International Journal of Astronomy 2014, 3(2): 31-34
DOI: 10.5923/j.astronomy.20140302.01

A Lagrangian Solution for the Precession of Mercury's Perihelion

Adrián G. Cornejo

Electronics and Communications Engineering from Universidad Iberoamericana, Col. Santa Mónica, Querétaro, Mexico

Abstract A Langrangian solution is proposed for the apsidal precession by considering the precession of a body orbiting and rotating in a rotating reference frame, deriving an analogous equation to the relativistic solution.

Keywords Rotating Reference Frame, Lagrangian Solutions, General Theory of Relativity, Precession of Mercury's Perihelion

1. Introduction

The rotating reference frame is a frame that rotates about a spin axis (fixed for simplicity) with a given angular velocity. Formalism to describe a rotating reference frame is commonly given by the Lagrangian mechanics. In this kind of reference frame, an angular velocity of rotation is present in the particles within the rotating reference frame (like the particles in a whirl as described by the fluid mechanics or the particles on a solid body in rotation as described by the classical mechanics). This additional angular velocity in the periodic motion causes a precession towards the sense of rotation, as it is the case of the gyroscope. This effect is also present in some other natural phenomena where a rotating frame is involved, such as the Coriolis effect and the Foucault's pendulum. However, could this kind of effect also be present in the apsidal precession?

A Langrangian solution is proposed for the apsidal precession (like the precession of Mercury's perihelion) by considering the precession of a body orbiting and rotating in a rotating reference frame, deriving an analogous equation to the relativistic solution.

2. Lagrangian Formulation for the Rotating Reference Frame

In this section, certain aspects of the rotating reference frame will be reviewed through the general case of a rotating reference frame and a fixed frame where a body is orbiting and rotating about a spin axis [1]. Let us consider the distance from the spin axis to the final position of a point P that according to the fixed frame of reference is the radius vector named $\mathbf{r}_f$ while some position of the same point according to the rotating reference frame is named $\mathbf{r}$, and

$$\mathbf{r}_f = \mathbf{r} + \mathbf{R}, \tag{1}$$

where $\mathbf{R}$ denotes the position from the origin of the rotating frame according to the fixed frame. Furthermore, the rotating reference frame is described by the Lagrangian [2] for the motion of a particle with mass m in a rotating frame (with its origin coinciding with the fixed-frame origin) in the presence of the potential $U(\mathbf{r})$, defined as

$$L(\mathbf{r},\dot{\mathbf{r}}) = \frac{m}{2}\left|\dot{\mathbf{r}} + \Omega \times \mathbf{r}\right|^2 - U(\mathbf{r}), \tag{2}$$

where Ω is the angular velocity vector of the rotating reference frame, and

$$\left|\dot{\mathbf{r}} + \Omega \times \mathbf{r}\right|^2 = \left|\dot{\mathbf{r}}\right|^2 + 2\Omega\left(\mathbf{r} \times \dot{\mathbf{r}}\right) + \left[\Omega^2 \mathbf{r}^2 - (\Omega \mathbf{r})^2\right]. \tag{3}$$

Moreover, the equation for the canonical momentum is given by

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\left(\dot{\mathbf{r}} + \Omega \times \mathbf{r}\right), \tag{4}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) = m\left(\ddot{\mathbf{r}} + \dot{\Omega} \times \mathbf{r} + \Omega \times \dot{\mathbf{r}}\right). \tag{5}$$

Then, the partial derivative is given by

$$\frac{\partial L}{\partial \mathbf{r}} = -\nabla U(\mathbf{r}) - m\left[\Omega \times \dot{\mathbf{r}} + \Omega \times \left(\Omega \times \mathbf{r}\right)\right]. \tag{6}$$

By applying Lagrangian from Eq. (2), we derive the general Euler-Lagrange equation for r term [3], defined as

* Corresponding author:
adriang.cornejo@gmail.com (Adrián G. Cornejo)
Published online at http://journal.sapub.org/astronomy

$$m\ddot{\mathbf{r}} = -\nabla U(\mathbf{r}) - m\left[\dot{\Omega}\times\mathbf{r} + 2\Omega\times\dot{\mathbf{r}} + \Omega\times(\Omega\times\mathbf{r})\right]. \quad (7)$$

