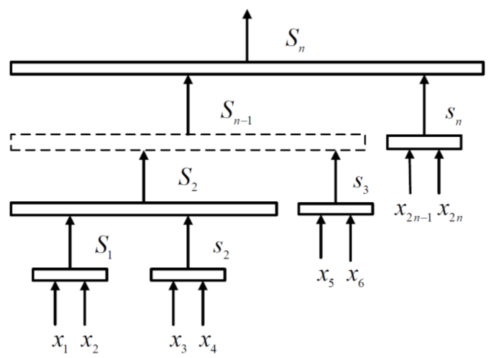

Figure 2. Hierarchical structure of sliding surfaces

First, let the unknown nonlinear functions $f_1(\mathbf{x}(t-\tau)), f_2(\mathbf{x}(t-\tau)), \cdots, f_n(\mathbf{x}(t-\tau)$, $b_1(\mathbf{x}), b_2(\mathbf{x}), \cdots, b_n(\mathbf{x})$, and $\rho_1(\mathbf{x}), \rho_2(\mathbf{x}), \cdots, \rho_n(\mathbf{x})$ can be approximated, over a compact set Γ, by using the fuzzy logic systems as follows:

$$\hat{f}_i(\mathbf{x}(t-\tau) \mid \hat{\boldsymbol{\theta}}_{f_i}) = \hat{\boldsymbol{\theta}}_{f_i}^T \boldsymbol{\xi}(\mathbf{x}(t-\tau)) \tag{23}$$

$$\hat{b}_i(\mathbf{x} \mid \hat{\boldsymbol{\theta}}_{b_i}) = \hat{\boldsymbol{\theta}}_{b_i}^T \boldsymbol{\xi}(\mathbf{x}) \tag{24}$$

$$\hat{\rho}_i(\mathbf{x} \mid \hat{\boldsymbol{\theta}}_{\rho_i}) = \hat{\boldsymbol{\theta}}_{\rho_i}^T \boldsymbol{\xi}(\mathbf{x}) \tag{25}$$

where $\boldsymbol{\xi}(\mathbf{x}(t-\tau))$ and $\boldsymbol{\xi}(\mathbf{x})$ are the fuzzy basis vector, $\hat{\boldsymbol{\theta}}_{f_i}, \hat{\boldsymbol{\theta}}_{b_i}$ and $\hat{\boldsymbol{\theta}}_{\rho_i}$ for $i = 1, 2, \ldots, n$, are the corresponding adjustable parameter vector of each fuzzy logic systems. It is assumed that $\hat{\boldsymbol{\theta}}_{f_i}$, $\hat{\boldsymbol{\theta}}_{b_i}$ and $\hat{\boldsymbol{\theta}}_{\rho_i}$ belong to compact sets $\Omega_{\hat{\boldsymbol{\theta}}_{f_i}}$, $\Omega_{\hat{\boldsymbol{\theta}}_{b_i}}$, $\Omega_{\hat{\boldsymbol{\theta}}_{\rho_i}}$, respectively, which are defined as:

$$\Omega_{\hat{\theta}_{f_i}} = \{\hat{\theta}_{f_i} \in R^M : \|\hat{\theta}_{f_i}\| \le N_{f_i} < \infty\},$$

$$\Omega_{\hat{\theta}_{b_i}} = \{\hat{\theta}_{b_i} \in R^M : \|\hat{\theta}_{b_i}\| \le N_{b_i} < \infty\},$$

$$\Omega_{\hat{\theta}_{\rho_i}} = \{\hat{\theta}_{\rho_i} \in R^M : \|\hat{\theta}_{\rho_i}\| \le N_{\rho_i} < \infty\},$$

where N_{f_i}, N_{b_i}, N_{ρ_i} for $i = 1,2,\cdots,n$, are the designed parameters, and M is the number of fuzzy inference rules. Let us define the optimal parameter vectors, $\theta^*_{f_i}$, $\theta^*_{b_i}$, $\theta^*_{\rho_i}$ for $i = 1,2,\cdots,n$, as follows:

$$\theta^*_{f_i} = \arg \min_{\theta_{f_i} \in \Omega_{\hat{\theta}_{f_i}}} \left\{ \sup_{\mathbf{x}\in\Gamma} \left| \hat{f}_i(\mathbf{x}(t-\tau) \middle| \hat{\theta}_{f_i}) - f_i(\mathbf{x}(t-\tau)) \right| \right\}$$

$$\theta^*_{b_i} = \arg \min_{\hat{\theta}_{b_i} \in \Omega_{\hat{\theta}_{b_i}}} \left\{ \sup_{\mathbf{x}\in\Gamma} \left| \hat{b}_i(\mathbf{x}(t) \middle| \hat{\theta}_{b_i}) - b_i(\mathbf{x}(t)) \right| \right\}$$

$$\theta^*_{\rho_i} = \arg \min_{\hat{\theta}_{\rho_i} \in \Omega_{\hat{\theta}_{\rho_i}}} \left\{ \sup_{\mathbf{x}\in\Gamma} \left| \hat{\rho}_i(\mathbf{x}(t) \middle| \hat{\theta}_{\rho_i}) - \rho_i(\mathbf{x}(t)) \right| \right\}$$

where $\theta^*_{f_i}$, $\theta^*_{b_i}$, $\theta^*_{\rho_i}$, for $i = 1,2,\cdots,n$, are bounded in the suitable closed sets $\Omega_{\hat{\theta}_{f_i}}$, $\Omega_{\hat{\theta}_{b_i}}$, $\Omega_{\hat{\theta}_{\rho_i}}$, respectively. The parameter estimation errors can be defined as:

