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Experimental Study of Synchronization & Anti-synchronization for Spin Orbit Problem of Enceladus

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Abstract In this paper, we have investigated the synchronization and anti-synchronization phenomenon of two identical spin orbit problem of enceladus evolving from different initial conditions using the active control technique based on the Lyapunov stability theory and Routh-Hurwitz criteria. The designed controller, with our own choice of the coefficient matrix of the error dynamics that satisfy the Lyapunov stability theory and Routh-Hurwitz criteria, are found to be effective in the stabilization of the error states at the origin, thereby, achieving synchronization and anti-synchronization between the states variables of two nonlinear dynamical systems under consideration. The results are validated by numerical simulations using *mathematica*.

Keywords Synchronization, Active Control, Anti-synchronization (AS)

1. Introduction

Synchronization in chaotic dynamical systems has been of major interest both from a fundamental point of view and due to its potential applications in a wide variety of systems. In recent times the phenomenon of synchronization in nonlinear dynamics has received considerable attention and has been used to understand a wide variety of topics in almost all fields of nonlinear sciences[1-5]. Synchronization techniques have been improved in recent years and many different methods are applied theoretically as well as experimentally to synchronize the chaotic systems[6-10]. Notable among these methods, chaos synchronization using active control scheme has recently been widely accepted as one of the efficient technique to synchronize the chaotic systems which is based on the Lyapunov stability theory and Routh-Hurwitz criteria to use active control in order to achieve stable synchronization has been applied to many practical systems successfully[11-24].

In this article, we have applied the active control technique based on the Lyapunov stability theory and Routh-Hurwitz criteria to study the synchronization and AS phenomenon of two identical spin orbit problem of enceladus (a satellite of Saturn) in elliptic orbit evolving from different initial conditions. The system under consideration is chaotic for some values of parameter involved in the system. In synchronization, the two systems (master & slave) are synchronized that starts with different initial conditions. The same problem may be treated as the design of control laws for full chaotic slave system using the known information of the master system so as to ensure that the controlled receiver synchronizes with the master system. Hence, the slave chaotic system completely traces the dynamics of the master system in the course of time. The aim of this study is to trace the chaotic dynamics of the spin orbit problem of enceladus, a satellite of Saturn in elliptic orbit based on synchronization and AS. To the best of my knowledge nobody studied this before.

2. Description of the Model

An approximate model for synchronous rotation in the spin-orbit problem of enceladus using the standard techniqu es of Hamiltonian perturbation theory was developed by Wisdom[25]. It is based on approximation for a fixed orbit the planet-to-satellite distance and the true anomaly is periodic. The Hamiltonian governing the rotational dynamics of an out of round satellite in a fixed elliptical orbit with spin axis perpendicular to the orbit plane is

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$$H(p,\theta,t)=\frac{p^2}{2C}-\frac{\varepsilon^2 n^2 C}{4}\cos(2\theta-2nt)+\frac{\varepsilon^2 n^2 Ce}{8}\left\{\cos(2\theta-nt)-7\cos(2\theta-3nt)\right\}. \tag{2.1}$$

Where $\varepsilon=\sqrt{3(B-A)/C}$, the moments of inertia are $A<B<C$, n is the orbital frequency, θ measures the orientation of the axis of minimum moment from the line to pericenter, p is the angular momentum conjugate to θ.

Using the Hamilton's equations $\left(\text{i.e. } \frac{d\theta}{dt}=\frac{\partial H}{\partial\dot{\theta}} \;\&\; \frac{d\dot{\theta}}{dt}=-\frac{\partial H}{\partial\theta}\right)$ and (2.1), the equation of motion of the system under study can be written as:

$$\frac{d^2\theta}{dt^2}=\frac{\varepsilon^2 n^2 e}{4}\sin(2\theta-nt)-\frac{7\varepsilon^2 n^2 e}{4}\sin(2\theta-3nt)-\frac{\varepsilon^2 n^2}{2}\sin(2\theta-2nt). \tag{2.2}$$