Here, we can see that potential energy generates the fixed-frame acceleration, -Rot$U$ = $m\mathbf{a}_f$, and that the Euler-Lagrange equation (7) takes the form

$$\mathbf{a}_r = \mathbf{a}_f - \mathbf{A} - \Omega\times(\Omega\times\mathbf{r}) - 2\Omega\times\mathbf{v}_r - \dot{\Omega}\times\mathbf{r}, \quad (8)$$

which represents the sum of the net acceleration ($\mathbf{a}_f$ -$\mathbf{A}$), where centrifugal acceleration is given by -$\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$, and $\mathbf{a}_c$ = -2$\mathbf{\Omega}\times\mathbf{v}_r$ is the Coriolis acceleration that only depends on velocity [4]. Being final velocity $\mathbf{v}_f$ equivalent to the Coriolis velocity $\mathbf{v}_c$, it is defined as

$$\mathbf{v}_c = \mathbf{v}_f = 2\Omega\times\mathbf{r} \therefore \Omega = \frac{\mathbf{v}_f}{2\mathbf{r}}. \quad (9)$$

In terms of velocity, from Eq. (8) yields

$$\mathbf{v}_r^2 = \mathbf{v}_f^2 - 2\Omega\times\mathbf{v}_f\mathbf{r} + \Omega^2\times\mathbf{r}^2 = \left(\mathbf{v}_f - \Omega\times\mathbf{r}\right)^2. \quad (10)$$

Simplifying and reordering, we can write this equation as

$$\mathbf{v}_f = \mathbf{v}_r + \Omega\times\mathbf{r}. \quad (11)$$

Then, final velocity (like the final orbital velocity v_o) of a body orbiting and rotating in a rotating reference frame is the addition of both, velocity of the body at the distance r from the centre and the angular velocity of the rotating reference frame at the same distance from the spin axis.

3. Total Energy in the Rotating Reference Frame

In the rotating coordinate system [5], total energy on the orbital motion is defined as

$$E = E_K + V(\mathbf{r}), \quad (12)$$

where E_K is the kinetic energy and $V(\mathbf{r})$ is the effective gravitational potential energy. From Eq. (10), the total energy for a rotating reference frame is given by

$$E = \frac{1}{2}m\mathbf{v}_r^2 + V(\mathbf{r}) = \frac{1}{2}m\left(\mathbf{v}_f - \Omega\times\mathbf{r}\right)^2 + V(\mathbf{r}). \quad (13)$$

Gravitational potential energy is described between a massive body M and another given mass m orbiting at a distance r. Then, the total energy can be written as

$$E = \frac{1}{2}m\left(\mathbf{v}_f^2 - 2\Omega\times\mathbf{v}_f\mathbf{r} + \Omega^2\times\mathbf{r}^2\right) - \frac{GMm}{\mathbf{r}}, \quad (14)$$

where G is the Newtonian constant of gravity. Rearranging this equation, we rewrite the new equation as

$$\frac{1}{2}m\mathbf{v}_f^2 = E - \left(-\frac{GMm}{\mathbf{r}} + \frac{1}{2}m\Omega^2\mathbf{r}^2 - m\Omega\times\mathbf{v}_f\mathbf{r}\right), \quad (15)$$

where second term is equivalent to the effective gravitational potential energy, given by

$$V(\mathbf{r}) = -\frac{GMm}{\mathbf{r}} + \frac{1}{2}m\Omega^2\mathbf{r}^2 - m\Omega\times\mathbf{v}_f\mathbf{r}. \quad (16)$$

This Lagrangian solution for gravitational potential energy can be written in terms of the angular momentum of a body orbiting in circular motion, which in polar coordinates is defined as

$$\mathbf{L} = m\mathbf{r}^2\boldsymbol{\omega}, \quad (17)$$

being $\boldsymbol{\omega}$ the angular velocity of the body in circular orbit. Nevertheless, this angular velocity does not consider any rotation from a rotating reference frame. Then, considering that the angular velocity of a body with respect to the rotating reference frame can be defined as

$$\mathbf{L}_\Omega = m\mathbf{r}^2\Omega, \quad (18)$$

being $\mathbf{\Omega}$ the angular velocity of the rotating reference frame. In addition, the second term in Eq. (16) corresponds to the angular velocity of the body in circular motion (as orbiting) around the centre, while the third term corresponds to the angular velocity of the body with respect to the rotating reference frame. Then, having this correspondence, Eq. (16) becomes

$$V(\mathbf{r}) = -\frac{GMm}{\mathbf{r}} + \frac{1}{2}m\boldsymbol{\omega}^2\mathbf{r}^2 - m\Omega\times\mathbf{v}_f\mathbf{r}. \quad (19)$$