$$\tilde{\theta}_{f_i} = \hat{\theta}_{f_i} - \theta^*_{f_i} \tag{26}$$

$$\tilde{\theta}_{b_i} = \hat{\theta}_{b_i} - \theta^*_{b_i} \tag{27}$$

$$\tilde{\theta}_{\rho_i} = \hat{\theta}_{\rho_i} - \theta^*_{\rho_i} \tag{28}$$

and

$$\omega \ge |\omega_1| + |\omega_2| + |\omega_3| \tag{29}$$

where

$$\omega_1 = \sum_{i=1}^{n} (\prod_{j=i}^{n} a_j)[f_i(\mathbf{x}(t-\tau)) - \hat{f}_i(\mathbf{x}(t-\tau)) \middle| \theta^*_{f_i}] \tag{30}$$

$$\omega_2 = \sum_{i=1}^{n} (\prod_{j=i}^{n} a_j)[b_i(\mathbf{x}(t)) - \hat{b}_i(\mathbf{x}(t) \middle| \theta^*_{b_i})] \times [mu(t) - md(u(t))] \tag{31}$$

$$\omega_3 = \sum_{i=1}^{n} \left(\left| \prod_{j=i}^{n} a_j \right| \right) [\rho_i(\mathbf{x}(t)) - \hat{\rho}_i(\mathbf{x}(t) \middle| \theta^*_{\rho_i})] \tag{32}$$

are the minimum approximation errors, which correspond to approximation errors obtained when optimal parameters are used.

Secondly, we define:

$$\tilde{\omega} = \hat{\omega} - \omega \tag{33}$$

$$\tilde{\varphi} = \hat{\varphi} - \varphi \tag{34}$$

where $\hat{\varphi}$ is an estimate of φ, which is defined as $\varphi = (m)^{-1}$, and $\hat{\omega}$ be as the estimate of ω.

Based on the fuzzy logic systems, the equation (22) can be replaced as the following controller:

$$\hat{u}_{eqi} = (\hat{b}_i(\mathbf{x}) \middle| \hat{\theta}_{b_i})^{-1} \hat{\varphi} \left[-c_i \dot{e}_{2i-1} + \dot{x}_{m2i} \right] \quad \text{for } i = 1,2,...,n \tag{35}$$

Next, the ith-layer sliding surface S_i and its control law u can be defined as follows.

$$S_i = \lambda_{i-1} S_{i-1} + s_i \tag{36}$$

$$u = \sum_{i=1}^{n} \left(\hat{u}_{eqi} \right) + \hat{u}_{sw} \tag{37}$$

where λ_{i-1} for $i = 1,2,\cdots,n$ are positive constants; $\lambda_0 = S_0 = 0$. From the recursive formulas (36), we have

$$S = \sum_{i=1}^{n} (\prod_{j=i}^{n} a_j) s_i \tag{38}$$

Here for a given n, $a_j = \lambda_j$ $(j \ne n)$ is a constant, and $a_j = 1$ $(j = n)$. $\hat{u}_{sw}$ is the switching control of sliding surface can be chosen as.

$$\begin{aligned}
\hat{u}_{sw} = &- [\sum_{i=1}^{n} (\prod_{j=i}^{n} a_j)(\hat{b}_i(\mathbf{x}) \middle| \hat{\theta}_{b_i})]^{-1} \\
&\times \left\{ \sum_{i=1}^{n} (\prod_{j=i}^{n} a_j) \hat{b}_i(\mathbf{x} \middle| \hat{\theta}_{b_i}) \times \sum_{\substack{l=1 \\ l \ne i}}^{n} \hat{u}_{eql} + \frac{1}{2} S \left(\frac{\sum_{i=1}^{n} (\prod_{j=i}^{n} a_j)}{m_{\min}} \right)^2 \right. \\
&+ \frac{1}{m_{\min}} \operatorname{sgn}(S) \sum_{i=1}^{n} (\prod_{j=i}^{n} a_j) \hat{\rho}_i(\mathbf{x} \middle| \hat{\theta}_{\rho_i}) \\
&+ \frac{1}{m_{\min}} \operatorname{sgn}(S) \hat{\omega} + \operatorname{sgn}(S) [\sum_{i=1}^{n} (\prod_{j=i}^{n} a_j)(\hat{b}_i(\mathbf{x}) \middle| \hat{\theta}_{b_i})] \times \varepsilon \\
&\left. + \frac{1}{2S} [\sum_{i=1}^{n} \hat{f}_i^2 (\mathbf{x}(t-\tau) \middle| \hat{\theta}_{f_i}) + KS \right\}
\end{aligned} \tag{39}$$

where K is a positive constant, and the parameter update laws as follows:

$$\dot{\hat{\theta}}_{f_i} = \gamma_{f_i} S \prod_{j=i}^{n} a_j \xi(\mathbf{x}(t-\tau)) \tag{40}$$

$$\dot{\hat{\theta}}_{b_i} = \gamma_{b_i} \left[S \prod_{j=i}^{n} a_j u(t) + |S| \left| \prod_{j=i}^{n} a_j \times \varepsilon \right| \right] \times \xi(\mathbf{x}) \tag{41}$$

$$\dot{\hat{\theta}}_{\rho_i} = \gamma_{\rho_i} |S| \left| \prod_{j=i}^{n} a_j \times \varepsilon \right| \times \xi(\mathbf{x}) \tag{42}$$

$$\dot{\hat{\omega}} = \gamma_{\omega} \frac{1}{m_{\min}} |S| \tag{43}$$

$$\dot{\hat{\varphi}} = \eta S\prod_{j=i}^{n} a_j(-c_i\dot{e}_{2i-1} + \dot{x}_{m2i}) \tag{44}$$

where γ_{f_i}, γ_{b_i}, γ_{ρ_i} for $i = 1,2,\cdots,n$, η and γ_ω are positive constants.

