

Exact Solution of Goursat Problem with Linear and Non-linear Partial Differential Equations by Double Elzaki Decomposition Method

Mousumi Datta, Md. Shah Alam, Umme Hahiba, Nigar Sultana, Md. Babul Hossain*

Department of Mathematics, Mawlana Bhashani Science and Technology University, Santosh, Tangail, Bangladesh

Abstract This article explores a new approach to achieving an exact solution of a special type of both linear and non-linear partial differential equation (LNLPDE) with mixture derivatives, namely Goursat problems by Double Elzaki Decomposition Method (DEDM). The process utilizes characteristics of Double Elzaki Transform and Adomian Polynomials to obtain an exact solution for Goursat problems. Moreover, Adomian polynomials are used to decompose the non-linear terms. The findings indicate the proposed method is very easy to use and more effective than other techniques because of seed and computational time. It can easily be expanded to other mixed-type LNLPDEs and the approximation task size can be simplified.

Keywords Linear and Non-linear PDEs, Double Elzaki Transform, Mixed Partial Derivatives, Adomian Polynomials, Goursat Problem

1. Introduction

LNLPDE describes many problems in different fields of applied mathematics and sciences. These equations arise in various scientific models such as the fluid dynamics, chemical reaction-diffusion, shallow water wave propagation, and Schrodinger wave equation models. LNLPDEs naturally occurs in various fields of study such as in engineering, physics, and applied mathematics. Many mathematicians and scientists have drawn too much attention to the extensive effectiveness of these equations. In the analysis of physical phenomena and applied sciences, the Goursat partial differential equation occurs in LNLPDEs including mixed-type derivatives and it is known as hyperbolic PDE of second order. The standard form of the Goursat problems with conditions [7-10] is

$$\frac{\partial^2 u}{\partial x \partial t} = f(x, t, u, u_x, u_t), \quad 0 \leq x \leq a, \quad 0 \leq t \leq b$$

$$u(x, 0) = f(x), \quad u(0, t) = g(t)$$

$$f(0) = g(0) = u(0, 0)$$

Several computational techniques, such as Variation Iteration Method (VIM) [7], Laplace Substitution-Variational Iteration Method (LSVIM) [15],

Adomian Decomposition Method (ADM) [11], Two-Dimensional Differential Transform Method (DTM) [8], Modified Variational Iteration Methods [12], Runge-Kutta Method [10] and Finite Difference Method (FDM) [13] and Homotopy Analysis Method (HAM) [14] have been developed to resolve the linear and nonlinear form of Goursat Problems. Previously, the ESM was used [16-17] to solve second-order linear and non-linear PDEs containing mixed partial derivatives. Recently, inspired and encouraged by research work in these fields, we introduce a new method named DEDM, based on Elzaki transform [1] and Double Elzaki Transform [2-6]. The proposed technique is combined the Double Elzaki Transform with the Adomian polynomials for solving non-linear Goursat problems. Here in section (2), definitions and partial derivatives of the Double Elzaki transform will be described. In Section (3), explanation of main advantage of DEDM applied directly without linearization or interruption to Goursat problems are presented. To clarify the applicability and effectiveness of the proposed method for several linear and non-linear problems are presented in section (4). We finally conclude the result at last section.

2. Preliminaries

2.1. Elzaki Transform

The Elzaki transform for exponential order of function f , in the set A is defined by:

* Corresponding author:

babul.math@mbstu.ac.bd (Md. Babul Hossain)

Received: Jan. 30, 2021; Accepted: Feb. 22, 2021; Published: Feb. 28, 2021

Published online at <http://journal.sapub.org/am>

$$A = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_1}}, \text{ if } t \in (-1)^j X[0, \infty) \right\} \quad (1)$$

where M is a finite constant and k_1, k_2 are finite or infinite in set A .

The Elzaki transform symbolized by the operator $E(.)$ and expressed by the integral equation-

$$E[f(t)] = T(v) = v \int_0^\infty f(t) e^{-\frac{t}{v}} dt, \quad t > 0, k_1 \leq v \leq k_2 \quad (2)$$

where v is used to factor the variable t in the argument of function f .