We can write Eq. (19) in terms of angular momentum throughout its transformation to an equivalent equation that includes terms of the angular momentum, for instance, by multiplying the second and third terms, respectively, by some complementary terms which are equals to unit, giving

$$V(\mathbf{r}) = -\frac{GMm}{\mathbf{r}} + \frac{1}{2}m\boldsymbol{\omega}^2\mathbf{r}^2\left(\frac{m\mathbf{r}^2}{m\mathbf{r}^2}\right) - m\Omega\times\mathbf{v}_f\mathbf{r}\left(\frac{m\boldsymbol{\omega}\times\mathbf{v}_f\mathbf{r}}{m\boldsymbol{\omega}\times\mathbf{v}_f\mathbf{r}}\right). \quad (20)$$

Having angular velocity for the Coriolis effect from Eq. (9), gravitational potential energy can be written as

$$V(\mathbf{r}) = -\frac{GMm}{\mathbf{r}} + \frac{\left(m\mathbf{r}^2\boldsymbol{\omega}\right)^2}{2m\mathbf{r}^2} - \left(\frac{\mathbf{v}_f}{2\mathbf{r}}\right)\frac{\mathbf{v}_f^2 m^2\mathbf{r}^2\boldsymbol{\omega}}{m\boldsymbol{\omega}\times\mathbf{v}_f\mathbf{r}}. \quad (21)$$

Now, we will apply some equivalences such as the equivalence between the escape velocity and the final orbital velocity [5], which is defined as

$$\mathbf{v}_e = \sqrt{\frac{2GM}{\mathbf{r}}} = \sqrt{2}\mathbf{v}_f, \quad (22)$$

and the square of velocity of a body in a circular orbit about a central mass [6], given by

$$\mathbf{v}_0^2 = \mathbf{v}_f^2 = \frac{GM}{\mathbf{r}}. \quad (23)$$

Then, considering that the velocity of a body in circular motion is given by $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, replacing Eq. (22) and reducing common terms, Eq. (21) becomes

$$V(\mathbf{r}) = -\frac{GMm}{\mathbf{r}} + \frac{\left(m\mathbf{r}^2\boldsymbol{\omega}\right)^2}{2m\mathbf{r}^2} - \frac{\mathbf{v}_f^2\mathbf{r}\left(m\mathbf{r}^2\boldsymbol{\omega}\right)^2}{m\mathbf{v}_e^2\mathbf{r}^3}. \quad (24)$$

According to the equations (17) and (23), we can write the gravitational potential energy equation from Eq. (24) in terms of the angular momentum of a body in circular motion, giving

$$V(\mathbf{r}) = -\frac{GMm}{\mathbf{r}} + \frac{\mathbf{L}^2}{2m\mathbf{r}^2} - \frac{GM\mathbf{L}^2}{m\mathbf{v}_e^2\mathbf{r}^3}, \quad (25)$$

and by the radius derivative on Eq. (25), while the angular momentum and the escape velocity are kept constants, the net (or absolute) force is given by

$$\frac{dV(\mathbf{r})}{dr} = -\mathbf{F} = \frac{GMm}{\mathbf{r}^2} - \frac{\mathbf{L}^2}{m\mathbf{r}^3} + \frac{3GM\mathbf{L}^2}{m\mathbf{v}_e^2\mathbf{r}^4}. \quad (26)$$

Thus, the magnitude of the net force is given by

$$\left|\mathbf{F}\right| = \frac{GMm}{\mathbf{r}^2} - \frac{\mathbf{L}^2}{m\mathbf{r}^3} + \frac{3GM\mathbf{L}^2}{m\mathbf{v}_e^2\mathbf{r}^4}. \quad (27)$$

The first and second terms are the gravitational force and the centrifugal force, respectively. The third term includes the Coriolis effect present in a rotating reference frame, that is inversely proportional to the radius to the fourth power ($\mathbf{r}^4$), which is also included in the relativistic solution [7].