Remark 1: Without loss of generality, the adaptive laws used in this paper are assumed that the parameter vectors are within the constraint sets or on the boundaries of the constraint sets but moving toward the inside of the constraint sets. If the parameter vectors are on the boundaries of the constraint sets but moving toward the outside of the constraint sets, we have to use the projection algorithm to modify the adaptive laws such that the parameter vectors will remain inside of the constraint sets. Readers can refer to reference [14]. The proposed adaptive laws (40)-(44) can be modified as the following form:

$$\dot{\hat{\boldsymbol{\theta}}}_{f_i} = \begin{cases} \gamma_{f_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t-\tau)), \text{ if } (\|\hat{\boldsymbol{\theta}}_{f_i}\| < N_{f_i}) \text{ or} \\ \qquad (\|\hat{\boldsymbol{\theta}}_{f_i}\| = N_{f_i} \text{ and } S\prod_{i=j}^{n} a_j \hat{\boldsymbol{\theta}}_{f_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t-\tau)) \le 0) \\ P\left\{\gamma_{f_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t-\tau))\right\}, \text{ if } (\|\hat{\boldsymbol{\theta}}_{f_i}\| = N_{f_i} \\ \qquad \text{and } S\prod_{i=j}^{n} a_j \hat{\boldsymbol{\theta}}_{f_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t-\tau)) > 0) \end{cases} \tag{45}$$

where $P\left\{\gamma_{f_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t-\tau))\right\}$ is defined as

$$\begin{aligned} P\left\{\gamma_{f_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t-\tau))\right\} &= \gamma_{f_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t-\tau)) \\ &\quad - \gamma_{f_i} S\prod_{i=j}^{n} a_j \frac{\hat{\boldsymbol{\theta}}_{f_i}\hat{\boldsymbol{\theta}}_{f_i}^{\mathrm{T}}}{\|\hat{\boldsymbol{\theta}}_{f_i}\|^2} \boldsymbol{\xi}(\mathbf{x}(t-\tau)) \end{aligned} \tag{46}$$

$$\dot{\hat{\boldsymbol{\theta}}}_{b_i} = \begin{cases} \gamma_{b_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t)), \qquad \text{if } (\|\hat{\boldsymbol{\theta}}_{b_i}\| < N_{f_i}) \text{ or} \\ \qquad (\|\hat{\boldsymbol{\theta}}_{b_i}\| = N_{f_i} \text{ and } S\prod_{i=j}^{n} a_j \hat{\boldsymbol{\theta}}_{b_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t)) \le 0) \\ P\left\{\gamma_{b_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t))\right\}, \\ \qquad \text{if } (\|\hat{\boldsymbol{\theta}}_{b_i}\| = N_{f_i} \text{ and } S\prod_{i=j}^{n} a_j \hat{\boldsymbol{\theta}}_{b_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t)) > 0) \end{cases} \tag{47}$$

where $P\left\{\gamma_{b_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t))\right\}$ is defined as

$$\begin{aligned} P\left\{\gamma_{b_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t))\right\} &= \gamma_{b_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t)) \\ &\quad - \gamma_{b_i} S\prod_{i=j}^{n} a_j \frac{\hat{\boldsymbol{\theta}}_{b_i}\hat{\boldsymbol{\theta}}_{b_i}^{\mathrm{T}}}{\|\hat{\boldsymbol{\theta}}_{b_i}\|^2} \boldsymbol{\xi}(\mathbf{x}(t)) \end{aligned} \tag{48}$$

$$\dot{\hat{\boldsymbol{\theta}}}_{\rho_i} = \begin{cases} \gamma_{\rho_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t)), \text{if } (\|\hat{\boldsymbol{\theta}}_{\rho_i}\| < N_{f_i}) \text{ or} \\ \qquad (\|\hat{\boldsymbol{\theta}}_{\rho_i}\| = N_{f_i} \text{ and } S\prod_{i=j}^{n} a_j \hat{\boldsymbol{\theta}}_{\rho_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t)) \le 0) \\ P\left\{\gamma_{\rho_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t))\right\}, \text{ if } (\|\hat{\boldsymbol{\theta}}_{\rho_i}\| = N_{f_i} \\ \qquad \text{and } S\prod_{i=j}^{n} a_j \hat{\boldsymbol{\theta}}_{\rho_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t)) > 0) \end{cases} \tag{49}$$

where $P\left\{\gamma_{\rho_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t))\right\}$ is defined as

$$\begin{aligned} P\left\{\gamma_{\rho_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t))\right\} &= \gamma_{b_i} S\prod_{i=j}^{n} a_j \boldsymbol{\xi}(\mathbf{x}(t)) \\ &\quad - \gamma_{b_i} S\prod_{i=j}^{n} a_j \frac{\hat{\boldsymbol{\theta}}_{b_i}\hat{\boldsymbol{\theta}}_{b_i}^{\mathrm{T}}}{\|\hat{\boldsymbol{\theta}}_{b_i}\|^2} \boldsymbol{\xi}(\mathbf{x}(t)) \end{aligned} \tag{50}$$

The main result of robust adaptive tracking control scheme is summarized on the following theorem.

Theorem 1: Consider the single-input-multi-output uncertain underactuated system (1). If *Assumptions 1-6* are satisfied, then the proposed model reference sliding-mode controller defined by (37) with adaptive laws (40)-(44) guarantees that all signals of closed-loop system are bounded.

