

# Full Averaging of Control Fuzzy Integrodifferential Inclusions with Terminal Criterion of Quality

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**Abstract** In this paper we consider the fuzzy control system with fuzzy terminal quality criterion (Mayer's fuzzy problem) when the behaviour of system is described by the fuzzy controlled integrodifferential inclusion with a small parameter. We use an averaging method of Krylov-Bogolyubov and set in correspondence to the given problem the Mayer's full-averaged fuzzy problem that is more simple for solving. In paper we receive the conditions when the optimal fuzzy solutions of these problems are close.

**Keywords** Fuzzy Control Integrodifferential Inclusion, Averaging Method, Mayer Fuzzy Problem

## 1. Introduction

Many important problems of analytical dynamics are described by the nonlinear mathematical models that as a rule are presented by the nonlinear differential or the integrodifferential equations. The absence of exact universal research methods for nonlinear systems has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

The averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. Averaging theory for ordinary differential equations has a rich history, dating to back to the work of N.M. Krylov and N.N. Bogoliubov[1], and has been used extensively in engineering applications[2-6]. Books that cover averaging theory for differential equations and inclusions include[7-11].

In recent years, the fuzzy set theory introduced by Zadeh[12] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of science as physical, mathematical, differential equations and engineering sciences. Recently there have been new advances in the theory control fuzzy integrodifferential equations[13-17] and control fuzzy integrodifferential

inclusions[18].

In work[19] full schemes of an average for the fuzzy integrodifferential inclusions have been considered. In this article we prove the substantiation of the method of full averaging for the control fuzzy integrodifferential inclusions with small parameter and terminal criterion of quality (Mayer fuzzy problem). Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging is a template.

## 2. Preliminaries

Let  $\text{comp}(\mathbb{R}^n)(\text{conv}(\mathbb{R}^n))$  be a set of all nonempty (convex) compact subsets from the space  $\mathbb{R}^n$ ,

$$h(A, B) = \min_{r \geq 0} \{S_r(A) \supset B, S_r(B) \supset A\}$$

be Hausdorff distance between sets  $A$  and  $B$ ,  $S_r(A)$  is  $r$ -neighborhood of set  $A$ .

Let  $E^n$  be the set of all  $u: \mathbb{R}^n \rightarrow [0, 1]$  such that  $u$  satisfies the following conditions:

1)  $u$  is normal, that is, there exists an  $x_0 \in \mathbb{R}^n$  such that  $u(x_0) = 1$ ;

2)  $u$  is fuzzy convex, that is,

$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$$

for any  $x, y \in \mathbb{R}^n$  and  $0 \leq \lambda \leq 1$ ;

3)  $u$  is upper semicontinuous;

4)  $[u]^0 = \text{cl}\{x \in \mathbb{R}^n : u(x) > 0\}$  is compact.

If  $u \in E^n$ , then  $u$  is called a fuzzy number, and  $E^n$  is said to be a fuzzy number space. For  $0 < \alpha \leq 1$ , denote  $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$ .

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Then from 1)-4), it follows that the  $\alpha$ -level set  $[u]^\alpha \in \text{conv}(\mathbb{R}^n)$  for all  $0 \leq \alpha \leq 1$ .

Let  $\hat{\theta}$  be the fuzzy mapping defined by  $\hat{\theta}(x) = 0$  if  $x \neq 0$  and  $\hat{\theta}(0) = 1$ .

Define  $D: E^n \times E^n \rightarrow [0, \infty)$  by the relation

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} h([u]^\alpha, [v]^\alpha),$$

where  $h$  is the Hausdorff metric defined in  $\text{comp}(\mathbb{R}^n)$ .

Then  $D$  is a metric in  $E^n$ .

Further we know that [20]:

- 1)  $(E^n, D)$  is a complete metric space,
- 2)  $D(u + w, v + w) = D(u, v)$  for all  $u, v, w \in E^n$ ,
- 3)  $D(\lambda u, \lambda v) = |\lambda| D(u, v)$  for all  $u, v \in E^n$  and  $\lambda \in \mathbb{R}$ .

