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Hyperchaotification and Synchronization of Chaotic Systems

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Abstract This paper addresses the hyperchaotification and the synchronization of chaotic systems. A nonlinear state feedback controller was designed to generate hyperchaos from the original chaotic system. The hyperchaos was identified by the existence of two positive Lyapunov exponents, bifurcation diagram and phase diagrams. Furthermore, effective active controllers are designed for synchronizing the obtained hyperchaotic system with different chaotic systems. To illustrate the effectiveness of the proposed approach, numerical simulation results obtained with the Lü and Qi chaotic systems are given.

Keywords Synchronisation, Active Control, Hyperchaotification

1. Introduction

Chaos has been extensively studied within the scientific, engineering and mathematical communities as an interesting complex dynamic phenomenon. Recently, the traditional trend of understanding and analysing chaos has evolved to a new phase of investigation: controlling and creating chaos. More specifically, when chaos is useful, it is generated intentionally. However, when chaos is harmful, it is controlled[1-4].

Indeed, several studies have showed that chaos can be useful or has great potential in many disciplines and most of the developed methods concern with the chaotic synchronization[5-8]. Pecora and Carroll[5] suggested that the phenomenon of chaos synchronism may serve as the basis for new ways for achieving secure communication. Since, many techniques have been proposed in order to hide the contents of a message by exploiting chaotic systems. Perez and Cerderia have proved that messages masked by simple chaotic processes can be easily extracted once intercepted[9].

After, Pecora proved that this problem can be solved by using higher dimensional hyperchaotic systems[10]. This consideration has led to the development of interesting techniques for hyperchaos synchronization[11,12].

Hyperchaotic systems have the characteristics of high security and high efficiency. They can be broadly applied in nonlinear circuits, secure communications, biological systems and many other fields.

In this paper, we are interested in the hyperchaotification and synchronization of chaotic systems. The hyperchaotification is obtained using a delay feedback control algorithm. It is based on the idea that hyperchaotic systems are usually defined as chaotic systems with more than one positive Lyapunov exponent. A closed form expression was provided for the controller, in terms of the system state vector and a set of Lyapunov exponents. Simultaneously, the controlled system is synchronized with a second chaotic system, by applying an algorithm based on the active control theory.

The paper is organized as follows: In section 2, we introduce the hyperchaotification algorithm and its application to the Lü system. In section 3, the proposed synchronization algorithm is detailed on chaotic and hyperchaotic systems and we provide some simulation results obtained with the Lü and Qi systems. These results illustrate the effectiveness of the proposed procedure.

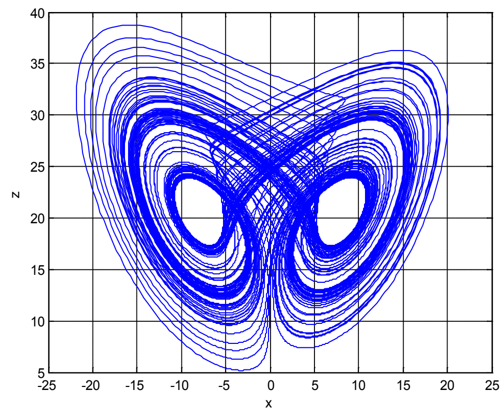

Figure 1. Attractors of the Lü system for $a = 36$, $b = 3$ and $c = 20$

2. Hyperchaotification of Lü System

The Lü system is described by the following equations:

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$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = -xz + cy \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

Where a, b and c represent the system parameters

The system of differential equations is integrated using the fourth order Runge-Kutta method.

For: $a = 36$, $b = 3$ and $c = 20$, the Lyapunov exponents, are $\lambda_1 = 1.33$, $\lambda_2 = 0.00$, $\lambda_3 = -20.10$.

We deduce that for these values, the system is chaotic since one of the exponents is positive and the system exhibits a chaotic behaviour, as shown in Fig. 1.

For the hyperchaotification of a chaotic system, the two following conditions must be verified:

The dimension of the system must be at least equal to 4 and the order of the state equation must be at least 2.

The system must have at least two positive Lyapunov exponents and the sum of all the exponents must be negative.