Proof: Consider the Lyapunov-Krasovskii function candidate

$$\begin{aligned} V &= V_1 + V_2 \\ V_1 &= \frac{1}{2}\left[\frac{1}{m}S^2 + \sum_{i=1}^{n}\frac{1}{\gamma_{f_i}}\tilde{\boldsymbol{\theta}}_{f_i}^T\tilde{\boldsymbol{\theta}}_{f_i} + \sum_{i=1}^{n}\frac{1}{\gamma_{b_i}}\tilde{\boldsymbol{\theta}}_{b_i}^T\tilde{\boldsymbol{\theta}}_{b_i} \right. \\ &\qquad \left. + \sum_{i=1}^{n}\frac{1}{\gamma_{\rho_i}}\tilde{\boldsymbol{\theta}}_{\rho_i}^T\tilde{\boldsymbol{\theta}}_{\rho_i} + \frac{1}{\eta}\tilde{\varphi}^2 + \frac{1}{\gamma_\omega}\tilde{\omega}^2\right] \\ V_2 &= \frac{1}{2}\int_{t-\tau}^{t}\sum_{i=1}^{n}\hat{f}_i^2(\mathbf{x}(t-\tau)\,|\,\hat{\boldsymbol{\theta}}_{f_i})\,dz \end{aligned} \tag{51}$$

Differentiating the Lyapunov function V_1 with respect to time, and by the fact $\dot{\tilde{\boldsymbol{\theta}}}_{fi} = \dot{\hat{\boldsymbol{\theta}}}_{fi}$, $\dot{\tilde{\boldsymbol{\theta}}}_{b_i} = \dot{\hat{\boldsymbol{\theta}}}_{b_i}$, $\dot{\tilde{\boldsymbol{\theta}}}_{\rho\mathbf{i}} = \dot{\hat{\boldsymbol{\theta}}}_{\rho\mathbf{i}}$, $\dot{\tilde{\varphi}} = \dot{\hat{\varphi}}$, $\dot{\tilde{\omega}} = \dot{\hat{\omega}}$, we can obtain.

$$\dot{V}_1 = \frac{1}{m} S\dot{S} + \sum_{i=1}^{n} \frac{1}{\gamma_{fi} m} \tilde{\boldsymbol{\theta}}_{fi}^T \dot{\hat{\boldsymbol{\theta}}}_{fi} + \sum_{i=1}^{n} \frac{1}{\gamma_{b_i}} \tilde{\boldsymbol{\theta}}_{b_i}^T \dot{\hat{\boldsymbol{\theta}}}_{b_i} + \sum_{i=1}^{n} \frac{1}{\gamma_{\rho_i} m} \tilde{\boldsymbol{\theta}}_{\rho_i}^T \dot{\hat{\boldsymbol{\theta}}}_{\rho_i} + \frac{1}{\eta} \tilde{\varphi}\dot{\hat{\varphi}} + \frac{1}{\gamma_\omega} \tilde{\omega}\dot{\hat{\omega}}$$

According to (38), and (21), we have

$$\begin{aligned} \dot{V}_1 = {} & \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \{ c_i \dot{e}_{2i-1} + [f_i(\mathbf{x}(t-\tau)) + b_i(\mathbf{x}) D(u(t)) \\ & + d_{2i}(\mathbf{x},t) - \dot{x}_{m2i}] + \dot{d}_{2i-1}(\mathbf{x},t) \} \\ & + \sum_{i=1}^{n} \frac{1}{\gamma_{fi} m} \tilde{\boldsymbol{\theta}}_{fi}^T \dot{\hat{\boldsymbol{\theta}}}_{fi} + \sum_{i=1}^{n} \frac{1}{\gamma_{b_i}} \tilde{\boldsymbol{\theta}}_{b_i}^T \dot{\hat{\boldsymbol{\theta}}}_{b_i} \\ & + \sum_{i=1}^{n} \frac{1}{\gamma_{\rho_i} m} \tilde{\boldsymbol{\theta}}_{\rho_i}^T \dot{\hat{\boldsymbol{\theta}}}_{\rho_i} + \frac{1}{\eta} \tilde{\varphi}\dot{\hat{\varphi}} + \frac{1}{\gamma_\omega} \tilde{\omega}\dot{\hat{\omega}} \end{aligned} \tag{52}$$

Substituting (21), and **Assumption 6** into (52) yields

$$\begin{aligned} \dot{V}_1 = {} & \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big\{ c_i \dot{e}_{2i-1} + \hat{f}_i(\mathbf{x}(t-\tau) | \hat{\boldsymbol{\theta}}_{fi}) \\ & + [f_i(\mathbf{x}(t-\tau)) - \hat{f}_i(\mathbf{x}(t-\tau) | \hat{\boldsymbol{\theta}}_{fi}^*)] \\ & + [\hat{f}_i(\mathbf{x}(t-\tau) | \hat{\boldsymbol{\theta}}_{fi}^* - \hat{f}_i(\mathbf{x}(t-\tau) | \hat{\boldsymbol{\theta}}_{fi})] \\ & + \hat{b}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{b_i}) D(u(t)) + [b_i(\mathbf{x}) - \hat{b}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{b_i}^*)] D(u(t)) \\ & + [\hat{b}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{b_i}^*) - \hat{b}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{b_i})] D(u(t)) - \dot{x}_{m2i} \Big\} \\ & + \frac{1}{m} |S| \sum_{i=1}^{n} \left| (\prod\nolimits_{j=i}^{n} a_j) \right| \hat{\rho}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{\rho_i}) \\ & + \frac{1}{m} |S| \sum_{i=1}^{n} \left| (\prod\nolimits_{j=i}^{n} a_j) \right| [\rho_i(\mathbf{x}) - \hat{\rho}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{\rho_i}^*)] \\ & + \frac{1}{m} |S| \sum_{i=1}^{n} \left| (\prod\nolimits_{j=i}^{n} a_j) \right| [\hat{\rho}_i(\mathbf{x} | \theta_{\rho_i}^*) - \hat{\rho}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{\rho_i})] \\ & + \sum_{i=1}^{n} \frac{1}{\gamma_{fi} m} \tilde{\boldsymbol{\theta}}_{fi}^T \dot{\hat{\boldsymbol{\theta}}}_{fi} + \sum_{i=1}^{n} \frac{1}{\gamma_{b_i}} \tilde{\boldsymbol{\theta}}_{b_i}^T \dot{\hat{\boldsymbol{\theta}}}_{b_i} + \sum_{i=1}^{n} \frac{1}{\gamma_{\rho_i} m} \tilde{\boldsymbol{\theta}}_{\rho_i}^T \dot{\hat{\boldsymbol{\theta}}}_{\rho_i} \\ & + \frac{1}{\eta} \tilde{\varphi}\dot{\hat{\varphi}} + \frac{1}{\gamma_\omega} \tilde{\omega}\dot{\hat{\omega}} \end{aligned} \tag{53}$$