3. Synchronization VIA Active Control

For a system of two coupled chaotic dynamical systems:

$$\text{Master System:}\begin{cases}\dot{x}_i=x_{i+1} & 1\le i\le n-1\\ \dot{x}_n=f(x,t) & x=[x_1,x_2...x_n]\in\Re^n\end{cases} \tag{3.1}$$

$$\text{Slave System:}\begin{cases}\dot{y}_i=y_{i+1}+u_i(t) & 1\le i\le n\\ \dot{y}_n=g(y,t)+u_n(t), & y=[y_1,y_2...y_n]\in\Re^n \;\&\; u=[u_1,u_2...u_n]\in\Re\end{cases}, \tag{3.2}$$

where $x(t)\in\Re^n$ & $y(t)\in\Re^n$ are the phase space (state variables), $f(x,t)$ & $g(y,t)$ are the corresponding nonlinear functions and $u(t)$ are the control functions to be determined, synchronization in a direct sense implies $\lim_{t\to\infty}|x(t)-y(t)|\to 0$. When this occurs the coupled systems are said to be completely synchronized. Since chaos synchronization is related to the observer problem in control theory [26], the problem may be treated as the design of control laws for full chaotic slave system using the known information of the master system so as to ensure that the controlled receiver synchronizes with the master system and hence, the slave chaotic system completely traces the dynamics of the master in the course of time.

Introducing the two variables: $\theta=x_1$ and $\dot{x}_1=x_2$, the system defined by (2.2) can be written as

$$\text{Master system:}\begin{cases}\dot{x}_1=x_2,\\ \dot{x}_2=\frac{\varepsilon^2 n^2 e}{4}\sin(2x_1-nt)-\frac{7\varepsilon^2 n^2 e}{4}\sin(2x_1-3nt)-\frac{\varepsilon^2 n^2}{2}\sin(2x_1-2nt).\end{cases} \tag{3.3}$$

$$\text{Slave System:}\begin{cases}\dot{y}_1=y_2+u_1(t),\\ \dot{y}_2=\frac{\varepsilon^2 n^2 e}{4}\sin(2y_1-nt)-\frac{7\varepsilon^2 n^2 e}{4}\sin(2y_1-3nt)-\frac{\varepsilon^2 n^2}{2}\sin(2y_1-2nt)+u_2(t).\end{cases} \tag{3.4}$$

Where $u_1(t)$ and $u_2(t)$ are control functions to be determined. Let $e_i(t)=y_i(t)-x_i(t)$ be the synchronization errors such that $\lim_{t\to\infty}e_i(t)\to 0$ for $i=1,2$. From (3.3) and (3.4), we have

$$\begin{aligned}\dot{e}_1(t)&=e_2(t)+u_1(t),\\ \dot{e}_2(t)&=\frac{\varepsilon^2 n^2 e}{4}\left[\sin(2y_1-nt)-\sin(2x_1-nt)\right]+\frac{7\varepsilon^2 n^2 e}{4}\left[\sin(2x_1-3nt)-\sin(2y_1-3nt)\right]\\ &\quad+\frac{\varepsilon^2 n^2}{2}\left[\sin(2x_1-2nt)-\sin(2y_1-2nt)\right]+u_2(t).\end{aligned} \tag{3.5}$$

In order to express (3.5) as only linear terms in $e_1(t)$ and $e_2(t)$, we redefine the control functions as follows:

$$\begin{aligned}u_1(t)&=v_1(t),\\ u_2(t)&=-\frac{\varepsilon^2 n^2 e}{4}\left[\sin(2y_1-nt)-\sin(2x_1-nt)\right]-\frac{7\varepsilon^2 n^2 e}{4}\left[\sin(2x_1-3nt)-\sin(2y_1-3nt)\right]\\ &\quad-\frac{\varepsilon^2 n^2}{2}\left[\sin(2x_1-2nt)-\sin(2y_1-2nt)\right]+v_2(t).\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we have

$$\dot{e}_1(t) = e_2(t) + v_1(t),$$
$$\dot{e}_2(t) = v_2(t). \quad (3.7)$$

Equation (3.7) is the error dynamics, which can be interpreted as a control problem where the system, to be controlled is a linear system with control inputs $v_i(t) = v_i\left(e_i(t), e_i(t)\right)$ for $i = 1,2$. As long as these feedbacks stabilize the system, $\lim_{t\to\infty} e_i(t) \to 0$ for $i = 1,2$. This simply implies that the two systems (3.3) and (3.4) evolving from different initial conditions are synchronized. As functions of $e_1(t)$ and $e_2(t)$, we choose $v_1(t)$ and $v_2(t)$ as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = D \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \quad (3.8)$$

where $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is a 2×2 constant feedback matrix to be determined. Hence the error system (3.7) can be written as:

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{pmatrix} = C \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \quad (3.9)$$

where $C = \begin{pmatrix} a & 1+b \\ c & d \end{pmatrix}$, is the coefficient matrix.

