International Journal of Astronomy 2014, 3(2): 35-39
DOI: 10.5923/j.astronomy.20140302.02

The Structure of Solitary Dipole Vortises in Rotating Gaseous Gravitating Disc

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Abstract The structure of a solitary dipole vortex (modon) in rotating gaseous gravitating disc has been investigated. The velocity field of modon always has a dipolar structure. The two types of distribution of mass in a modon have been found. The first type is characterized by anti-symmetrically located one round condensation and one rarefaction. The second type is characterized by anti-symmetrically located two consolidations and two rarefactions, and the second pair condensation-rarefaction is crescent. The surfaces of constant-density of modon that are much smaller than the Jeans length λ_J coincide with stream-lines; the constant-density surfaces of larger modons do not coincide with stream-lines. We discuss the possible astrophysical manifestations of modons in astrophysical objects.

Keywords Vortex, Dipole, Structure, Gravitating disc

1. Introduction

The nonlinear fluid equations describing the dynamics of two-dimensional vortices are of great importance in oceanic and atmospheric physics, in plasma physics and in astrophysics. All these vortical structures are described by the same type of non-linear equations. In hydrodynamics this is known as the Hasegawa-Mima equation [1]

$$\frac{\partial}{\partial t}(1-\Delta)\psi - v_0\frac{\partial \psi}{\partial y} - (\boldsymbol{e}_z \times \nabla\psi)\nabla\Delta\psi \ , \qquad (1)$$

describing the nonlinear Ross by waves in the atmosphere [2] and nonlinear drift waves in plasma [3]. Here $\Phi(x,y,z)$ is the stream function, i.e. the velocity is given by $\mathbf{v} = \mathbf{e}_z\times\nabla\Phi$. In the case of plasma Φ is also the electrostatic potential; the constant v_0 is determined by the steady state density gradient.

An exact solution of this equation, describing a stationary solitary dipole vortex (modon), travelling along the y axis on the shallow rotating water was obtained by Larichev and Reznik [4]. The same kind of solution later has been found a large number of similar equations [5-10]. A more complicated vortex structure has been investigated by different authors numerically and by laboratory experiments [11-18].

Rotating gas is kept in equilibrium by a gravitational field (external or self-induced). The non-linear equation that describes modons in a self-gravitating rotating gaseous medium was derived by Dolotin and Fridman [9]. Both in the case of short-wave perturbations (wavelength much smaller than the Jeans wavelength, i.e. $\lambda \ll \lambda_J$) and in the opposite limiting case of $\lambda \gg \lambda_J$, this equation reduces to the Hasegawa-Mima equation.

The existence of modons in rotating gravitating systems, fulfilment of the following three conditions was assumed: 1) uniform rotation of the axisymmetric equilibrium system 2) validity of the epicycle approximation: $d/\Omega dt \ll 1$; 3) isentropic perturbations.

The present paper is devoted to the investigation of the structure of solitary dipolar vortices, or 'modons', that form in gaseous gravitating disc of uniform rotation. The paper [9] gives the velocity field of two-dimensional modons in a gaseous system. For astrophysical applications it is important to know also the density structure of modons, and calculating this is the main aim of the present paper. In particular, we show that the constant-density contours coincide with stream lines when the modon size a is small compared to λ_J but not in the limit $a>\lambda_J$.

2. The Model and the Basic Equations

Let there be a gravitating barotropic gaseous disc of density $\rho_0(r)$ rotating uniform with angular velocity Ω about the z-axis. We will examine isentropic two-dimensional in plane of disc perturbations, neglecting its vertical structure, and presenting any perturbed function f as

$$f \rightarrow f_0(r) + f(r,\varphi,t) \qquad (2)$$

where $f_0(r)$ describes the steady state and $f(r, \varphi, t)$ describes the perturbations.