**Definition 1.** [21] A fuzzy mapping  $F: [0, T] \rightarrow E^n$  is measurable if for all  $\alpha \in [0, 1]$  the set-valued map  $F_\alpha: [0, T] \rightarrow \text{conv}(\mathbb{R}^n)$  defined by  $F_\alpha(t) = [F(t)]^\alpha$  is Lebesgue measurable.

**Definition 2.** [21] A fuzzy mapping  $F: [0, T] \rightarrow E^n$  is said to be integrably bounded if there is an integrable function  $h(t)$  such that  $\|x(t)\| \leq h(t)$  for every  $x(t) \in F_0(t)$ .

**Definition 3.** [21] The integral of a fuzzy mapping

$$F: [0, T] \rightarrow E^n \text{ is defined levelwise by } \left[ \int_0^T F(t) dt \right]^\alpha =$$

$$\int_0^T F_\alpha(t) dt. \text{ The set } \int_0^T F_\alpha(t) dt \text{ of all } \int_0^T f(t) dt \text{ such that}$$

$f: [0, T] \rightarrow \mathbb{R}^n$  is a measurable selection for  $F_\alpha: [0, T] \rightarrow \text{conv}(\mathbb{R}^n)$  for all  $\alpha \in [0, 1]$ .

**Definition 4.** [21] A measurable and integrably bounded fuzzy mapping  $F: [0, T] \rightarrow E^n$  is said to be integrable over

$$[0, T] \text{ if } \int_0^T F(t) dt \in E^n.$$

Note that if  $F: [0, T] \rightarrow E^n$  is measurable and integrably bounded, then  $F$  is integrable. Further if  $F: [0, T] \rightarrow E^n$  is continuous, then it is integrable.

Now we consider following fuzzy integrodifferential inclusion

$$\dot{x} \in F(t, x) + \int_0^t \Phi(t, s, x(s)) ds, \quad x(0) = x_0, \quad (1)$$

where  $\dot{x}$  means  $\frac{dx}{dt}$ ;  $x \in \mathbb{R}^n$  is the state;  $t, s \in \mathbb{R}_+$ ;

$F: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow E^n$  is a fuzzy mapping;

$\Phi: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \rightarrow E^n$  is a fuzzy mapping;  $x_0 \in \mathbb{R}^n$ .

We interpret [22-25] the fuzzy integrodifferential inclusion (1) as a family of integrodifferential inclusions

$$\dot{x}_\alpha \in [F(t, x_\alpha)]^\alpha + \int_0^t [\Phi(t, s, x_\alpha(s))]^\alpha ds, \quad x_\alpha(0) = x_0, \quad (2)$$

where the subscript  $\alpha$  indicates that the  $\alpha$ -level set of a fuzzy set is involved (the system (2) can only have any significance as a replacement for (1) if the solutions generate fuzzy sets (fuzzy R-solution) [25]).

Let  $X(t)$  denotes the fuzzy R-solution of the fuzzy integrodifferential inclusion (1).

Now we consider following control integrodifferential equations with the fuzzy parameter

$$\dot{x} = f(t, x, v_1) + \int_0^t \phi(t, s, x(s), v_2(s)) ds + r(t, v_3, w), \quad (3)$$

$$x(0) = x_0,$$

where  $w \in \mathbb{R}^m$  is the control;  $v_i \in V_i \in E^k$ ,  $i = 1, 2, 3$  are

fuzzy parameters;  $f: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ ,

$\phi: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ ,  $r: \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ .

**Definition 5.** The set  $LW$  of all measurable single-valued branches of the set  $W \in \text{conv}(\mathbb{R}^m)$  is the set of the admissible controls.