For this, the hyperchaotification of the Lü chaotic system consists to increase the dimension of the system by adding an equation representing a state feedback controller in the state equations system. Some authors suggested to construct a hyperchaotic attractor of the Lü system by adding the state feedback controller u on the state x. In our case, we choose to introduce it on y, as follows:

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = -xz + cy + u \\ \dot{z} = xy - bz \\ \dot{u} = -yz + du \end{cases} \tag{2}$$

The fixed points are obtained by solving:

$$\begin{cases} a(y-x) = 0 \\ -xz + cy + u = 0 \\ xy - bz = 0 \\ -yz + du = 0 \end{cases} \tag{3}$$

And the jacobian matrix of the system is given by:

$$J = \begin{bmatrix} -a & a & 0 & 0 \\ -z & c & -x & 1 \\ y & x & -b & 0 \\ 0 & -z & -y & d \end{bmatrix} \tag{4}$$

This implies that the stability around the fixed points is function of (a, b, c) and also d.

Assume that the Lyapunov exponents of the controlled chaotic Lü system are λ_i (i = 1,...,4) satisfying $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, then the dynamical behaviours of the system can be classified as follows:

For $\lambda_1 > 0$; $\lambda_2 = 0$, $\lambda_4 < \lambda_3 < 0$ and $\lambda_1 + \lambda_3 + \lambda_4 < 0$, the system is chaotic.

For $\lambda_1 > \lambda_2 > 0$; $\lambda_3 = 0$, $\lambda_4 < 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 0$, the system is hyperchaotic.

For $\lambda_1 = 0$; $\lambda_4 < \lambda_3 < \lambda_2 < 0$, the system is periodic

The analysis of the dynamic behaviour of this system is done using the bifurcation diagram and the Lyapunov exponents computed with a=36, b=36, c=20, and by increasing the parameter d.

Table 1. Lyapunov Exponents for different values of d

d	λ_1	λ_2	λ_3	λ_4
-5	1.189	0.005	-4..321	-20.810
-1.5	0.001	-0.120	-0.157	-20.202
0.1	0.785	-0.001	-0.342	-19.305
1.8	0.978	0.197	-0.004	-18.330

Table 1 gives the obtained Lyapunov exponents. We note that:

For d=-5 and d= 0.1, we have $\lambda_1 > 0$; $\lambda_2 \cong 0$, $\lambda_4 < \lambda_3 < 0$ and $\lambda_1 + \lambda_3 + \lambda_4 < 0 \Rightarrow$ the system is chaotic.

For d=-1.5; $\lambda_1 \cong 0$; $\lambda_4 < \lambda_3 < \lambda_2 < 0 \Rightarrow$ the system is periodic

For d=1.8, $\lambda_1 > \lambda_2 > 0$; $\lambda_3 \cong 0$, $\lambda_4 < 0$ and $\lambda_1 + \lambda_2 + \lambda_4 < 0 \Rightarrow$ the system becomes hyperchaotic

The bifurcation diagram in the y-d plan, where $Y(nT)$ represents the Poincaré section, is given in Fig. 2. It shows that the system evolves from chaotic state to periodic orbit and from periodic orbit to chaotic, then hyperchaotic state.

We can conclude that for some values of d, this method of control stabilizes the initial chaotic system. However, if this parameter continues to increase, a hyperchaotic behaviour appears.

Fig 3. to Fig. 5. show the behaviour in the phase plane, of the controlled Lü system for different values of d.

In Fig. 5 we can notice that the obtained hyperchaotic attractor has a similar form that the original Lü system with the appearance of additional branches which characterizes the hyperchaotic dynamic