We assume the perturbations isentropic, i.e. S = const., and therefore the enthalpy $H = H(P)$ and $dH = \rho^{-1}dP$, where P

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Published online at http://journal.sapub.org/astronomy

is the pressure. The stationary quantities we denote with the subscript zero and the prime denotes differentiation with respect to the radial coordinate r: $\rho_0'(r)<0$.

In a uniformly rotating with angular velocity Ω cylindrical coordinate system the perturbed state of disc describes by hydrodynamic and Poisson equations:

$$d\mathbf{v}/dt = 2\mathbf{v}\times\mathbf{\Omega} - \nabla\Phi, \quad (3)$$

$$d(\rho_0+\rho)/dt + (\rho_0+\rho)\nabla\mathbf{v} = 0 \quad (4)$$

$$\Delta U = 4\pi G\rho, \quad (5)$$

$$\Phi = U + H, \quad (6)$$

where

$$d/dt = \partial/\partial t + \mathbf{v}\nabla; \quad \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}, \quad (7)$$

G, U- respectively the gravitational constant and the potential. In steady state, the disc is axially symmetric and has no radial flow. In writing (3) the equilibrium condition of system in the radial direction, and the notation (6) has been used:

$$\Omega^2 r = d\Phi_0/dr. \quad (8)$$

Crossing equation (3) through by the unit vector $\mathbf{e}_z$, we obtain

$$\mathbf{v} = \mathbf{v}_\Phi + \mathbf{v}_I, \quad (9)$$

where

$$\mathbf{v}_\Phi \equiv (1/2\Omega)\, \mathbf{e}_z\times\nabla\Phi, \quad (10)$$

$$\mathbf{v}_I = (1/2\Omega)\, \mathbf{e}_z\times d\mathbf{v}/dt. \quad (11a)$$

Here a subscript Φ denotes the dependence of the velocity on the function Φ, a subscript index I denotes the 'inertial' part of the velocity.

Substituting (9) into (11a) and taking into account (10) and (11a), in approximation $d/\Omega dt \ll 1$, i.e. slow varying perturbations, the term $\mathbf{v}_I\nabla$ can be omitted in the expression d/dt, we obtain:

$$\mathbf{v}_I = (1/4\Omega^2)\mathbf{e}_z\times d[\mathbf{e}_z\nabla\Phi]dt. \quad (11b)$$

Now using expressions (10) and (11') we obtain

$$\nabla\mathbf{v}_\Phi = 0, \quad (12)$$

$$\nabla\mathbf{v}_I = -(1/4\Omega^2)L\Delta\Phi. \quad (13)$$

where

$$L \equiv \partial/\partial t + (1/2\Omega)\,(\nabla\Phi\times\nabla)_z. \quad (14)$$

So, the continuity equation takes the form

$$d(\rho+\rho_0)/dt + (\rho+\rho_0)\nabla\mathbf{v}_I = 0, \quad (15)$$

or, using (7), (13) and (14), we have

$$L(\rho+\rho_0)+(\rho_0/4\Omega^2)\, L\Delta\Phi = 0, \quad (16)$$

In equation (16) we keep terms that are second order in the perturbed amplitude and neglect higher-order terms. In view of (16), we estimate

$$(\rho/4\Omega^2)L\Delta\Phi \approx O(\rho^2 d\rho^3/\Omega dt) \ll O(\rho^2). \quad (17)$$

Consequently, we omit this term.

Using the Poisson equation (9), we obtain from (16) the basic non-linear equation [9]

$$L\Delta U - \tfrac{1}{2}\alpha L\Delta\Phi - \alpha'\Omega\partial\Phi/r\partial\varphi = 0, \quad (18)$$

where

$$\alpha \equiv \omega_J^2/2\Omega^2; \quad \alpha' \equiv d\alpha/dr; \omega_J^2 \equiv 4\pi G\rho_0 \quad (19)$$