Further we consider following control fuzzy integrodifferential inclusions

$$\dot{x} \in F(t, x) + \int_0^t \Phi(t, s, x(s)) ds + R(t, w), \quad x(0) = x_0, \quad (4)$$

where  $F: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow E^n$ ,  $\Phi: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \rightarrow E^n$ ,  $R: \mathbb{R}_+ \times \mathbb{R}^m \rightarrow E^n$  are fuzzy maps such that  $F(t, x) \equiv f(t, x, V_1)$ ,  $\Phi(t, s, x) \equiv \phi(t, s, x, V_2)$ ,  $R(t, w) \equiv r(t, w, V_3)$ .

Obviously, the control fuzzy integrodifferential inclusion (4) turns into the ordinary fuzzy integrodifferential inclusion

$$\dot{x} \in G(t, x) + \int_0^t \Phi(t, s, x(s)) ds, \quad x(0) = x_0, \quad (5)$$

if the control  $w(\cdot) \in LW$  is fixed and  $G(t, x) \equiv F(t, x) + R(t, w(t))$ .

Let  $X(t)$  denotes the fuzzy R-solution of the fuzzy integrodifferential inclusion (5), then  $X(t, w)$  denotes the fuzzy R-solution of the control fuzzy integrodifferential inclusion (4) for the fixed  $w(\cdot) \in LW$ .

**Definition 6.** The set  $Z(T) = \{X(T, w): w(\cdot) \in LW\}$  be called the attainable set of the fuzzy system (4).

### 3. Main Result

In this section we consider the fuzzy control problem with small parameter

$$\dot{x} \in \varepsilon \left[ F(t, x) + \int_0^t \Phi(t, s, x(s)) ds + R(t, w) \right], \quad x(0) = x_0, \quad (6)$$

where  $\varepsilon > 0$  is a small parameter.

In this section we associate with the equation (6) the following averaged integrodifferential equation

$$\dot{y} \in \varepsilon \left[ \bar{F}(y) + \int_0^t \bar{\Phi}(t, y(s)) ds + \nu(t) \right], \quad y(0) = x_0, \quad (7)$$

where

$$\lim_{T \rightarrow \infty} D \left( \frac{1}{T} \int_0^T F(t, x) dt, \bar{F}(x) \right) = 0, \quad (8)$$

$$\bar{\Phi}(t, x) = \begin{cases} \frac{1}{t} \Phi_1(t, x), & t > 0, \\ \lim_{t \rightarrow 0+} \frac{1}{t} \Phi_1(t, x), & t = 0, \end{cases} \quad \Phi_1(t, x) = \int_0^t \Phi(t, s, x) ds, \quad (9)$$

$$\nu \in P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T R(t, W) dt. \quad (10)$$

**Remark.** In this paper we will consider a case when the limits (8), (9) and (10) exist.

We will set correspondence between the control  $w(\cdot)$  of the initial inclusion (6) and by the control  $\nu(\cdot)$  of averaged inclusion (7).

We will put the control  $w(\cdot)$  in correspondence a control  $\nu_w(\cdot)$  as it follows:

1) We calculate

$$\nu_i = \frac{1}{T_1} \int_{iT_1}^{(i+1)T_1} R(t, w(t)) dt, \quad i = 0, 1, 2, \dots,$$

where  $T_1 > 0$  is constant.

2) Now we build the control  $\nu_w(\cdot)$  as it follows:

$$\nu_w(t) = \{\nu_i(t), \quad iT_1 \leq t < (i+1)T_1, \quad i = 0, 1, 2, \dots\},$$

where

$$\min_{\bar{\nu}(t) \in P} D(\nu_i, \bar{\nu}(t)) = D(\nu_i, \nu_i(t)).$$

We will put the control  $\nu(\cdot)$  in correspondence a control  $\nu_w(\cdot)$  as it follows:

1) We calculate

$$\nu_i = \frac{1}{T_1} \int_{iT_1}^{(i+1)T_1} \nu(t) dt, \quad i = 0, 1, 2, \dots,$$

where  $T_1 > 0$  is constant.