4. The Precession in a Rotating Reference Frame

Precession results from the angular velocity of rotation and the angular velocity that is produced by the torque. Let us consider that a body in circular orbit at the distance **r** from a central point. Let us also consider that such a body is in a rotating reference frame where the whole system is rotating with a given angular velocity. In a rotating reference frame, the body orbiting undergoes a precession expressed by the angular velocity of precession ω_φ as the described for a gyroscope of radius **r** in rotation [8], with some equivalences defined as

$$\omega_\varphi = \frac{d\varphi}{dt} = \frac{M_0}{\mathbf{L}} = \frac{M_0}{I\boldsymbol{\omega}} = \frac{mg\mathbf{r}}{I\boldsymbol{\omega}} = \frac{\mathbf{r}\times F(\mathbf{r})}{m\mathbf{r}^2\boldsymbol{\omega}}, \quad (28)$$

where $d\varphi$ is the differential of angle of precession, dt is the time differential, M_0 is the module of angular momentum, I is the momentum of inertia for a ring of radius **r**, **ω** is the angular velocity of spin about the spin axis and g is the gravitational acceleration. When a gyroscope rotates about its spin axis and being under external forces (given by $F(\mathbf{r}) = m\cdot g$), momentum of external forces is not null and the angular momentum is not conservative ($\mathbf{L} \neq 0$). Thus, the angular momentum changes direction having a precession which describes a circular motion.

Furthermore, the third term of Eq. (27) includes the Coriolis force, given by

$$F(\mathbf{r}) = \frac{3GM\mathbf{L}^2}{m\mathbf{v}_e^2\mathbf{r}^4} = \frac{3GM\left(m\mathbf{r}^2\boldsymbol{\omega}\right)^2}{m\mathbf{v}_e^2\mathbf{r}^4}, \quad (29)$$

having that velocity of the body in rotation is given by $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ and reducing common terms, the angular momentum of inertia is written as

$$M_0 = \mathbf{r}\times F(\mathbf{r}) = \frac{3GMm\mathbf{v}\boldsymbol{\omega}}{\mathbf{v}_e^2}. \quad (30)$$

The angular velocity of precession for a body rotating about a central point can be calculated by replacing Eq. (30) in Eq. (28), given by

$$\frac{d\varphi}{dt} = \left(\frac{3GMm\mathbf{v}\boldsymbol{\omega}}{\mathbf{v}_e^2}\right)\frac{1}{m\mathbf{r}^2\boldsymbol{\omega}}. \quad (31)$$

Replacing the velocity of the body in rotation by $\boldsymbol{\omega}\times\mathbf{r}$ and reducing common terms, yields

$$d\varphi = \frac{3GM\boldsymbol{\omega}dt}{\mathbf{r}\mathbf{v}_e^2} = \frac{3GMd\theta}{\mathbf{r}\mathbf{v}_e^2}, \quad (32)$$

where $d\theta$ is the differential of the radial angle covered by the rotation of the body in the rotating reference frame.

5. Analogous Equation to the Relativistic Solution

Now, considering a body in elliptical orbit around a central point in a rotating reference frame (Figure 1), distance from the centre changes according to the ellipse, where semi-latus rectum is represented by $\rho = a(1 - e^2)$, being a the semi-major axis of the ellipse and e the eccentricity. For an ellipse of zero eccentricity, ρ is tending to the radius r of a circle. Thus, from Eq. (32), the rate of angle of precession $d\varphi$ for the elliptical motion, while the body moves one revolution (θ completing 2π radians) advancing through an angle is given as

$$d\varphi = \frac{3GM(2\pi)}{a\left(1-e^2\right)\mathbf{v}_e^2} = \frac{6\pi GM}{a\left(1-e^2\right)\mathbf{v}_e^2}. \quad (33)$$

In addition, having the Newtonian equivalence of GM in terms of period T for an elliptical orbit, defined as

$$GM = \frac{(2\pi)^2 a^3}{T^2}, \quad (34)$$

and replacing in Eq. (33), we get

$$d\varphi = \frac{3(2\pi)^3 a^2}{\mathbf{v}_e^2 T^2\left(1-e^2\right)} = \frac{24\pi^3 a^2}{\mathbf{v}_e^2 T^2\left(1-e^2\right)}. \quad (35)$$