By applying the fuzzy logic system (13) and (26)-(28) into (53), we obtain

$$\begin{aligned} \dot{V}_1 \le {} & \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[c_i \dot{e}_{2i-1} + \hat{f}_i(\mathbf{x}(t-\tau) | \hat{\boldsymbol{\theta}}_{fi}) \\ & + \hat{b}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{b_i}) D(u(t)) - \dot{x}_{m2i} \Big] \\ & + \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[f_i(\mathbf{x}(t-\tau)) - \hat{f}_i(\mathbf{x}(t-\tau) | \hat{\boldsymbol{\theta}}_{fi}^*) \Big] \\ & + \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[b_i(\mathbf{x}) - \hat{b}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{b_i}^*) \Big] D(u(t)) \\ & + \frac{1}{m} |S| \sum_{i=1}^{n} \left| (\prod\nolimits_{j=i}^{n} a_j) \right| \Big[\rho_i(\mathbf{x}) - \hat{\rho}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{\rho_i}^*) \Big] \\ & + \frac{1}{m} |S| \sum_{i=1}^{n} \left| (\prod\nolimits_{j=i}^{n} a_j) \right| \hat{\rho}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{\rho_i}) + \frac{1}{\eta} \tilde{\varphi}\dot{\hat{\varphi}} + \frac{1}{\gamma_\omega} \tilde{\omega}\dot{\hat{\omega}} \\ & + \left\{ \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[-\tilde{\boldsymbol{\theta}}_{fi}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t-\tau)) \Big] + \sum_{i=1}^{n} \frac{1}{\gamma_{fi} m} \tilde{\boldsymbol{\theta}}_{fi}^T \dot{\hat{\boldsymbol{\theta}}}_{fi} \right\} \\ & + \left\{ \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[-\tilde{\boldsymbol{\theta}}_{b_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t)) \Big] D(u(t)) + \sum_{i=1}^{n} \frac{1}{\gamma_{b_i}} \tilde{\boldsymbol{\theta}}_{b_i}^T \dot{\hat{\boldsymbol{\theta}}}_{b_i} \right\} \\ & + \left\{ \frac{1}{m} |S| \sum_{i=1}^{n} \left| (\prod\nolimits_{j=i}^{n} a_j) \right| \Big[-\tilde{\boldsymbol{\theta}}_{\rho_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t)) \Big] + \sum_{i=1}^{n} \frac{1}{\gamma_{\rho_i} m} \tilde{\boldsymbol{\theta}}_{\rho_i}^T \dot{\hat{\boldsymbol{\theta}}}_{\rho_i} \right\} \end{aligned} \tag{54}$$

By employing the adaptive laws (40), (42), and (3), we can get

$$\begin{aligned} \dot{V}_1 \le {} & \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[c_i \dot{e}_{2i-1} + \hat{f}_i(\mathbf{x}(t-\tau) | \hat{\boldsymbol{\theta}}_{fi}) \\ & + \hat{b}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{b_i}) m [u(t) - d(u(t))] - \dot{x}_{m2i} \Big] \\ & + \frac{1}{m} |S| \sum_{i=1}^{n} \left| (\prod\nolimits_{j=i}^{n} a_j) \right| \hat{\rho}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{\rho_i}) \\ & + \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[f_i(\mathbf{x}(t-\tau)) - \hat{f}_i(\mathbf{x}(t-\tau) | \hat{\boldsymbol{\theta}}_{fi}^*) \Big] \\ & + \frac{1}{m} S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[b_i(\mathbf{x}) - \hat{b}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{b_i}^*) \Big] D(u(t)) \\ & + \frac{1}{m} |S| \sum_{i=1}^{n} \left| (\prod\nolimits_{j=i}^{n} a_j) \right| \Big[\rho_i(\mathbf{x}) - \hat{\rho}_i(\mathbf{x} | \hat{\boldsymbol{\theta}}_{\rho_i}^*) \Big] \\ & + \Big\{ S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \Big[-\tilde{\boldsymbol{\theta}}_{b_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t)) \Big] u(t) \\ & - S \sum_{i=1}^{n} (\prod\nolimits_{j=i}^{n} a_j) \times \Big[-\tilde{\boldsymbol{\theta}}_{b_i}^{\mathrm{T}} \boldsymbol{\xi}(\mathbf{x}(t)) \Big] d(u(t)) \\ & + \sum_{i=1}^{n} \frac{1}{\gamma_{b_i}} \tilde{\boldsymbol{\theta}}_{b_i}^T \dot{\hat{\boldsymbol{\theta}}}_{b_i} \Big\} + \frac{1}{\eta} \tilde{\varphi}\dot{\hat{\varphi}} + \frac{1}{\gamma_\omega} \tilde{\omega}\dot{\hat{\omega}} \end{aligned}$$