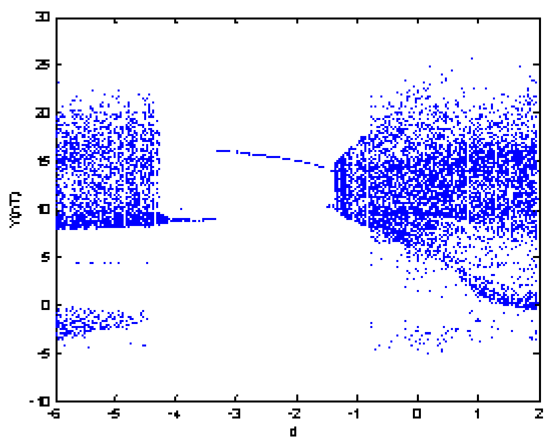


Figure 2. Bifurcation diagram of the controlled Lü system

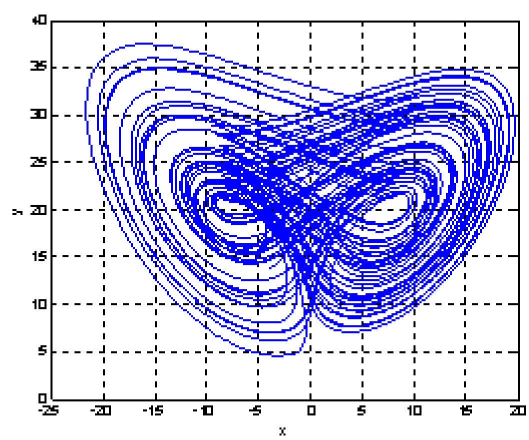


Figure 3. Chaotic attractor of the modified Lü system for d=-5

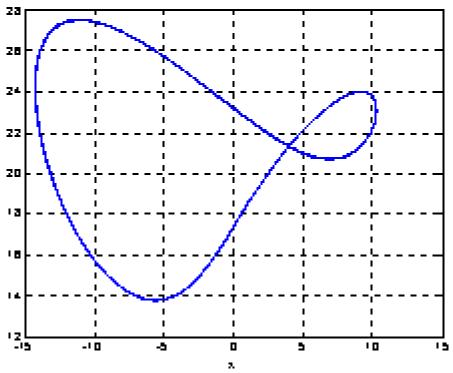


Figure 4. Periodic orbit of the controlled Lü system for $d=-1.5$

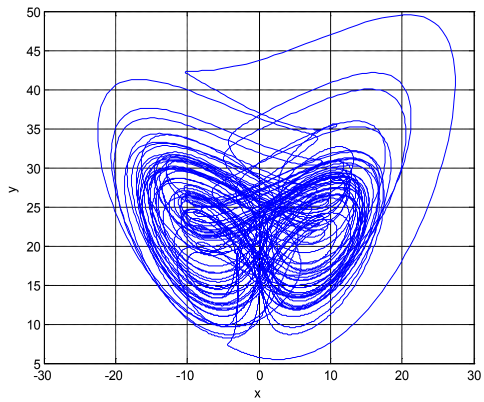


Figure 5. Hyperchaotic attractor of the controlled Lü system for $d=1.8$

3. Synchronization of Hyperchaotic and Chaotic Systems

Many effective methods have been presented to synchronize chaotic systems. Synchronization is always done between a system designed as master and another as slave. The principle of synchronization is to apply on the slave a control function, such as the error between the two systems tends to zero. The problem can then be expressed as a problem of control that consists of minimizing the error between the master and the slave by applying the control law. In our case we use an active control algorithm.

In the following, we consider the synchronization of the hyperchaotic Lü system with a chaotic system, by applying an algorithm based on the active control theory. The hyperchaotic system is considered as master and the chaotic system as slave.

3.1. Synchronization of the Hyperchaotic LÜ System and the 4D Chaotic LÜ System

The master system is given by:

$$\begin{cases} \dot{x}_1 = a_1(y_1 - x_1) \\ \dot{y}_1 = -x_1 z_1 + c_1 y_1 + u_1 \\ \dot{z}_1 = x_1 y_1 - b_1 z_1 \\ \dot{u}_1 = -y_1 z_1 + d_1 u_1 \end{cases} \tag{5}$$

And the slave by:

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) + s_1 \\ \dot{y}_2 = -x_2 z_2 + c y_2 + u_2 + s_2 \\ \dot{z}_2 = x_2 y_2 - b z_2 + s_3 \\ \dot{u}_2 = -y_2 z_2 + d_2 u_2 + s_4 \end{cases} \tag{6}$$

Where: $S=[s_1, s_2, s_3, s_4]^T$ represents the active control function to be designed.

The control function is composed of two parts: one non linear for eliminating non linear term and a linear part to ensure the stability of the obtained system.