From the equivalence between the final orbital velocity and the escape velocity [9], we can consider the proportional equivalence with a central mass and a given radius such that its escape velocity is tending to the speed of light, where

$$v_e^2 = v_f^2\left(\frac{2r}{r_e}\right) \rightarrow c^2 = v_f^2\left(\frac{2r}{r_S}\right), \tag{36}$$

being r_e the escape radius, r_s the Schwarzschild radius [10], and c is the speed of light. Then, replacing Eq. (36) in (35) we can write the equivalent equation given as

$$d\varphi = \frac{24\pi^3 a^2}{v_f^2\left(\frac{2r}{r_S}\right)T^2\left(1-e^2\right)} = \frac{24\pi^3 a^2}{c^2 T^2\left(1-e^2\right)}, \tag{37}$$

which is an analogous equation to the relativistic solution for the advance of Mercury's perihelion [11]. Replacing the known data of Mercury [12] in Eq. (37) and having the Schwarzschild radius is about 2.95 Km in the case of the Sun, advance of the Mercury's perihelion is calculated resulting 43.0133" of arc per century, thus matching well with the actual observations.

It is known that all the planets precess and this equivalence can be applied to the other planets matching as well with the actual observations.

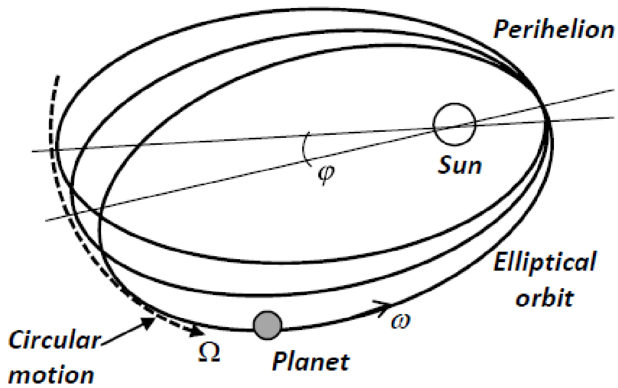

Figure 1. Precession of Mercury's perihelion in a rotating reference frame with angular velocity Ω

6. Conclusions

This work aims to analyze an alternative way to derive an approximation to the celebrated Einstein solution to predict the precession of Mercury's perihelion, proposing an equivalent equation derived from the Lagrangian mechanics by considering an additional angular velocity which is included in a rotating reference frame.

The here proposed solution for the apsidal precession is equivalent to the precession as described for a gyroscope in rotation, which could indicate that the apsidal precession may be related with this rotational effect.

ACKNOWLEDGEMENTS

The author would like to thank Professor Sergio S. Cornejo for the review and comments for this paper.

REFERENCES

[1] Tocaci, E., 1984, Relativistic Mechanics, Time, and Inertia, C. W. Kilmister Ed., Cambridge University Press, Cambridge, UK, pp. 183-251.

[2] Ramos, J., de Montigny, M. and Khanna, F. C., 2010, On a Lagrangian formulation of gravitoelectromagnetism, Gen Relativ Gravit, Vol. 42, No. 10, pp. 2403-2420.

[3] Goldstein, Herbert; Poole, Charles P.; Safko, John L., 2002, Classical Mechanics, 3rd ed., Addison-Wesley Publishing Company, Inc. Reading, Massachusetts, p. 21.

[4] Marion, J. B., 1970, Classical Dynamics of Particles and Systems, 2nd ed., New York, Academic Press.

[5] Tan, A., 2008, Theory of Orbital Motion, Singapore: World Scientific Publishing Co. Pte. Ltd., pp. 90-106.

[6] Danby, J. M. A., 1988, Fundamentals of Celestial Mechanics, 2nd ed., Richmond: Willmann-Bell, Inc., pp. 59–63.

[7] Einstein, A., 1956, The Meaning of Relativity, 5th ed., Princeton, NJ: Princeton University Press, pp 157-159.

[8] Scarborough, J. B., 1958, The Gyroscope: Theory and Applications, New York: Interscience Publishers, Inc., pp. 37-52.

[9] Cornejo, A. G., 2010, The equivalent expressions between escape velocity and orbital velocity, Lat. Am. J. Phys. Educ., Vol. 4, No. 3, pp. 578-579.

[10] Schwarzschild, K., 1916, Uber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, Sitzungsberichte der Deutschen Akademie der Wissenschaften zu Berlin, Klasse fur Mathematik, Physik, und Technik, p. 424.

[11] Einstein, A., 1915, Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin) 47(2) pp. 831-839.

[12] Beatty, J. K., Petersen, C. C., Chaikin, A. and Chaikin, A. L., 1990, The New Solar System, 4th ed., Cambridge: Cambridge University Press, pp. 193–198.