According to (27), the above equation can be rewritten as

$$\begin{aligned}\dot{V}_1 \le & \frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\Big[c_i\dot{e}_{2i-1} + \hat{f}_i(\mathbf{x}(t-\tau)|\hat{\boldsymbol{\theta}}_{f_i}) \\ & +\hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{b_i})m\times u(t) - \dot{x}_{m2i}\Big] \\ & +\frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\Big[f_i(\mathbf{x}(t-\tau)) - \hat{f}_i(\mathbf{x}(t-\tau)|\hat{\boldsymbol{\theta}}^*_{f_i})\Big] \\ & + \frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\Big[b_i(\mathbf{x}) - \hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}^*_{b_i})\Big]D(u(t)) \\ & +\frac{1}{m}|S| \sum_{i=1}^{n}\Big|(\prod_{j=i}^{n} a_j)\Big| \Big[\rho_i(\mathbf{x}) - \hat{\rho}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}^*_{\rho_i})\Big] \\ & +\frac{1}{m}|S| \sum_{i=1}^{n}\Big|(\prod_{j=i}^{n} a_j)\Big| \hat{\rho}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{\rho_i}) \\ & +\Big\{S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\Big[-\tilde{\boldsymbol{\theta}}^{\mathrm{T}}_{b_i}\boldsymbol{\xi}(\mathbf{x}(t))\Big]u(t) \\ & +|S|\sum_{i=1}^{n}(\Big|\prod_{j=i}^{n} a_j\Big|)\times\hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}^*_{b_i})\times|d(u(t))| \\ & +\sum_{i=1}^{n}\frac{1}{\gamma_{b_i}}\tilde{\boldsymbol{\theta}}^T_{b_i}\dot{\hat{\boldsymbol{\theta}}}_{b_i}\Big\} + \frac{1}{\eta}\tilde{\varphi}\dot{\hat{\varphi}} + \frac{1}{\gamma_\omega}\tilde{\omega}\dot{\hat{\omega}}\end{aligned} \tag{55}$$

By using the adaptive law (41) and (5) we have

$$\begin{aligned}\dot{V}_1 \le & \frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\Big[c_i\dot{e}_{2i-1} + \hat{f}_i(\mathbf{x}(t-\tau)|\hat{\boldsymbol{\theta}}_{f_i}) \\ & +\hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{b_i})m\times u(t) - \dot{x}_{m2i}\Big] \\ & +\frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\Big[f_i(\mathbf{x}(t-\tau)) - \hat{f}_i(\mathbf{x}(t-\tau)|\hat{\boldsymbol{\theta}}^*_{f_i})\Big] \\ & + \frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\Big[b_i(\mathbf{x}) - \hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}^*_{b_i})\Big]D(u(t)) \\ & +\frac{1}{m}|S| \sum_{i=1}^{n}\Big|(\prod_{j=i}^{n} a_j)\Big| \Big[\rho_i(\mathbf{x}) - \hat{\rho}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}^*_{\rho_i})\Big] \\ & +\frac{1}{m}|S| \sum_{i=1}^{n}\Big|(\prod_{j=i}^{n} a_j)\Big| \hat{\rho}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{\rho_i}) \\ & +|S|\sum_{i=1}^{n}(\Big|\prod_{j=i}^{n} a_j\Big|)\times\hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{b_i})\times\varepsilon + \frac{1}{\eta}\tilde{\varphi}\dot{\hat{\varphi}} + \frac{1}{\gamma_\omega}\tilde{\omega}\dot{\hat{\omega}}\end{aligned}$$

According to (29)-(33) and (43), we obtain

$$\begin{aligned}\dot{V}_1 \le & \frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)[c_i\dot{e}_{2i-1} - \dot{x}_{m2i}] + \frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j) \\ & \times\hat{f}_i(\mathbf{x}(t-\tau)|\hat{\boldsymbol{\theta}}_{f_i}) + S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\times\hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{b_i})\times u(t) \\ & + \frac{1}{m}|S| \sum_{i=1}^{n}\Big|(\prod_{j=i}^{n} a_j)\Big| \hat{\rho}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{\rho_i}) + \frac{1}{m}|S|\hat{\omega} \\ & +|S|\sum_{i=1}^{n}(\Big|\prod_{j=i}^{n} a_j\Big|)\times\hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{b_i})\times\varepsilon + \frac{1}{\eta}\tilde{\varphi}\dot{\hat{\varphi}}\end{aligned} \tag{56}$$

With the use of $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ for scalars a and b, we obtain

$$\frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\times\hat{f}_i(\mathbf{x}(t-\tau)|\hat{\boldsymbol{\theta}}_{f_i}) \le \frac{1}{2}\left(\frac{S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)}{m}\right)^2 + \sum_{i=1}^{n}\frac{1}{2}\hat{f}_i^2((\mathbf{x}(t-\tau)|\hat{\boldsymbol{\theta}}_{f_i})^2 \tag{57}$$

