

A Short Note on Converting a Series into a Form of Fraction

Ameha Tefera Tessema

Strategic Planning, Commercial Bank of Ethiopia, Addis Ababa, Ethiopia

Abstract Let for all $n \in N$ and $m_n, R_n \neq 1 \in R$. If the sum of a series defined as $R_1 + R_2 + R_3 + \dots + R_n$ and if

$$m_n = \left(\sum_{i=1}^n \left(\sum_{a=i}^{n-i} m_{n-a} R_{n-a} (-1)^{a+1} \right)^{-1} \right)^{-1} \quad \text{where} \quad \sum_{a=i}^{n-i} m_{n-a} R_{n-a} (-1)^{a+1} = (-1)^{a+1} \quad \text{for} \quad n=i, \quad m_n \neq 0, R_n \neq 0,$$

$m_n R_n - m_{n-1} R_{n-1} + \dots + (-1)^{n+1} m_1 R_1 \neq 0$ then the sum of a series can be converted into fraction as follows

$$\sum_{i=1}^n R_i = (n-1) + \frac{1}{m_n} \left[\sum_{i=1}^n m_{n+1-i} R_{n+1-i} (-1)^{i+1} \right].$$

Keywords Fractional series, Continued fraction, Series

1. Introduction

Continued fraction can be defined as dividing the numerator by successive real numbers. Continued fraction algorithm is a general factoring method which has received a great deal of attention in recent years and in order to implement it on a highly parallel computer, like the massively parallel processor, it is necessary to be able to compute certain numbers which occur at widely-spaced intervals within the continued fraction expansion of \sqrt{N} where N is the number to be factored [8]. The Massively Parallel Processor (MPP) is a highly parallel scientific computer which was originally intended for image processing and analysis applications but it is also suitable for a large range of other scientific applications [6]. Continued fractions are used in analysing Frieze patterns [5], in R_dseth's formula for Frobenius numbers [2], in computing the Jacobi symbol [4], in stream ciphers [1], in pseudo-random number generators [7], in cryptanalysis of the RSA public-key cryptosystem with small private exponents [3]. There are numerous beautiful continued fractions developed by Euler, Lagrange, Gauss, Stieltjes, Ramanujan and so on using various methods. However, according to this paper the sum of real numbers can be written as

$$\sum_{i=1}^n R_i = (n-1) + \frac{1}{m_n} \left(m_n R_n - m_{n-1} R_{n-1} + \dots + m_1 R_1 (-1)^{n+1} \right)$$

where $m_1 = 1, m_2 = \frac{1}{\frac{1}{R} - 1}, m_3 = \frac{1}{\frac{1}{\frac{R_2}{R_1} - R_1} - \frac{1}{R_1} + 1},$ and

$$\frac{1}{\frac{1}{R_1} - 1}$$

so on.

Still nowadays there is no any stated general formula which serves to transform every kind of series into a form of fraction since the formulas existed function valid only for specific values of rational integers. However, on this paper I want to show that how every kind of series can be converted into a form of fraction. Transforming a series into a form of fraction will help to solve various mathematical problems since any series can be converted into fractional series or rational fraction series. Since rational function, which can be defined by rational fraction, can be used in the series, converting a series into form of fraction can be used to approximate or model more complex equations in science and engineering including fields and forces in physics, spectroscopy in analytical chemistry, enzyme kinetics in biochemistry, electronic circuitry, aerodynamics, medicine concentrations in vivo, wave functions for atoms and molecules, optics and photography to improve image resolution, and acoustics and sound [9].