Table 1

Sr.No	$f(t)$	$E[f(t)] = T(v)$
1.	1	v^2
2.	t	v^3
3.	$t^n, n = 0, 1, 2, \dots$	$n! v^{n+2}$
4.	e^{at}	$\frac{v^2}{1 - av}$
5.	te^{at}	$\frac{v^3}{(1 - av)^2}$
6.	$\frac{t^{n-1} e^{at}}{(n-1)!}, n = 1, 2, \dots$	$\frac{v^{n+1}}{(1 - av)^n}$
7.	$\sin at$	$\frac{av^3}{1 + a^2 v^2}$
8.	$\cos at$	$\frac{v^2}{1 + a^2 v^2}$
9.	$\sinh at$	$\frac{av^3}{1 - a^2 v^2}$
10.	$\cosh at$	$\frac{av^2}{1 - a^2 v^2}$

2.2. Inverse Elzaki Transform

If $T(v)$ is the Elzaki transforms of $f(t)$ then $f(t)$ is called the inverse Elzaki transform of $T(v)$ and in mathematical terms, it can be expressed as $f(t) = E^{-1}\{T(v)\}$, where E^{-1} is an operator and is called inverse Elzaki transform.

2.3. Double Elzaki Transform

The function $f(x, t), t, x \in R^+$ which can be expressed as an infinite convergent series, as a consequence Double

Elzaki Transform is:

$$E_2[f(x, t), u, v] = T(u, v) = uv \int_0^\infty \int_0^\infty f(x, t) e^{-\left(\frac{x}{u} + \frac{t}{v}\right)} dx dt, \quad x, t > 0 \quad (3)$$

where u, v are complex.

Table 2

Sr. No	$f(x, t)$	$E_2[f(x, t)] = T(u, v)$
1.	1	$u^2 v^2$
2.	x	$u^3 v^2$
3.	t	$u^2 v^3$
4.	xt	$u^3 v^3$
5.	x^n	$n! u^{n+2} v^2, n \geq 0$
6.	t^n	$n! u^{n+2} v^2, n \geq 0$
7.	$x^n t^n$	$n! u^{n+2} v^2, n \geq 0$
8.	e^{ax}	$\frac{u^2 v^2}{(1 - au)}$
9.	e^{bt}	$\frac{u^2 v^2}{(1 - bv)}$
10.	e^{ax+bt}	$\frac{u^2 v^2}{(1 - au)(1 - bv)}$
11.	$\sin ax$	$\frac{au^3 v^2}{(1 + u^2 a^2)}$
12.	$\cos bt$	$\frac{u^2 v^2}{(1 + b^2 v^2)}$

2.4. Partial Derivatives of Double Elzaki Transform

Using Integration by parts to produce double Elzaki transforms of partial derivatives of x and t -

$$I. E_2 \left[\frac{\partial f(x, t)}{\partial x} \right] = \frac{1}{u} T(u, v) - u T(0, v)$$

$$II. E_2 \left[\frac{\partial f(x, t)}{\partial t} \right] = \frac{1}{v} T(u, v) - v T(u, 0)$$

$$III. E_2 \left[\frac{\partial^2 f(x, t)}{\partial x^2} \right] = \frac{1}{u^2} T(u, v) - T(0, v) - u \frac{\partial}{\partial x} T(0, v)$$

$$IV. E_2 \left[\frac{\partial^2 f(x, t)}{\partial t^2} \right] = \frac{1}{v^2} T(u, v) - T(u, 0) - v \frac{\partial}{\partial t} T(u, 0)$$

$$V. E_2 \left[\frac{\partial^2 f(x, t)}{\partial x \partial t} \right] = \frac{1}{uv} T(u, v) - \frac{v}{u} T(u, 0) - \frac{u}{v} T(0, v) + uv T(0, 0)$$

Proof:

By the definition of double Elzaki transform-

$$\begin{aligned} I. \quad E_2 \left[\frac{\partial f(x,t)}{\partial x} \right] &= uv \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u} + \frac{t}{v}\right)} \frac{\partial}{\partial x} f(x,t) dx dt \\ &= v \int_0^\infty e^{-\frac{t}{v}} \left\{ u \int_0^\infty e^{-\frac{x}{u}} \frac{\partial}{\partial x} f(x,t) dx \right\} dt \end{aligned}$$

Using first partial derivatives of Elzaki transform, the inner integral provides

$$u \int_0^\infty e^{-\frac{x}{u}} \frac{\partial}{\partial x} f(x,t) dx = \frac{1}{u} T(u,t) - u f(0,t)$$