According to the Lyapunov stability theory and the Routh-Hurwitz criteria, if

$$a + d < 0,$$
$$c(1+b) - ad < 0 \quad (3.10)$$

then the eigen values of the coefficient matrix of error system (3.7) must be real or complex with negative real parts and, hence, stable synchronized dynamics between systems (3.3) and (3.4) is guaranteed. Let

$$a + d = c(1+b) - ad = -E, \quad (3.11)$$

Where $E > 0$ is a real number which is usually set equal to 1. There are several ways of choosing the constant elements *a, b, c, d* of matrix *D* in order to satisfy the Lyapunov stability theory and the Routh-Hurwitz criteria (3.10).

4. Anti-synchronization VIA Active Control

AS of two coupled systems (3.1) and (3.2) means $\lim_{t\to\infty} |x(t) + y(t)| \to 0$. This phenomenon has been investigated both experimentally and theoretically in many physical systems by many researchers[19, 20, 24, 27-30]. A recent study of the AS phenomenon in non-equilibrium systems suggests that it could be used as a technique for particle separation in a mixture of interacting particles[20]. It has been shown that AS was working faster than synchronization in the study of Shahzad[24].

In order to formulate the active controllers for AS, we need to redefine the error functions as $e_i(t) = y_i(t) + x_i(t)$, where $e_i(t)$ are called the AS errors such that $\lim_{t\to\infty} e_i(t) \to 0$ for $i = 1,2$. From (3.3) and (3.4), error dynamics for AS can be written as:

$$\dot{e}_1(t) = e_2(t) + u_1(t),$$
$$\dot{e}_2(t) = \frac{\varepsilon^2 n^2 e}{4}\left[\sin\left(2y_1 - nt\right) + \sin\left(2x_1 - nt\right)\right]$$
$$- \frac{7\varepsilon^2 n^2 e}{4}\left[\sin\left(2x_1 - 3nt\right) + \sin\left(2y_1 - 3nt\right)\right]$$
$$- \frac{\varepsilon^2 n^2}{2}\left[\sin\left(2x_1 - 2nt\right) + \sin\left(2y_1 - 2nt\right)\right] + u_2(t) \quad (4.1)$$

In order to express (4.1) as only linear terms in $e_1(t)$ and $e_2(t)$, we redefine the control functions as follows:

$$u_1(t) = v_1(t),$$
$$u_2(t) = -\frac{\varepsilon^2 n^2 e}{4}\left[\sin\left(2y_1 - nt\right) - \sin\left(2x_1 - nt\right)\right]$$
$$- \frac{7\varepsilon^2 n^2 e}{4}\left[\sin\left(2x_1 - 3nt\right) - \sin\left(2y_1 - 3nt\right)\right]$$
$$- \frac{\varepsilon^2 n^2}{2}\left[\sin\left(2x_1 - 2nt\right) - \sin\left(2y_1 - 2nt\right)\right] + v_2(t) \quad (4.2)$$

From (4.1) and (4.2), we have

$$\dot{e}_1(t) = e_2(t) + v_1(t),$$
$$\dot{e}_2(t) = v_2(t). \quad (4.3)$$

Equation (4.3) is the error dynamics, which can be interpreted as a control problem where the system, to be controlled is a linear system with control inputs $v_i(t) = v_i\left(e_i(t), e_i(t)\right)$ for $i = 1,2$. As long as these feedbacks stabilize the system, $\lim_{t\to\infty} e_i(t) \to 0$ for $i = 1,2$. This simply implies that the two systems (3.3) and (3.4) evolving from different initial conditions are AS. As functions of $e_1(t)$ and $e_2(t)$, we choose $v_1(t)$ and $v_2(t)$ as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = D \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \quad (4.4)$$

where $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is a 2×2 constant feedback matrix to be determined. Hence the error system (4.3) can be written as:

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{pmatrix} = C \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \quad (4.5)$$

where $C = \begin{pmatrix} a & 1+b \\ c & d \end{pmatrix}$, is the coefficient matrix.