The order of magnitude of the ratio |H/U| is

$$|H/U| \approx k^2 c_s^2/\omega_J^2, \quad (20)$$

where we used the definition $c_s^2 = (dP_0/d\rho_0)_s$ and the relationship $|\Delta\Phi| = k^2\Phi$, where k is the wave number. From (20) we can see that the case $|H| \gg |U|$ corresponds to the condition $k^2c_s^2 \gg \omega_J^2$, or $\lambda \ll \lambda_J$ to within a factor 2π. In this limit $\lambda \ll \lambda_J$, equation (18) takes the form

$$\left(\frac{\partial}{\partial t}+\frac{1}{2\Omega}\left(\nabla H\times\nabla\right)_z\right)\Delta H + 2\Omega\frac{\rho_0'}{\rho_0}\frac{1}{r}\frac{\partial H}{\partial\varphi} = 0. \quad (21)$$

Conversely, the limit $|H| \ll |\Phi|$ corresponds to the long-wave disturbances: $\lambda \gg \lambda_J$, and the equation (18) takes the form

$$\left(\frac{\partial}{\partial t}+\frac{1}{2\Omega}\left(\nabla U\times\nabla\right)_z\right)\Delta U + 2\Omega\frac{\alpha'}{\alpha-2}\frac{1}{r}\frac{\partial U}{\partial\varphi} = 0 \quad (22)$$

The linear approximation of the last equation gives the gravitating Ross by waves [9].

Equations (21) and (22) have the same structure (differ only in their coefficients) and are of the Hasegava-Mima (1) equation type.

3. Modons and Their Velocity Field

Let's introduce local Cartesian coordinates (x,y) defined such that (Fig.1)

$$\partial/\partial x = \partial/\partial r, \quad \partial/\partial y = \partial/r\partial\varphi.$$

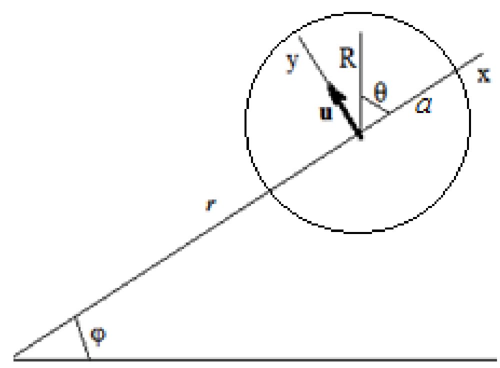

Figure 1.

We look for stationary solutions of eq. (22) (the same for (23)), propagating in the y-direction with the constant velocity u, which is equivalent to the introduction of a wave variable

$$\eta = y - ut. \quad (23)$$

Equation (22) can now be rewritten as

$$\{\partial/\partial\eta \ -A(\nabla U\times\nabla)_z\}\Delta U = -\Lambda\partial U/\partial\eta, \tag{24}$$

or in the form of Jacobean in respect to (x, η)

$$J(U - x/A,\ \Delta U + \Lambda x/A) = 0, \tag{25}$$

where

$$A^{-1} = 2u\Omega, \Lambda = -\ 4\Omega^2 A(\ln|\alpha-2|)'. \tag{26}$$

Hence

$$\Delta U + \Lambda x/A = F(U-x/A), \tag{27}$$

where F is an arbitrary function. As we are interested in localized solutions, so in the limit of very large η, the solution U must vanish at arbitrary x, hence

$$F(-x/A) = -\Lambda x/A. \tag{28}$$

We assume that the function F in equation (27) is linear not just at large η but throughout the entire (x, η) plane. Then in a general case F can be represented as ∝(U-x/A). Introducing polar coordinates R, θ through x = Rcosθ, η = Rsinθ we can (27) rewrite in the form

$$(\Delta + k^2)U = A^{-1}(k^2 - \Lambda)R\cos\theta, \quad r < a, \tag{29}$$

$$(\Delta - p^2)U = 0, \qquad r > a, \tag{30}$$

where k and p are real constants. Soon it will be obvious the motivation for splitting the (R, θ) plane into the interior and exterior of the circle of radius a around the guiding center. Equation (30) turns out to be homogeneous, since we must require U→ 0 as R→∞. This condition implies