The error between the master and the slave is given by:

$$\begin{cases} e_x(t) = x_2(t) - x_1(t) \\ e_y(t) = y_2(t) - y_1(t) \\ e_z(t) = z_2(t) - z_1(t) \\ e_u(t) = u_2(t) - u_1(t) \end{cases} \tag{7}$$

And its dynamic by:

$$\begin{cases} \dot{e}_x = -a e_x + a e_y + s_1 \\ \dot{e}_y = c e_y + e_u - x_2 z_2 + x_1 z_1 + s_2 \\ \dot{e}_z = -b e_z + x_2 y_2 - x_1 y_1 + s_3 \\ \dot{e}_u = (d_1 + d_2) e_u - y_2 z_2 + y_1 z_1 + d_2 u_1 - d_1 u_2 + s_4 \end{cases} \tag{8}$$

For the synchronization of the two systems, we must have:

In order $\lim e_i (i = 1,\ldots,4) \to 0$. to eliminate the nonlinear terms, we choose:

$$\begin{cases} s_1 = v_1 \\ s_2 = x_2 z_2 - x_1 z_1 + v_2 \\ s_3 = -x_2 y_2 + x_1 y_1 + v_3 \\ s_4 = y_2 z_2 - y_1 z_1 - d_2 u_1 + d_1 u_2 + v_4 \end{cases} \tag{9}$$

So, the system to be controlled is a linear system with a control input v_1, v_2, v_3, and v_4

$$\begin{cases} \dot{e}_x = -a e_x + a e_y + v_1 \\ \dot{e}_y = c e_y + e_u + v_2 \\ \dot{e}_z = -b e_z + v_3 \\ \dot{e}_u = (d_1 + d_2) e_u + v_4 \end{cases} \tag{10}$$

These equations converge to zero, if we choose a control law, which ensures that all the eigenvalues will be in the left part of the complex plan. So, we assume that:

$$v_i = A e_i \tag{11}$$

Several choices are possible, in particular:

$$\begin{cases} v_1 = (a-1) e_x - a e_y \\ v_2 = -(c+1) e_y - e_u \\ v_3 = (b-1) e_z \\ v_4 = -(d_2 + d_1 + 1) e_u \end{cases} \tag{12}$$

In this case, all the eigenvalues are equals to -1 and the error $e \to 0$ when $t \to +\infty$.

For the simulation, we choose the master system as the new hyperchaotic Lü system with d_1=1.8. The same system with d_2=-5 is considered as the chaotic slave system

For the others parameters, we assume that:

$$a_1 = a_2 = a = 36,\ b_1 = b_2 = b = 3,\ c_1 = c_2 = c = 20.$$

Fig.(6-a) to Fig.(6-d) represent the results of the synchronization when the control is actived at t=2s and initial conditions for the master and the slave, as follows:

x1(0)=0.1, y1(0)=1, z1(0)=0.2, u1(0)=0.1

x2(0)=-20, y2(0)=30, z2(0)=-8, u2(0)=5

We notice that the trajectories converge rapidly and become identical in a short time