According to the Lyapunov stability theory and the Routh-Hurwitz criteria, if

$$a + d < 0,$$
$$c(1+b) - ad < 0 \quad (4.6)$$

then the eigen values of the coefficient matrix of error system (4.3) must be real or complex with negative real parts and, hence, stable anti-synchronized dynamics between systems (3.3) and (3.4) is guaranteed. Let

$$a+d=c(1+b)-ad=-E, \quad (4.7)$$

Where $E>0$ is a real number which is usually set equal to 1. There are several ways of choosing the constant elements a, b, c, d of matrix D in order to satisfy the Lyapunov stability theory and the Routh-Hurwitz criteria (4.6).

5. Numerical Simulation

For the constant elements of feedback matrix, choosing $a=d=-0.5$ and for the parameters involved in system under investigation, $e=0.5$, $n=0.5$, $\varepsilon=0.5$ and together with the initial conditions $[x_1(0), y_1(0)]=[0,0]$ and $[x_2(0), y_2(0)]=[0.1,1]$, we have simulated the system under consideration using *mathematica* for both synchronization as well as AS phenomenon. The obtained results show that the system under consideration achieved synchronization & AS. It can be easily seen in figures 1 & 2 that the time series of the states variables are started to synchronize as $t\to 10$ and figure 3 is the witness of the synchronization errors that are approaching towards zero as $t\to 10$ between the states variables of the master & slave systems given by (3.3) & (3.4). For the same master & slave systems, AS phenomenon can be seen in the figures 4-6. Further, it also has been confirmed by the convergence of the synchronization and AS quality defined by

$$e(t)=\sqrt{e_1^2(t)+e_2^2(t)} \quad (5.1)$$

Figure (7) confirms that the convergence quality in both synchronization and AS phenomenon is almost same for the simulated system.