$$p^2 = -\Lambda. \tag{31}$$

Equations (31), (32) have the following stationary solutions

$$U(R,\theta) = \Omega u a \begin{cases} \left[\left(1-\dfrac{s^2}{g^2}\right)\dfrac{R}{a} + \dfrac{s^2}{g^2}\dfrac{J_1(gR/a)}{J_1(g)}\right]\cos\theta, & R \le a \\ \dfrac{K_1(sR/a)}{K_1(s)} cos\theta, & R \ge a \end{cases} \tag{32}$$

where J_1, K_1 are Bessel & McDonald functions, $g = ka$ and $s = pa$ are solutions of the following transcendental equation

$$J_1(g)K_3(s) + J_3(g)K_1(s) = 0. \tag{33}$$

For the long-wave perturbations

$$s^2 = (2\Omega a^2/u)\ (\ln|\alpha-2|)', \tag{34}$$

while for the short-wave perturbations

$$U \rightarrow H = c_s^2\rho/\rho_0, \text{ and} s^2 = (2\Omega\ a^2/u)\ (\ln\alpha)', \tag{35}$$

From (32) and (10) we can obtain the following velocity field

$$v_R = u \begin{cases} \left[1-\dfrac{s^2}{g^2}\left(1-\dfrac{aJ_1(gR/a)}{RJ_1(g)}\right)\right]\sin\theta, & R \le a \\ \dfrac{aK_1'(sR/a)}{RK_1(s)}\sin\theta, & R \ge a \end{cases} \tag{36}$$

$$v_\theta = u \begin{cases} \left[1-\dfrac{s^2}{g^2}\left(1-g\dfrac{J_1'(gR/a)}{J_1(g)}\right)\right]\cos\theta, & R \le a \\ \dfrac{sK_1'(sR/a)}{K_1(s)} cos\theta, & R \ge a \end{cases} \tag{37}$$

The streamlines plotted in Fig.2 are determined by

$$dR/v_R = Rd\theta/v_\theta, \tag{38}$$

which gives

$$const. = \begin{cases} \left[\left(1-\dfrac{s^2}{g^2}\right)\dfrac{R}{a} + \dfrac{s^2}{g^2}\dfrac{J_1(gR/a)}{J_1(g)}\right]\sin 2\theta, & R \le a \\ \dfrac{K_1(sR/a)}{K_1(s)}\sin 2\theta, & R \ge a \end{cases} \tag{39}$$

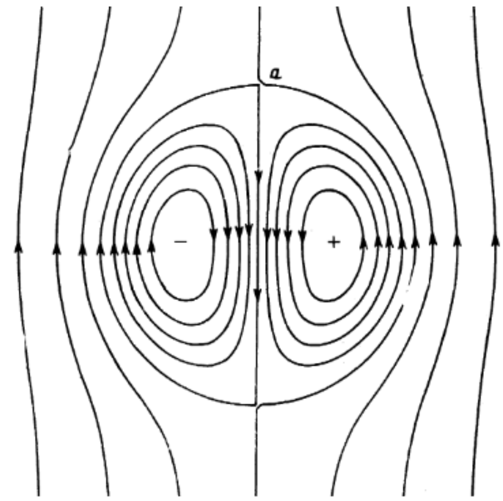

Figure 2. The modon stream lines

4. The Constant-Density Contours

By the condition d/Ωdt≪ 1 we can omit the first term in equation (3), and obtain

$$2\mathbf{v}\times\mathbf{\Omega} = \nabla\Phi. \tag{40}$$

From (40) we find

$$v_x = -\frac{1}{2\Omega}\frac{\partial\Phi}{\partial y}, \quad v_y = \frac{1}{2\Omega}\frac{\partial\Phi}{\partial x}. \tag{41}$$

Hence

$$\psi \equiv \Phi/2\Omega \tag{42}$$

is the stream function.

In case of short-wave perturbations Φ = H. Then (42) becomes ψ = H(ρ)/2Ω, and the isodense contours ρ = const. coincide with the stream lines ψ = const.: the mater flows along the constant-density contours in the opposite directions.