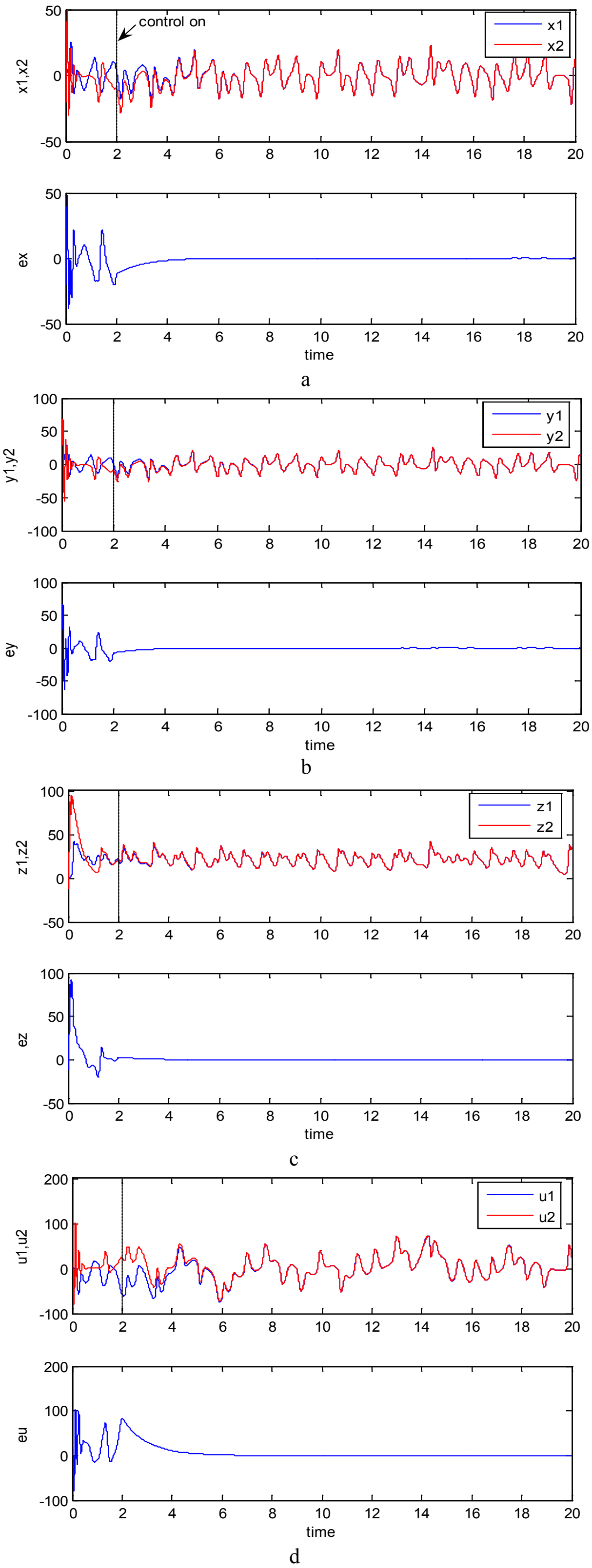


Figure 6. Synchronization of different state variables

3.2. Synchronization of the Hyperchaotic LÜ System and the 3D Chaotic LÜ System

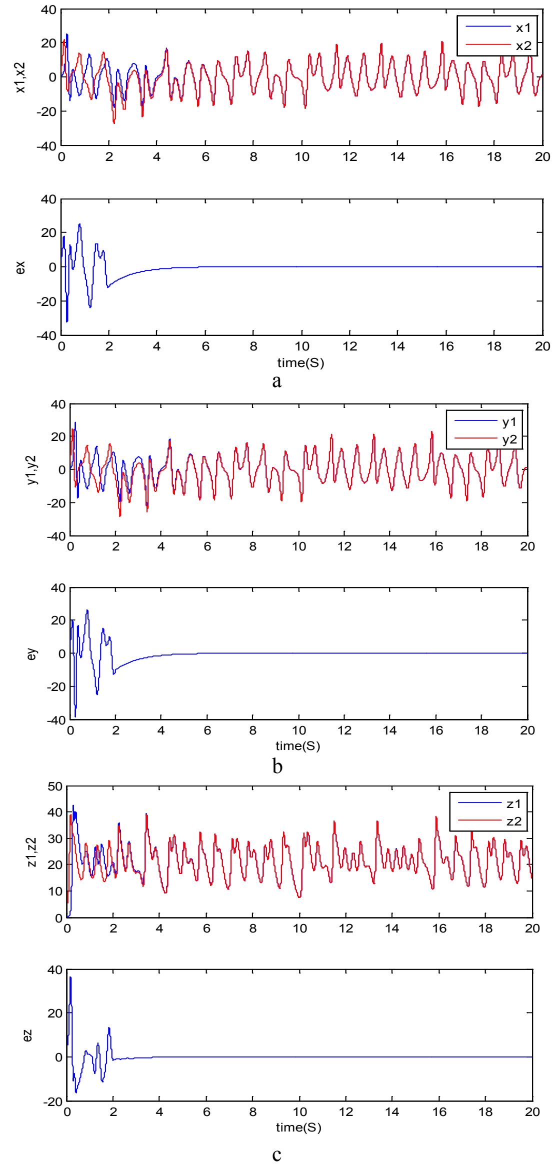


Figure 7. Synchronization of different state variables

The master system is the same as for the previous case and the slave system is given by:

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) + S_1 \\ \dot{y}_2 = -x_2 z_2 + c y_2 + S_2 \\ \dot{z}_2 = x_2 y_2 - b z_2 + S_3 \end{cases} \tag{13}$$

Where $S = [s_1, s_2, s_3]^T$ represents the active control function to design.