2) Now we build the control  $w_\nu(\cdot)$  as it follows:

$$w_\nu(t) = \{w_i(t), \quad iT_1 \leq t < (i+1)T_1, \quad i = 0, 1, 2, \dots\},$$

where

$$\begin{aligned} \min_{\bar{w}(t) \in W} D \left( \frac{1}{T_1} \int_{iT_1}^{(i+1)T_1} R(t, \bar{w}(t)) dt, \nu_i \right) \\ = D \left( \frac{1}{T_1} \int_{iT_1}^{(i+1)T_1} R(t, w_\nu(t)) dt, \nu_i \right) \end{aligned}$$

Now we consider Mayer fuzzy problem. As is generally known, the Mayer problem, is to maximize, over all

solutions to control system defined on fixed time interval, a functional depending on the final position [26, 27].

**Mayer fuzzy problem:** let  $\varphi: E^n \rightarrow E^1$  be a continuous fuzzy map and  $T > 0$ . For example  $\varphi(x)$  such that  $[\varphi(X)]^\alpha = [\zeta_{\min}^\alpha, \zeta_{\max}^\alpha]$  for all  $\alpha \in [0, 1]$ , where

$$\zeta_{\min}^\alpha = \min_{\zeta \in [X]^\alpha} (\zeta, \psi), \quad \zeta_{\max}^\alpha = \max_{\zeta \in [X]^\alpha} (\zeta, \psi) \quad \psi \in R^n$$

is constant,

$$(\zeta, \psi) = \zeta_1 \psi_1 + \dots + \zeta_n \psi_n.$$

**Definition 6.** An admissible control  $w^*(\cdot)$  is said to be an optimal control for problem (6) if

$$m[\varphi(X(T, w^*))]^0 \geq m[\varphi(X(T, w))]^0 \quad (11)$$

for all admissible control  $w(\cdot)$ , where  $mA = \min\{a: a \in A\}$ ,  $A \in \text{conv}(R^1)$ .

Now we obtain the main result of this article.

**Theorem.** Let in domain

$$Q = \{(t, x, w) \mid t \geq 0, x \in G \subset R^n, w \in W\}$$

the following hold:

- 1)  $F(t, x)$  is continuous in  $(t, x) \in R_+ \times G$ ;
- 2)  $\Phi(t, s, x)$  is continuous in  $(t, s, x) \in R_+ \times R_+ \times G$ ;
- 3) there exist constants  $\chi, M_1$  such that

$$D(F(t, x), \theta) \leq M_1,$$

$$D(F(t, x_1), F(t, x_2)) \leq \chi \|x_1 - x_2\|,$$

for all  $(t, x) \in Q$ , and any  $x_1, x_2 \in G$ ;

4) there exist continuous functions  $\mu(t, s)$ ,  $K(t, s)$ , and constant  $M_2$  such that

$$D(\Phi(t, s, x_1), \Phi(t, s, x_2)) \leq \mu(t, s) \|x_1 - x_2\|,$$

$$D(\Phi(t, s, x), \theta) \leq K(t, s), \quad \int_{t_1}^{t_2} \int_0^t K(t, s) ds dt \leq M_2 (t_2 - t_1),$$

for any  $0 \leq t_1 \leq t_2 < \infty$ , and any  $x_1, x_2 \in G$ ;

5) there exist constants  $M_3$ , and  $M_4$  such that

$$\int_{t_1}^{t_2} \int_0^t \mu(t, s) ds dt \leq M_3 (t_2 - t_1),$$

$$\int_{t_1}^{t_2} \int_0^t \mu(t, s) (t - s) ds dt \leq M_4 (t_2 - t_1),$$

for any  $0 \leq t_1 \leq t_2 < \infty$ ;