Therefore,

$$E_2 \left[\frac{\partial f(x,t)}{\partial x} \right] = \frac{1}{u} \left\{ v \int_0^\infty e^{-\frac{t}{v}} T(u,t) dt \right\} - u \left\{ v \int_0^\infty e^{-\frac{t}{v}} f(0,t) dt \right\}$$

Taking Elzaki transform with respect to t the above integrals is converted to

$$E_2 \left[\frac{\partial f(x,t)}{\partial x} \right] = \frac{1}{u} T(u,v) - u T(0,v)$$

$$II. \text{ Like-wise, } E_2 \left[\frac{\partial f(x,t)}{\partial t} \right] = \frac{1}{v} T(u,v) - v T(u,0)$$

$$\begin{aligned} III. \quad E_2 \left[\frac{\partial^2 f(x,t)}{\partial x^2} \right] &= uv \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u} + \frac{t}{v}\right)} \frac{\partial^2 f(x,t)}{\partial x^2} dx dt \\ &= v \int_0^\infty e^{-\frac{t}{v}} \left\{ u \int_0^\infty e^{-\frac{x}{u}} \frac{\partial^2 f(x,t)}{\partial x^2} dx \right\} dt \end{aligned}$$

By second order partial derivatives of Elzaki transform, the inner integral becomes-

$$u \int_0^\infty e^{-\frac{x}{u}} \frac{\partial^2 f(x,t)}{\partial x^2} dx = \frac{1}{u^2} T(u,t) - f(0,t) - u \frac{\partial f(0,t)}{\partial x}$$

Therefore,

$$\begin{aligned} E_2 \left[\frac{\partial^2 f(x,t)}{\partial x^2} \right] &= \frac{1}{u^2} \left\{ v \int_0^\infty e^{-\frac{t}{v}} T(u,t) dt \right\} \\ &\quad - \left\{ v \int_0^\infty e^{-\frac{t}{v}} f(0,t) dt \right\} - u \left\{ v \int_0^\infty e^{-\frac{t}{v}} \frac{\partial f(0,t)}{\partial x} dt \right\} \end{aligned}$$

Taking Elzaki transform with respect to t

$$E_2 \left[\frac{\partial^2 f(x,t)}{\partial x^2} \right] = \frac{1}{u^2} T(u,v) - T(0,v) - u \frac{\partial T(0,v)}{\partial x}$$

IV. Similarly the following formula can be established

$$E_2 \left[\frac{\partial^2 f(x,t)}{\partial t^2} \right] = \frac{1}{v^2} T(u,v) - T(u,0) - v \frac{\partial T(u,0)}{\partial t}$$

$$\begin{aligned} V. \quad E_2 \left[\frac{\partial^2 f(x,t)}{\partial x \partial t} \right] &= uv \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u} + \frac{t}{v}\right)} \frac{\partial^2 f(x,t)}{\partial x \partial t} dx dt \\ &= v \int_0^\infty e^{-\frac{t}{v}} \left\{ u \int_0^\infty e^{-\frac{x}{u}} \frac{\partial}{\partial x} \left(\frac{\partial f(x,t)}{\partial t} \right) dx \right\} dt \end{aligned}$$

The inner integral reduces to

$$u \int_0^\infty e^{-\frac{x}{u}} \frac{\partial}{\partial x} \left(\frac{\partial f(x,t)}{\partial t} \right) dx = \frac{1}{u} \frac{\partial f(u,t)}{\partial t} - u \frac{\partial f(0,t)}{\partial t}$$

Therefore,

$$\begin{aligned} E_2 \left[\frac{\partial^2 f(x,t)}{\partial x \partial t} \right] &= \frac{1}{u} \left\{ v \int_0^\infty e^{-\frac{t}{v}} \frac{\partial f(u,t)}{\partial t} dt \right\} - u \left\{ v \int_0^\infty e^{-\frac{t}{v}} \frac{\partial f(0,t)}{\partial t} dt \right\} \\ E_2 \left[\frac{\partial^2 f(x,t)}{\partial x \partial t} \right] &= \frac{1}{u} \left[\frac{1}{v} T(u,v) - v T(u,0) \right] - u \left[\frac{1}{v} T(0,v) - v T(0,0) \right] \\ E_2 \left[\frac{\partial^2 f(x,t)}{\partial x \partial t} \right] &= \frac{1}{uv} T(u,v) - \frac{v}{u} T(u,0) - \frac{u}{v} T(0,v) + uv T(0,0) \end{aligned}$$

3. DEDM for Goursat Problem

In order to explain the solution procedure for DEDM, considering a general Goursat problem rolling mixed partial derivatives with conditions is

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \quad (4)$$

$$\begin{aligned} u(x,0) &= f(x), \quad u(0,t) = g(t) \\ f(0) &= g(0) = u(0,0) \end{aligned} \quad (5)$$

where $L = \frac{\partial^2}{\partial x \partial t}$, $Ru(x,t)$, $Nu(x,t)$ and $h(x,t)$ are linear term, non-linear differential term and source term respectively.