The error between the master and the slave is given by:

$$\begin{cases} e_x = x_2 - x_1 \\ e_y = y_2 - y_1 \\ e_z = z_2 - z_1 \end{cases} \tag{14}$$

And its dynamic, by:

$$\begin{cases} \dot{e}_x = a e_y - a e_x + S_1 \\ \dot{e}_y = c e_y - x_2 z_2 + x_1 z_1 - u_1 + S_2 \\ \dot{e}_z = -b e_z + x_2 y_2 - x_1 y_1 + S_3 \end{cases} \tag{15}$$

The control law consists of two parts: A part to eliminate the nonlinear terms and another to stabilize the resulting linear system

$$\begin{cases} S_1 = v_1 \\ S_2 = x_2 z_2 - x_1 z_1 + v_2 \\ S_3 = x_1 y_1 - x_2 y_2 + v_3 \end{cases} \tag{16}$$

So, the controlled system is as follows:

$$\begin{cases} \dot{e}_x = a e_y - a e_x + v_1 \\ \dot{e}_y = c e_y - u_1 + v_2 \\ \dot{e}_z = -b e_z + v_3 \end{cases} \tag{17}$$

With:

$$\begin{cases} v_1 = (a-1)e_x - a e_y \\ v_2 = -(c+1)e_y + u_1 \\ v_3 = (b-1)e_z \end{cases} \tag{18}$$

In Fig. 7(a-c), the trajectories of the two systems are initially completely different due to the sensitivity to the initial conditions. Once activated the control at t=2s, the two systems take a short time to be perfectively synchronized. The trajectory of the chaotic slave system become the same as that of the master hyperchaotic system

3.3. Synchronization of the Hyperchaotic Lü System and the Qi Chaotic System

In this example, the master system is the hyperchaotic Lü system, and the slave system is the Qi chaotic system. This implies that when the two systems are synchronized, the Qi chaotic system will follow the same trajectories as those of the hyperchaotic Lü system.

The Qi system is defined by:

$$\begin{cases} \dot{x}_2 = a_2(y_2 - x_2) + y_2 z_2 u_2 \\ \dot{y}_2 = b_2(x_2 + y_2) - x_2 z_2 u_2 \\ \dot{z}_2 = -c_2 z_2 + x_2 y_2 u_2 \\ \dot{u}_2 = -d_2 u_2 + x_2 y_2 z_2 \end{cases} \tag{19}$$

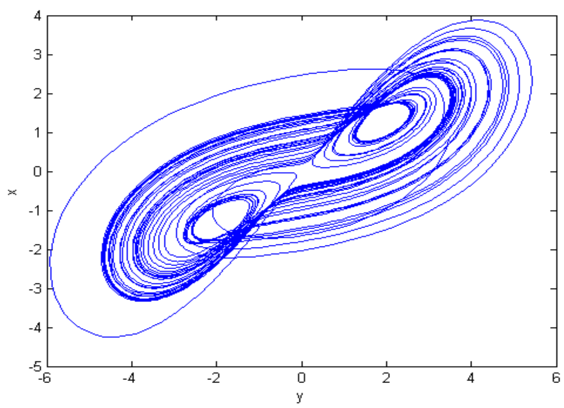


Figure 8. Chaotic Attractor of the Qi system

For a_2=30, b_2=10, c_2=1 and d_2=10 the system has a chaotic behaviour.

Fig. 8. shows the chaotic attractor obtained from the following initial conditions: $x_2(0)$ = 0.15, $y_2(0)$ = 1.1, $z_2(0)$ = 0.25, $u_2(0)$=0.15.

As for the previous example, we define the master system:

$$\begin{cases} \dot{x}_1 = a_1(y_1 - x_1) \\ \dot{y}_1 = -x_1 z_1 + c_1 y_1 + u_1 \\ \dot{z}_1 = x_1 y_1 - b_1 z_1 \\ \dot{u}_1 = -y_1 z_1 + d_1 u_1 \end{cases} \tag{20}$$

And the slave system

$$\begin{cases} \dot{x}_2 = a_2(y_2 - x_2) + y_2 z_2 u_2 + s_1 \\ \dot{y}_2 = b_2(x_2 + y_2) - x_2 z_2 u_2 + s_2 \\ \dot{z}_2 = -c_2 z_2 + x_2 y_2 u_2 + s_3 \\ \dot{u}_2 = -d_2 u_2 + x_2 y_2 z_2 + s_4 \end{cases} \tag{21}$$