6) there exist continuous functions  $\varpi(t)$ ,  $\bar{K}(t)$ , and constants  $\varpi_1$ ,  $\varpi_2$ ,  $K_1$  such that

$$D(\Phi(t, x), \theta) \leq \bar{K}(t),$$

$$D(\Phi(t, x_1), \Phi(t, x_2)) \leq \varpi(t) \|x_2 - x_1\|,$$

$$\int_{t_1}^{t_2} t \bar{K}(t) dt \leq K_1 (t_2 - t_1), \quad \int_{t_1}^{t_2} t \varpi(t) dt \leq \varpi_1 (t_2 - t_1),$$

$$\int_{t_1}^{t_2} t^2 \varpi(t) dt \leq \varpi_2 (t_2 - t_1),$$

for any  $0 \leq t_1 \leq t_2 < \infty$  and  $x_1, x_2 \in G$ ;

7) the limits (8),(9) exist uniformly in  $x \in G$ ;

8)  $R(t, w)$  is continuous in  $(t, w) \in R_+ \times W$ ;

9) there exists constant  $M_5$  such that

$$D(R(t, w), \theta) \leq M_5$$

for all  $(t, w) \in R_+ \times W$ ;

10) the limit (10) exists;

11) for any  $w(\cdot) \in LW$  and  $t \geq 0$  the fuzzy R-solution of the system (7) together with a  $\sigma$ -neighbourhood belong to the domain  $G$ , i.e.  $[X(t, w)]^0 \subset G$ ;

12) there exists a constant  $\lambda > 0$  such that  $D(\varphi(X_1), \varphi(X_2)) \leq \lambda D(X_1, X_2)$  for all  $X_1, X_2 \in E^n$ ;

Then for any  $\eta > 0$  and  $L > 0$  there exists  $\varepsilon^0(\eta, L) \in (0, \sigma]$  such that for all  $\varepsilon \in (0, \varepsilon^0]$  and  $T \in [0, L\varepsilon^{-1}]$  the following statements hold:

a) for an optimal control  $w^*(\cdot)$  of Mayer fuzzy problem (6) there exists an admissible control  $v_{w^*}(\cdot)$  of the fuzzy system (7) such that

$$h([\varphi(X(T, w^*))]^0, [\varphi(Y(T, v_{w^*}))]^0) < \eta; \quad (12)$$

b) for an optimal control  $v^*(\cdot)$  of Mayer fuzzy problem (7) there exists an admissible control  $w_{v^*}(\cdot)$  of the fuzzy system (6) such that

$$h([\varphi(Y(T, v^*))]^0, [\varphi(X(T, w_{v^*}))]^0) < \eta; \quad (13)$$

$$c) h([X(T, w^*)]^0, [Y(T, v^*)]^0) < \eta, \quad (14)$$

where  $w^*(\cdot), v^*(\cdot)$  are an optimal control of Mayer fuzzy problems (6) and (7);

**Proof.** Let  $w^*(\cdot)$  and  $v^*(\cdot)$  are optimal controls of Mayer fuzzy problems (6) and (7).

Let  $X(\cdot, w^*)$  and  $Y(\cdot, v^*)$  are fuzzy R-solutions of following fuzzy systems

$$\dot{x} \in \varepsilon[F(t, x) + \int_0^t \Phi(t, s, x(s))ds + R(t, w^*(t))], \quad x(0) = x_0,$$

$$\dot{y} \in \varepsilon[\bar{F}(y) + \int_0^t \bar{\Phi}(t, y(s))ds + v^*(t)], \quad y(0) = x_0.$$

Let  $v_{w^*}(\cdot)$  is an admissible control of the fuzzy system (7) corresponding to the optimal control of  $w^*(\cdot)$ , and  $w_{v^*}(\cdot)$  is an admissible control of the fuzzy system (6) corresponding to the optimal control of  $v^*(\cdot)$ .