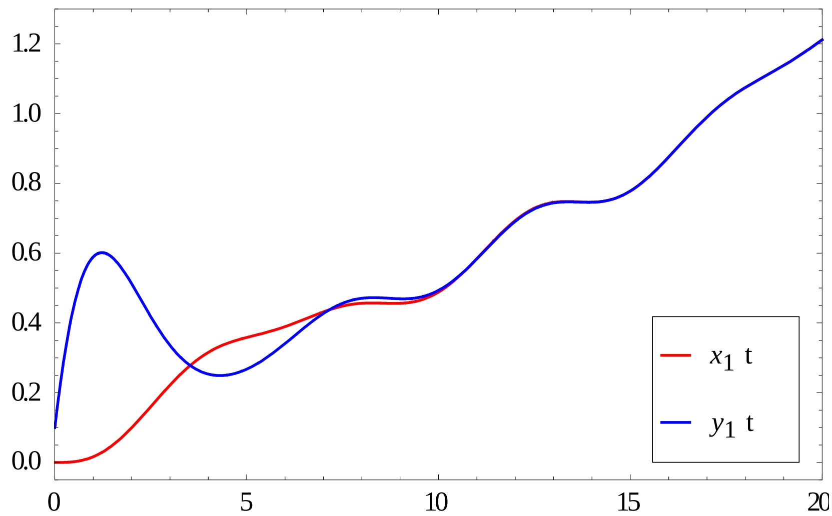

Figure 1. Time Series of x_1 & y_1 for Synchronization

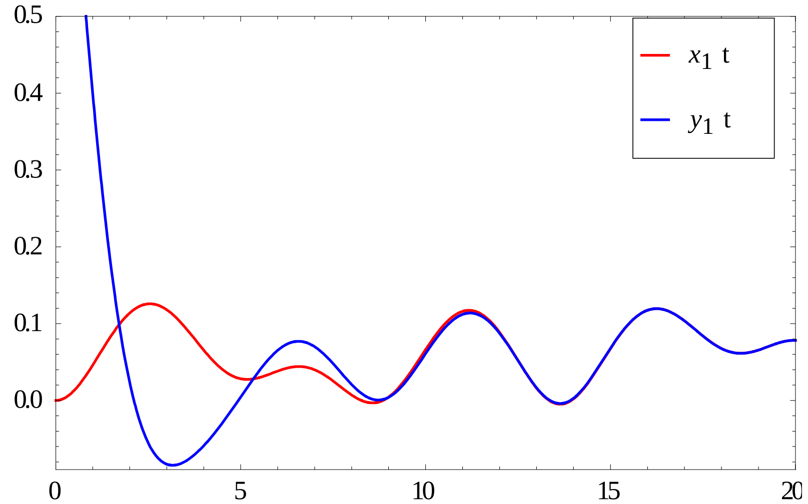

Figure 2. Time Series of x_2 & y_2 for Synchronization

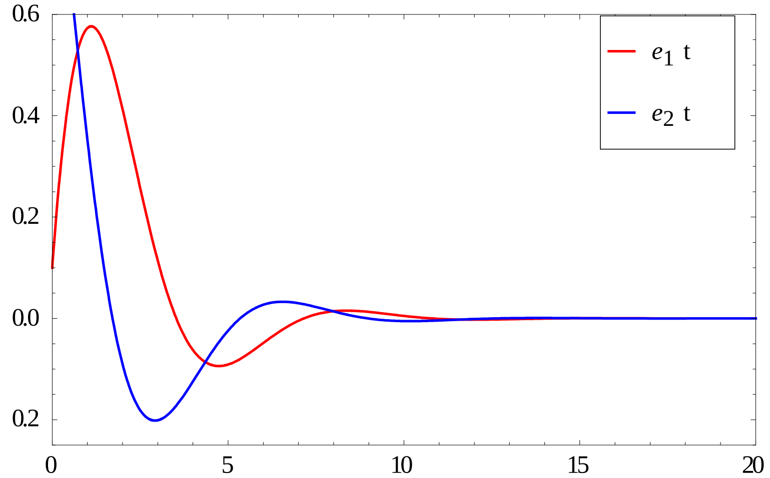

Figure 3. Time Series of e_2 & e_2 for Synchronization

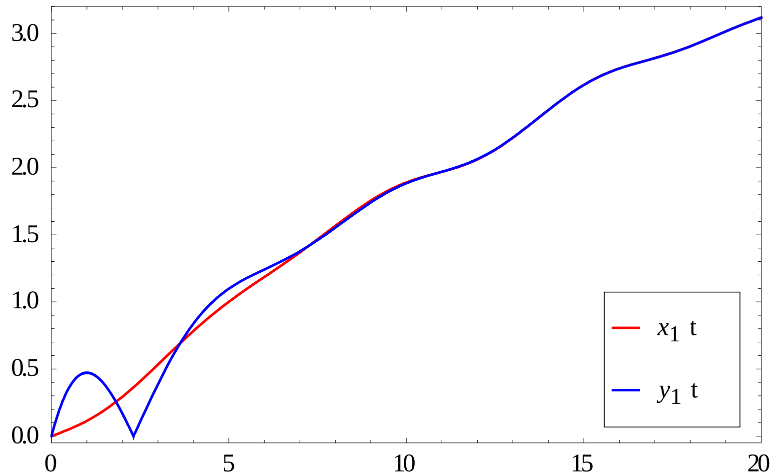

Figure 4. Time Series of x_2 & y_2 for A Synchronization

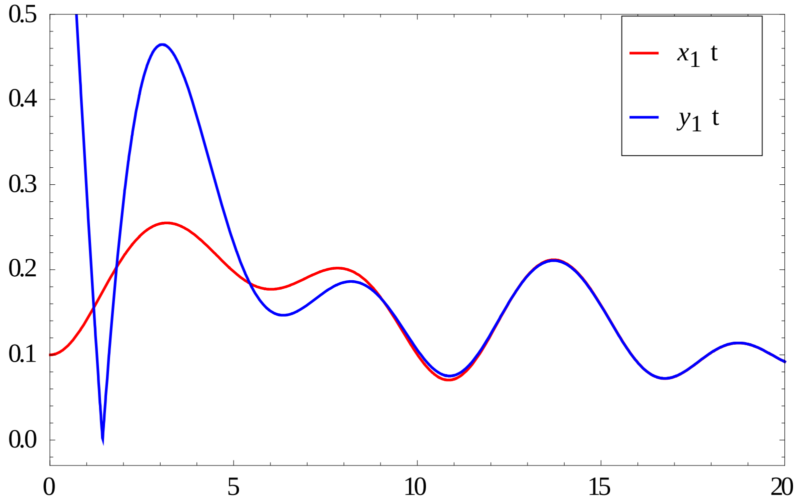

Figure 5. Time Series of x_2 & y_2 for A Synchronization

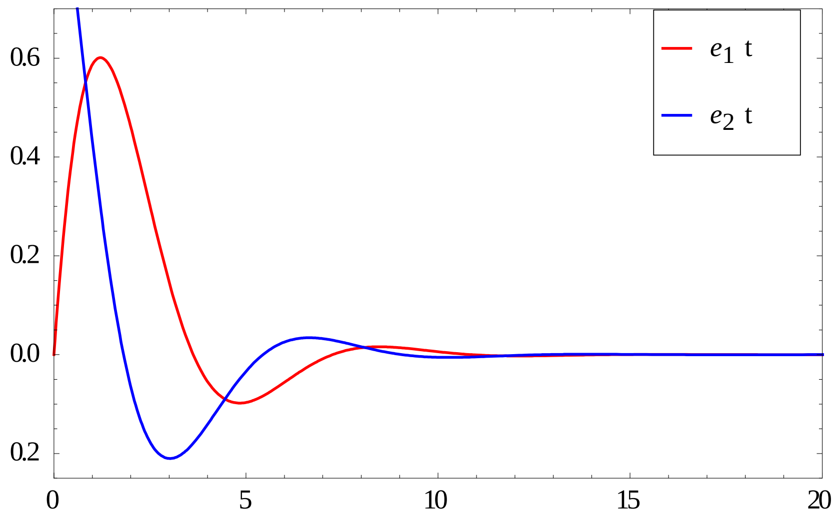

Figure 6. Time Series of e_2 & e_2 for Synchronization

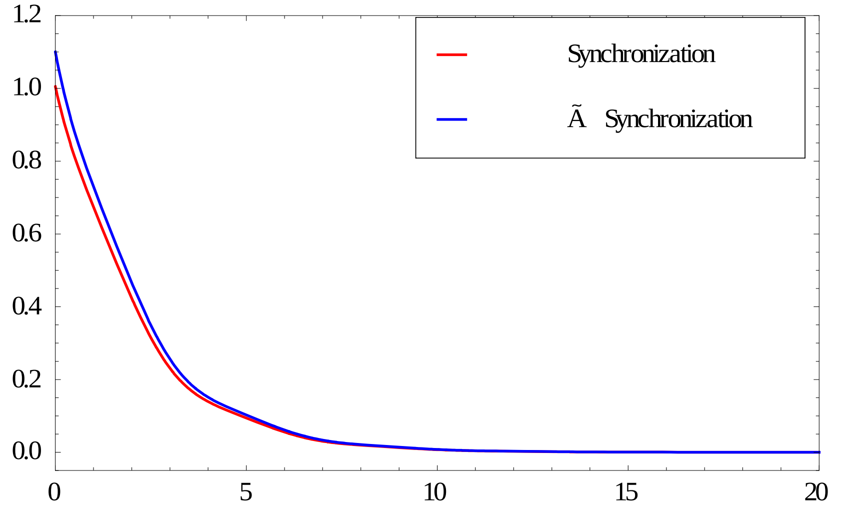

Figure 7. Convergence of Errors

6. Conclusions

In this paper, we have investigated the synchronization and AS behaviour of the two identical spin orbit problem of a satellite in elliptic orbit evolving from different initial conditions via the active control technique based on the Lyapunov stability theory and the Routh-Hurwitz criteria. The results were validated by numerical simulations using *mathematica*. For the errors in synchronization and AS behavior of the system under study, we have observed that the rate of convergence of errors is almost same in synchronization as well AS phenomenon that has been shown in figure (7).

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