In case of long-wave perturbations $\Phi = U$ and (42) becomes $\psi = \Phi/2\Omega$. Using the Poisson equation (4) this yields

$$\psi = 2\pi G\rho/\Omega \tag{43}$$

so to within a constant coefficient, the stream function ψ coincides with the gravitational potential U. Since equipotential curves U = constant generally do not coincide with isodense contours, it follows that the stream lines ψ = constant in general do not coincide with constant-density contours.

Let us calculate the isodeusity contours of a modon in the limiting short- and long-wave cases. The relative density of a perturbation in the short-wave limit is

$$\sigma\equiv\rho/\rho_0 = \beta_{sw}H/a u\Omega, \text{ where } \beta_{sw} \equiv a u\Omega/c_s^2. \tag{44}$$

The perturbed density in the long-wave limit is found from Poisson's equation to be

$$\sigma = \frac{1}{\omega_J^2}\Delta\psi = \beta_{lw}\begin{cases}\dfrac{J_1(gR/a)}{J_1(g)}\cos\theta, & R \le a\\ \dfrac{K_1(sR/a)}{K_1(s)}\cos\theta, & R \le a\end{cases} \tag{45}$$

where

$$\beta_{lw} \equiv -u\Omega s^2/a\omega_J^2 = -a(\ln|\alpha-2|)'/\alpha \tag{46}$$

Let's consider disc model, describing in a steady state by logarithmic potential, mass density and angular velocity

$$U_0(r) = ½v_0^2\ln(R_c^2 + r^2),\ \rho_0(r) = v_0^2R_c^2/2\pi G(R_c^2 + r^2),\ \Omega^2 = v_0^2/(R_c^2 + r^2), \tag{47}$$

where R_c and v_0 are constants. The small part of the disc ($a \ll R_c$) rotation can be considered as nearly homogeneously. The parameter β_{lw} for this model is equal

$$\beta_{lw}\approx 8ar/3R_c^2\approx 8a(R_c + R\sin\theta)/3R_c^2, \tag{48}$$

where we are using the relation $r^2\approx R_c^2 + 2RR_c\cos\theta$, putting the guiding center on distance R_c.

We illustrate the perturbed density distribution of a dipole vortex given by (38) using the following solutions of the transcendental equation (35)

g = 4.0, s = 1.52

g=4.2, s=2.90

g =4.5, s=6.00

g =4.7, s= 10.00

The curves in Fig. 3 show the perturbed density as a function of dimensionless distance R/*a*from the guiding center in both the short-wave approximation (curves that increase at the origin) and in the long-wave one (curves that decrease at the origin). Fig.4 presents the relative constant-density contours of short-wave modon.

Perturbed density is anti-symmetrical rather leading center. Depending on a choice of area of a dispersive curve (33), it is possible to allocate two types of distribution of mass in a modon. The first type is characterized by anti-symmetrically located one round condensation and one rarefaction (Fig. 4). The second type is characterized by anti-symmetrically located two consolidations and two rarefactions, and the second pair condensation-rarefaction is crescent (the Fig. 5). In the area of small g and s values short-wave modon has the second type, with obviously expressed two condensation (see the Fig. 4b), long-wave modon - the first type. In an average part of a dispersive curve short-wave and long-wave modons have approximately identical structure. They have anti- symmetrical located appreciable pair almost round condensation-rarefaction and rather small crescent pair condensation-rarefaction. In the big g and s part of dispersive curve short-wave modon is the first type and has character a cyclone - anti-cyclonic pair, long-wave - has the second type and it is characterized almost round and crescent condensations (the Fig. 4b). Laboratory experiments in which lonely dipolar whirlwinds on shallow water turn out, obviously, are the first type short-wave modons with asymmetry between the high and low centers of pressure [17].