Let  $X(\cdot, w_{v^*})$  and  $Y(\cdot, v_{w^*})$  are fuzzy R-solutions of following fuzzy systems

$$\dot{x} \in \varepsilon[F(t, x) + \int_0^t \Phi(t, s, x(s))ds + R(t, w_{v^*}(t))], \quad x(0) = x_0,$$

$$\dot{y} \in \varepsilon[\bar{F}(y) + \int_0^t \bar{\Phi}(t, y(s))ds + v_{w^*}(t)], \quad y(0) = x_0.$$

By conditions 1)-12) and [44] so that

$$D(X(t, w^*), Y(t, v_{w^*})) < \theta, \quad D(X(t, w_{v^*}), Y(t, v^*)) < \theta$$

for all  $t \in [0, T]$ .

Then we get

$$\begin{aligned} & |m[\varphi(X(T, w^*))]^0 - m[\varphi(Y(T, v_{w^*}))]^0| \leq \\ & \leq h([\varphi(X(T, w^*))]^0, [\varphi(Y(T, v_{w^*}))]^0) \leq \\ & \leq D(\varphi(X(T, w^*)), \varphi(Y(T, v_{w^*}))) \leq \\ & \leq \lambda D(X(T, w^*), Y(T, v_{w^*})) \leq \lambda \theta, \end{aligned} \quad (15)$$

$$\begin{aligned} & |m[\varphi(Y(T, v^*))]^0 - m[\varphi(X(T, w_{v^*}))]^0| \leq \\ & \leq h([\varphi(Y(T, v^*))]^0, [\varphi(X(T, w_{v^*}))]^0) \leq \\ & \leq D(\varphi(Y(T, v^*)), \varphi(X(T, w_{v^*}))) \leq \\ & \leq \lambda D(Y(T, v^*), X(T, w_{v^*})) \leq \lambda \theta. \end{aligned} \quad (16)$$

Let  $\eta = \lambda \theta$  then we obtain (15) and (16).

Since  $w^*(\cdot), v^*(\cdot)$  are optimal controls of Mayer fuzzy problems (6) and (7), and  $w_{v^*}(\cdot), v_{w^*}(\cdot)$  are admissible controls of the fuzzy systems (6) and (7), we have

$$m[\varphi(X(T, w^*))]^0 \geq m[\varphi(X(T, w_{v^*}))]^0,$$

$$m[\varphi(Y(T, v^*))]^0 \geq m[\varphi(Y(T, v_{w^*}))]^0.$$

Also, we obviously have

$$m[\varphi(Y(T, v_{w^*}))]^0 + \eta \geq m[\varphi(X(T, w^*))]^0 >$$

$$> m[\varphi(Y(T, v^*))]^0 \geq m[\varphi(Y(T, v_{w^*}))]^0$$

or

$$m[\varphi(X(T, w_{v^*}))]^0 + \eta \geq m[\varphi(Y(T, v^*))]^0 \geq$$

$$\geq m[\varphi(X(T, w^*))]^0 \geq m[\varphi(X(T, w_{v^*}))]^0$$

Hence, we obtain (12) - (14). The theorem is proved.

**Remark.** If we replace (11) on

$$m[\varphi(X(T, w^*))]^1 \geq m[\varphi(X(T, w))]^1$$

or

$$M[\varphi(X(T, w^*))]^0 \geq M[\varphi(X(T, w))]^0$$

or

$$M[\varphi(X(T, w^*))]^1 \geq M[\varphi(X(T, w))]^1,$$

where  $MA = \max\{a : a \in A\}$ ,  $A \in \text{conv}(R^1)$ , then a theorem will be just.

## 4. Conclusions

In this article we prove the substantiation of the method of full averaging for the control fuzzy integrodifferential inclusions with small parameter and terminal criterion of quality (Mayer fuzzy problem). This result generalize the results of A.V. Plotnikov [28, 29] for the control ordinary differential inclusions with small parameter and terminal criterion of quality.

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