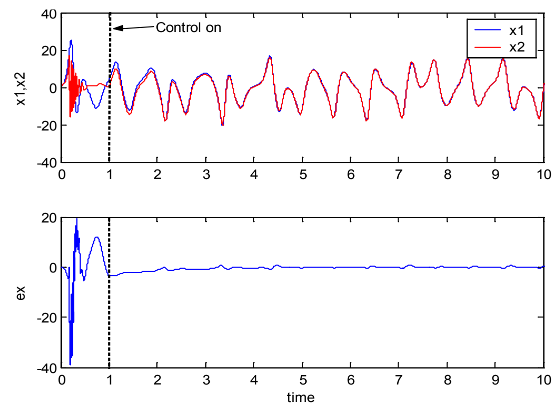
The active control function applied on the slave is:

$$\begin{cases} s_1 = -y_2 z_2 u_2 - a_2 y_1 - a_1 x_2 + a_2 x_1 + v_1 \\ s_2 = c_1 y_2 - b_2 y_1 - b_2 x_2 + x_2 z_2 u_2 - x_1 z_1 + u_1 + v_2 \\ s_3 = c_2 z_1 - b_1 z_2 - x_2 y_2 u_2 + x_1 y_1 + v_3 \\ s_4 = d_2 u_1 + d_1 u_2 - x_2 y_2 z_2 - y_1 z_1 + v_4 \end{cases}$$

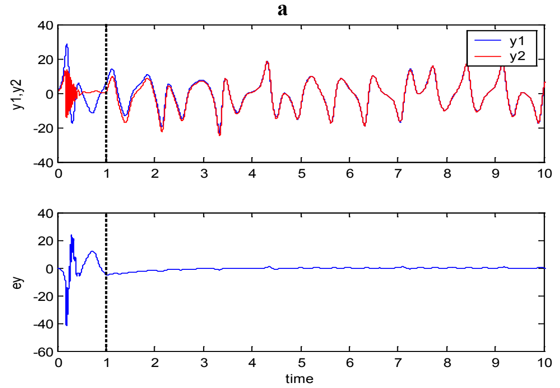
With:

$$\begin{cases} v_1 = (a_1 + a_2 - 1)e_x - (a_1 + a_2)e_y \\ v_2 = -(b_2 + c_1 + 1)e_y \\ v_3 = (c_2 + b_1 - 1)e_z \\ v_4 = (d_2 - d_1 - 1)e_u \end{cases} \tag{22}$$

Fig.(9-a) to (9-d) represent the simulation results for this case. The chaotic slave system becomes hyperchaotic although it is different from the master system. These results demonstrate the effectiveness of the described algorithm even when the two systems are different and the slave system is strongly nonlinear. The controller drives the chaotic system to hyperchaotic trajectories



a



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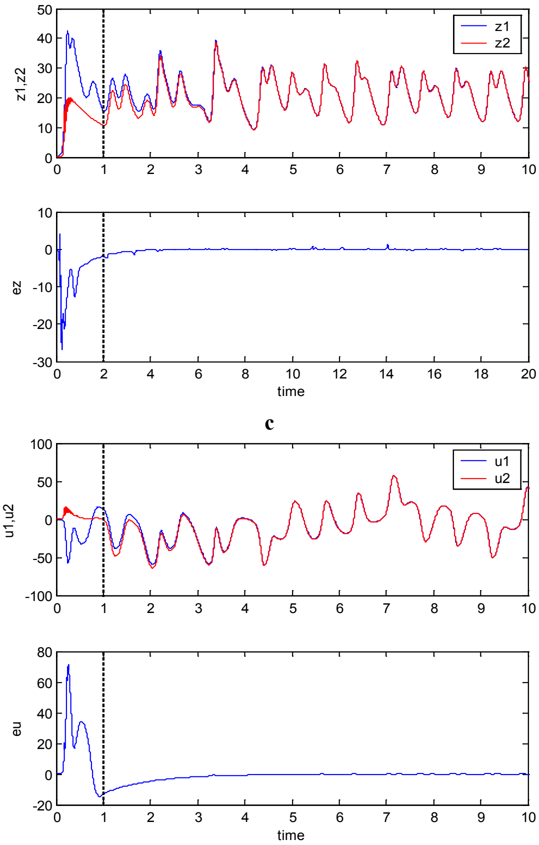

d

Figure 9. Synchronization of different state variables

4. Conclusions

In this paper, we introduced the generation of new hyperchaotic Lü system. Dynamical behaviours of the system are explored by calculating the Lyapunov exponents and the phase diagram. The synchronization of the obtained hyperchaotic system with a chaotic system is possible using an active control algorithm. In addition to its efficiency, this method is easy to implement and achieves the synchronization of two systems completely different, in a reduced time. The stability is guaranteed since the control law ensures that the eigenvalues of the system are always in the left part of the complex plan.

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