Let us estimate the perturbed mass m of two condensations of a long-wave modon:

$$m_1 = \frac{2\pi h\rho_0}{J_1(g)}\beta_{lw}\int_{\pi/2}^{3\pi/2}\cos\theta d\theta\int_0^{x_1}J_1(gx)xdx = \frac{4\pi a^2 h\rho_0}{J_1(g)}\beta_{lw}J_0(gx_1)H_1(gx_1) \tag{49}$$

where x = R/a, h is the thickness of the gas disk, $H_1(gx)$ is a first-order Struve function and x_1 satisfies $J_1(gx_1) = 0$. Similarly,

$$m_2 = \frac{2\pi a^2 h\rho_0}{J_1(g)}\beta_{lw}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta\int_{x_1}^{1}J_1(gx)xdx + \frac{2\pi a^2 h\rho_0}{K_1(g)}\beta_{lw}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta\int_1^{\infty}K_1(sx)xdx. \tag{50}$$

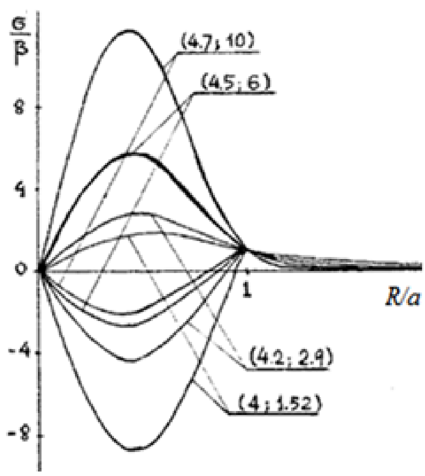

Figure 3.

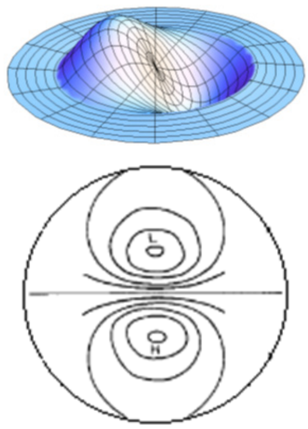

Figure 4.

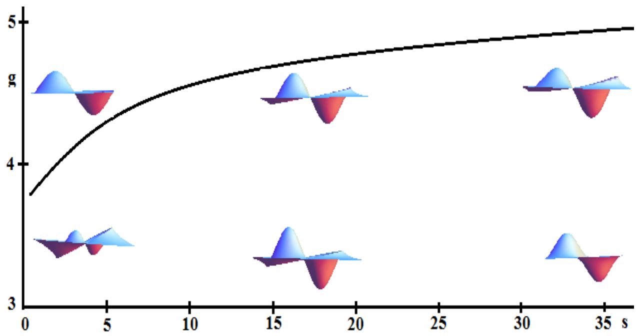

Figure 4b. A dispersive curve (33) in the (s,g) plane. 3D density profile of long-wave (upper row) modons and short-wave modons. Blue color indicate condensation, red – rarefaction

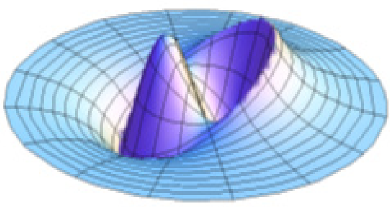

Figure 5. The second type modon is characterized by two condensations and two rarefactions

Numerical evaluation of these integrals shows that the relative mass of condensations in the second type modon, depending on values of g and s, can change in diapason: $m_1/m_2 \sim 2 - 30$.

5. Conclusions

So we would like to pay attention on the existence of double vortices in rotating gaseous gravitating discs and their nontrivial structure. It is difficult to say about a way of a modon evolution: if it could transform to well-known double objects such as double stars, double nuclei in galaxies (like Mrk 266 [19], see Fig.6) as well as in giant molecular clouds etc. or not. The importance of such origin of observed double objects would be the subject of discovery in forthcoming papers.

Figure 6. The isodense picture of the galaxy Markaryan 266 with two nuclei, rotating in the opposite direction [19]

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