* Corresponding author:
 ambet22002@yahoo.com (Ameha Tefera Tessema)
 Received: Apr. 8, 2021; Accepted: Apr. 24, 2021; Published: Apr. 26, 2021
 Published online at <http://journal.sapub.org/am>

Theorem 1: Let for all $n \in N$ and $m_n, R_n \neq 1 \in R$. If the sum of a series defined as $R_1 + R_2 + R_3 + \dots + R_n$ and if

$$m_n = \left(\sum_{i=1}^n \left(\sum_{a=i}^{n-i} m_{n-a} R_{n-a} (-1)^{a+1} \right)^{-1} \right)^{-1} \quad \text{where} \quad \sum_{a=i}^{n-i} m_{n-a} R_{n-a} (-1)^{a+1} = (-1)^{a+1} \quad \text{for } n=i$$

$m_n \neq 0, R_n \neq 0, m_n R_n - m_{n-1} R_{n-1} + \dots + (-1)^{n+1} m_1 R_1 \neq 0$ then the sum of a series can be converted into fraction as follows

$$\sum_{i=1}^n R_i = (n-1) + \frac{1}{m_n} \left[\sum_{i=1}^n m_{n+1-i} R_{n+1-i} (-1)^{i+1} \right]$$

Proof. Suppose $n \in N$ and $m_n, R_n \neq 1 \in R$ such that $m_n = \left(\sum_{i=1}^n \left(\sum_{a=i}^{n-i} m_{n-a} R_{n-a} (-1)^{a+1} \right)^{-1} \right)^{-1}$ where

$$\sum_{a=i}^{n-i} m_{n-a} R_{n-a} (-1)^{a+1} = (-1)^{a+1} \quad \text{for } n=i$$

$m_n \neq 0, R_n \neq 0, m_n R_n - m_{n-1} R_{n-1} + \dots + (-1)^{n+1} m_1 R_1 \neq 0$ and m_n can be calculated as follows

$$m_1 = (-1)^{1+1}$$

$$m_2 = \frac{1}{\frac{1}{(m_1 R_1)} + (-1)^{2+1}}$$

$$m_3 = \frac{1}{\frac{1}{(m_2 R_2 - m_1 R_1)} + \frac{1}{(-m_1 R_1)} + (-1)^{3+1}}$$

$$m_4 = \frac{1}{\frac{1}{(m_3 R_3 - m_2 R_2 + m_1 R_1)} + \frac{1}{(-m_2 R_2 + m_1 R_1)} + \frac{1}{(m_1 R_1)} + (-1)^{4+1}}$$

⋮
⋮
⋮

$$m_n = \frac{1}{\frac{1}{(m_{n-1} R_{n-1} - m_{n-2} R_{n-2} + \dots + (-1)^n m_1 R_1)} + \frac{1}{(-m_{n-2} R_{n-2} + m_{n-3} R_{n-3} + \dots + (-1)^n m_1 R_1)} + \dots + \frac{1}{((-1)^n m_1 R_1)} + (-1)^{n+1}}$$

The sum of real numbers can be represented as

$$\sum_{i=1}^n R_i = R_1 + R_2 + R_3 + \dots + R_n \quad \text{which can also calculate as follows}$$

$$\sum_{i=1}^n R_i = \frac{1}{m_1} (m_1 R_1) + \frac{1}{m_2} (m_2 R_2) + \frac{1}{m_3} (m_3 R_3) + \dots + \frac{1}{m_n} (m_n R_n)$$

In order to get cancellation one, split the terms in brackets in the following form

$$\begin{aligned} \sum_{i=1}^n R_i &= \frac{1}{m_1} (0 + m_1 R_1) + \frac{1}{m_2} (m_1 R_1 + (m_2 R_2 - m_1 R_1)) + \frac{1}{m_3} ((m_2 R_2 - m_1 R_1) + (m_3 R_3 - m_2 R_2 + m_1 R_1)) + \\ &\dots + \frac{1}{m_n} \left(\left(\sum_{i=1}^{n-1} m_{n-i} R_{n-i} (-1)^{i+1} \right) + \left(\sum_{i=1}^n m_{n+1-i} R_{n+1-i} (-1)^{i+1} \right) \right) \end{aligned}$$