Differentiating the V_2 with respect to time, we can obtain

$$\dot{V}_2 = \frac{1}{2}\sum_{i=1}^{n}\Big[\hat{f}_i^2(\mathbf{x}(t)|\hat{\boldsymbol{\theta}}_{f_i}) - \hat{f}_i^2(\mathbf{x}(t-\tau)|\hat{\boldsymbol{\theta}}_{f_i})\Big]$$

$$\begin{aligned}\dot{V} = & \dot{V}_1 + \dot{V}_2 \\ \le & \frac{1}{m} S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)[c_i\dot{e}_{2i-1} - \dot{x}_{m2i}] \\ & + \frac{1}{m}|S| \sum_{i=1}^{n}\Big|(\prod_{j=i}^{n} a_j)\Big| \hat{\rho}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{\rho_i}) \\ & + S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)\times\hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{b_i})\times u(t) + \frac{1}{2}\left(\frac{S\sum_{i=1}^{n}(\prod_{j=i}^{n} a_j)}{m}\right)^2 \\ & + \frac{1}{m}|S|\hat{\omega} + |S|\sum_{i=1}^{n}(\Big|\prod_{j=i}^{n} a_j\Big|) \\ & \times\hat{b}_i(\mathbf{x}|\hat{\boldsymbol{\theta}}_{b_i})\times\varepsilon + \sum_{i=1}^{n}\hat{f}_i^2(\mathbf{x}(t)|\hat{\boldsymbol{\theta}}_{f_i}) \\ & + \frac{1}{\eta}\tilde{\varphi}\dot{\hat{\varphi}}\end{aligned}$$

According to (34) (37) and (44), we have

$$\dot{V} \le S \times \sum_{i=1}^{n} (\prod_{j=i}^{n} a_j) \hat{b}_i(\mathbf{x}|\hat{\theta}_{b_i}) \times \left[\sum_{\substack{l=1 \\ l\neq i}}^{n} \hat{u}_{eql} + \hat{u}_{sw} \right] + \frac{1}{m_{\min}} |S| \hat{\omega} + \frac{1}{m_{\min}} |S| \sum_{i=1}^{n} \left| (\prod_{j=i}^{n} a_j) \right| \hat{\rho}_i(\mathbf{x}|\hat{\theta}_{\rho_i}) + \frac{1}{2} \left(\frac{S \sum_{i=1}^{n} (\prod_{j=i}^{n} a_j)}{m_{\min}} \right)^2 + |S| \sum_{i=1}^{n} (\left| \prod_{j=i}^{n} a_j \right|) \times \hat{b}_i(\mathbf{x}|\hat{\theta}_{b_i}) \times \varepsilon + \sum_{i=1}^{n} \hat{f}_i^2(\mathbf{x}(t)|\hat{\theta}_{f_i}) + \frac{1}{\eta} \tilde{\varphi} \dot{\hat{\varphi}}$$

Using the switching control laws (37), the above equation can be rewritten as

$$\dot{V} \le -KS^2 \le 0$$

Therefore, the hierarchical sliding surface S is stable, and the all signals of the closed-loop system are bounded based on the proposed controller. This completes the proof.

4. An Example and Simulation Results

In this section, a mass-spring-damper system [15] in the presence of uncertain parameter and exogenous disturbances is considered as our simulation example Fig. 3. The corresponding mathematical model is described as follows:

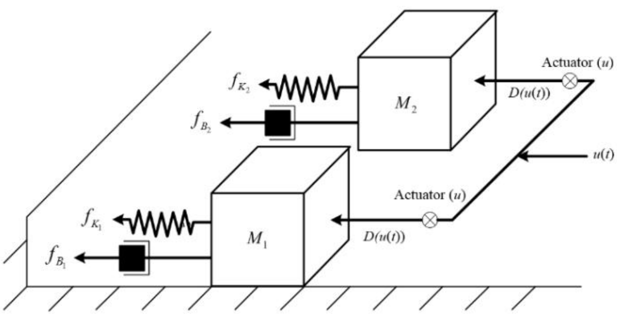

Figure 3. The mass-spring-damper system

$$\dot{x}_1 = x_2 + d_1(t, \mathbf{x})$$
$$\dot{x}_2 = \frac{-f_{K_1}(t-\tau, \mathbf{x}) - f_{B_1}(t-\tau, \mathbf{x}) + u}{M_1} + d_2(t, \mathbf{x})$$
$$\dot{x}_3 = x_4 + d_3(t, \mathbf{x})$$
$$\dot{x}_4 = \frac{-f_{K_2}(t-\tau, \mathbf{x}) - f_{B_2}(t-\tau, \mathbf{x}) + u}{M_2} + d_4(t, \mathbf{x})$$

where $\boldsymbol{y} = [x_1, x_3]^{\mathrm{T}}$ is the displacement of the mass, x_2, x_4 is the velocity of the mass, $f_{K_1}(t-\tau, \mathbf{x}) = x_1^2(t-\tau)$, $f_{K_2}(t-\tau, \mathbf{x}) = x_3^2(t-\tau)$ are the spring force, $f_{B_1}(t-\tau, \mathbf{x}) = 0.5x_2^3$, $f_{B_2}(t-\tau, \mathbf{x}) = 0.5x_4^3$ are the friction force, $m_1 = 1kg$, $m_2 = 1.05kg$ is the body mass, and u is the applied force. The structures of spring force and friction force are assumed to be known. The exogenous disturbance is assumed to be $d_1(t, \mathbf{x}) = 0.1x_1 \sin t$, $d_2(t, \mathbf{x}) = 0.1x_2 \sin t$, $d_3(t, \mathbf{x}) = 0.1x_3 \sin t$, $d_4(t, \mathbf{x}) = 0.1x_4 \sin t$. $\tau = 0.1$ sec is time delay. In the implementation, six fuzzy sets are defined over interval [-3,3] for x_1, x_2, x_3 and x_4, with labels *NB*,*NM*,*NS*,*PS*,*PM*, and *PB*, and their membership functions are

$$\mu_{NB}(x_i) = \frac{1}{1+\exp(-5(x_i+2))},$$
$$\mu_{NM}(x_i) = \frac{1}{1+\exp(-(x_i+1.5)^2)}$$
$$\mu_{NS}(x_i) = \frac{1}{1+\exp(-(x_i+0.5)^2)},$$
$$\mu_{PS}(x_i) = \frac{1}{1+\exp(-(x_i-0.5)^2)},$$
$$\mu_{PM}(x_i) = \frac{1}{1+\exp(-(x_i-1.5)^2)},$$
$$\mu_{PB}(x_i) = \frac{1}{1+\exp(-5(x_i-2))}, i = 1,2,3,4.$$

We apply the robust model reference sliding mode control approach in Section 3 to deal with control problem. The system matrices of reference model are given as follows:

$$A_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -3.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4.9 & -5.9 \end{bmatrix}, \quad B_m = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

and the reference input $r(t) = 9\sin(t)$.

The control object is to maintain the system output $\boldsymbol{y}(t)$ to follow the reference model $\boldsymbol{y}_m = [x_{m1}, x_{m3}]^{\mathrm{T}}$. In the case, the first level sliding surface $s_1 = c_1e_1 + \dot{e}_1$ and $s_2 = c_2e_3 + \dot{e}_3$, where $c_1 = c_2 = 0.9$, the hierarchical sliding surface is constructed as $S = \lambda_1 s_1 + s_2$, where $\lambda_1 = 1.1$. The initial values are chosen as $\mathbf{x}_m(0) = [0,0,0,0]^{\mathrm{T}}$, $\mathbf{x}(0) = [1,0,0,0]^{\mathrm{T}}$, $\boldsymbol{\theta}_{fK_1}(0) = 0$, $\boldsymbol{\theta}_{fK_2}(0) = 0$, $\boldsymbol{\theta}_{fB_1}(0) = 0$, $\boldsymbol{\theta}_{fB_2}(0) = 0$, $\hat{\omega}(0) = 0$, $\hat{h}(0) = 0$, $k = 11.5$, $\gamma_{fK_1} = 8.5$, $\gamma_{fK_2} = 9.5$, $\gamma_{b_1} = 0.1$, $\gamma_{b_2} = 0.1$, $\gamma_{\rho_1} = 0.1$ $\gamma_{\rho_2} = 0.1$, $\gamma_{\omega} = 0.015$, $\eta = 5.2$, and the boundary layer $\varepsilon = 0.01$.

The simulation results are shown in Figs. 4-6. Fig.4 and Fig.5 reveal that the state trajectories, respectively. The control signal is shown in Figs. 6. The simulation results verify the usefulness of the proposed adaptive model reference hierarchical sliding-mode controller.

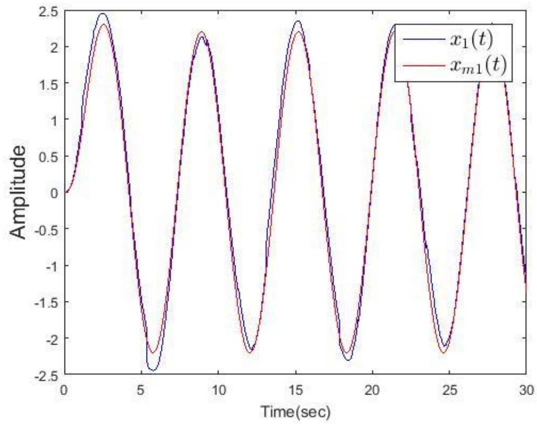

Figure 4. The trajectories of state x_1 and state of reference model x_{m1}

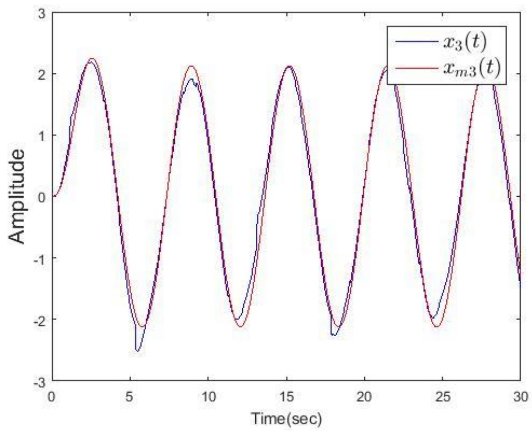

Figure 5. The trajectories of state x_3 and state of reference model x_{m3}

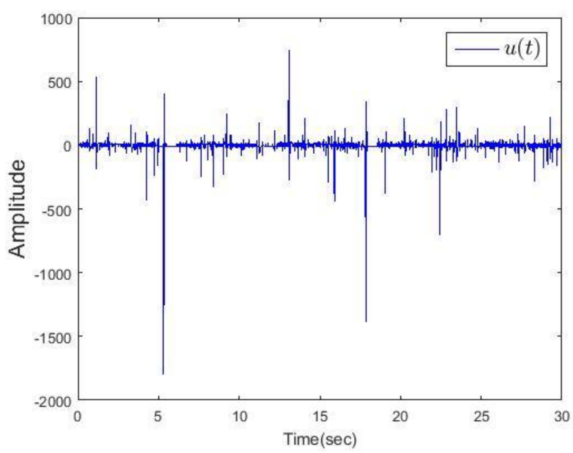

Figure 6. The control signal u

5. Conclusions

This paper proposes an adaptive model reference hierarchical sliding mode control scheme for a class of uncertain underactuated systems that guarantees the closed-loop stability in presence of unknown input dead-zone and time delay. Fuzzy logic systems are used to approximate the unknown nonlinear functions by some adaptive laws. Based on the Lyapunov-Krasovskii stability theorem, the proposed incremental hierarchical structure sliding-mode controller not only guarantees the stability of the uncertain nonlinear state time-delay systems with input dead-zone, but also ensures good tracking performance. Finally, some simulation results of a practical example are illustrated to show the effectiveness of the proposed control method.

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