Collecting like terms, we have the following series

$$\sum_{i=1}^n R_i = m_1 R_1 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) + (m_2 R_2 - m_1 R_1) \left(\frac{1}{m_2} + \frac{1}{m_3} \right) + (m_3 R_3 - m_2 R_2 + m_1 R_1) \left(\frac{1}{m_3} + \frac{1}{m_4} \right) + \dots + (m_{n-1} R_{n-1} - m_{n-2} R_{n-2} + \dots + (-1)^n m_1 R_1) \left(\frac{1}{m_{n-1}} + \frac{1}{m_n} \right) + (m_n R_n - m_{n-1} R_{n-1} + \dots + (-1)^{n+1} m_1 R_1) \left(\frac{1}{m_n} \right)$$

And putting the value of m_n in the above series, we have

$$= (m_1 R_1) \frac{1}{(m_1 R_1)} + (m_2 R_2 - m_1 R_1) \frac{1}{(m_2 R_2 - m_1 R_1)} + (m_3 R_3 - m_2 R_2 + m_1 R_1) \frac{1}{(m_3 R_3 - m_2 R_2 + m_1 R_1)} + \dots + (m_n R_n - m_{n-1} R_{n-1} + m_{n-2} R_{n-2} + \dots + (-1)^{n+1} m_1 R_1) \left[\left(\sum_{a=1}^n m_{n-a} R_{n-a} (-1)^{a+1} \right)^{-1} + \left(\sum_{a=2}^{n-2} m_{n-a} R_{n-a} (-1)^{a+1} \right)^{-1} + \left(\sum_{a=3}^{n-3} m_{n-a} R_{n-a} (-1)^{a+1} \right)^{-1} + \dots + \left(\sum_{a=n-1}^1 m_{n-a} R_{n-a} (-1)^{a+1} \right)^{-1} + (-1)^{n+1} \right]$$

Thus we have a series converted in the form of fraction as follow

$$\sum_{i=1}^n R_i = (n-1) + \frac{1}{m_n} \left[\sum_{i=1}^n m_{n+1-i} R_{n+1-i} (-1)^{i+1} \right]$$

2. Conclusions

Since R_n is arbitrary real number, the sum of R_n can be represented as a form of fraction. The difference between consecutive fractional terms, $\frac{m_{n-i} R_{n-i} (-1)^{i+2}}{m_n}$ where $i=0,1,2,\dots$, diminishes without limit for each R_n is divided by R_{n-1} up to R_1 and since the form of fraction are finite in number, the fraction is said to be definite. Therefor the formula developed by this paper serves as a general formula to convert every kind of series into a form of fraction.

REFERENCES

- [1] A. M. Kane, "On the use of continued fractions for stream ciphers," in Proceedings of the 2009 International Conference on Security & Management, 2009, pp. 583-589.
- [2] A. V. Ustinov, "Geometric proof of R_dseth's formula for

Frobenius numbers," Proceedings of the Steklov Institute of Mathematics, vol. 276, pp. 275-282, 2012.

- [3] D. Boneh and G. Durfee, "Cryptanalysis of RSA with private key d less than $N^{0.292}$," IEEE Transactions on Information Theory, vol. 46, pp. 1339-1349, 2000.
- [4] E. Bach and J. Shallit, Algorithmic number theory, Volume I: Efficient algorithms. MIT Press, 1996.
- [5] H. S. M. Coxeter, "Frieze patterns," Acta Arithmetica, vol. 18, pp. 297-310, 1971.
- [6] J. L. Potter, The Massively Parallel Processor, MIT Press (1985). Google Scholar.
- [7] L. Blum, M. Blum, and M. Shub, "A simple unpredictable pseudo-random number generator," SIAM Journal on Computing, vol. 15, no. 2, pp. 364-383, 1986.
- [8] Williams, H.C. and Wundertich M.C. (1987). On The Parallel Generation of the Residues for the Continued fraction Factoring Algorithm. Mathematics of Computation Journal, Volume 48, Number 177, the American mathematical Society press, U.S.A. PP.405-423.
- [9] https://en.wikipedia.org/wiki/Rational_function