Consequently, equation (4) becomes-

$$\frac{\partial^2 u}{\partial x \partial t} + Ru(x,t) + Nu(x,t) = h(x,t)$$

By double Elzaki transform the above equation is converted to-

$$\begin{aligned} E_2 \left(\frac{\partial^2 u}{\partial x \partial t} \right) + E_2 (Ru(x,t)) + E_2 (Nu(x,t)) &= E_2 (h(x,t)) \\ \frac{1}{uv} T(u,v) - \frac{v}{u} T(u,0) - \frac{u}{v} T(0,v) + uv T(0,0) &+ E_2 (Ru(x,t)) + E_2 (Nu(x,t)) = E_2 (h(x,t)) \end{aligned} \quad (6)$$

Again taking single Elzaki transform in equation (5), then the equation reduces to

$$\begin{aligned}
E(u(x,0)) &= E(f(x)) = T(u,0), \\
E(u(0,t)) &= E(g(t)) = T(0,v) \\
E(f(0)) &= E(g(0)) = E(u(0,0)) = T(0,0)
\end{aligned} \tag{7}$$

Substituting these values in equation (6), implies that

$$\begin{aligned}
T(u,v) &= v^2 E(f(x)) + u^2 (g(t)) - u^2 v^2 E(u(0,0)) \\
&+ uv E_2(h(x,t)) - uv E_2(Ru(x,t)) - uv E_2(Nu(x,t))
\end{aligned}$$

Now, on both sides of the above equation using inverse double Elzaki transform, finally-

$$\begin{aligned}
u(x,t) &= G(x,t) - E_2^{-1} [uv E_2(Ru(x,t))] \\
&- E_2^{-1} [uv E_2(Nu(x,t))]
\end{aligned} \tag{8}$$

where initial conditions are prescribed and $G(x,t)$ reflects the resulting source term.

Consider an infinite series solution to solve non-linear mixed type PDEs by DEDM is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{9}$$

Then the non-linear term is reduced to

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(u) \tag{10}$$

where $A_n(u)$ are Adomian polynomials with components $u_0(x,y), u_1(x,y), \dots, \dots, u_n(x,y), n \geq 0$ are defined by

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\gamma^n} [N[\sum_{i=0}^{\infty} \gamma^i u_i]] \right]_{\gamma=0}, \quad n = 0, 1, 2, 3, \dots \tag{11}$$

Substituting $u(x,y)$ and $Nu(x,y)$ in equation (8), the equation is converted to-

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x,t) &= G(x,t) - E_2^{-1} \left[uv E_2 \left(R \sum_{n=0}^{\infty} u_n(x,t) \right) \right] \\
&- E_2^{-1} \left[uv E_2 \sum_{n=0}^{\infty} A_n \right]
\end{aligned} \tag{12}$$

Comparing on both sides of the above equation, obtained relationships are

$$u_0(x,t) = G(x,t)$$

$$u_1(x,t) = -E_2^{-1} [uv E_2(Ru_0(x,t))] - E_2^{-1} [uv E_2(A_0)]$$

$$u_2(x,t) = -E_2^{-1} [uv E_2(Ru_1(x,t))] - E_2^{-1} [uv E_2(A_1)]$$

In general, the recursive relations are

$$\left. \begin{aligned}
u_0(x,t) &= G(x,t) \\
u_{n+1}(x,t) &= -E_2^{-1} [uv E_2(Ru_n(x,t))] - E_2^{-1} [uv E_2(A_n)]
\end{aligned} \right\}$$

The values of all $u_n(x,t), n \geq 0$ are obtained from the above recursive relation and substitute in equation (9), then the series solution for the given equation is found.

4. Applications

The DEDM for LNL PDEs with mixed type derivatives will be explained by considering Goursat problems in a linear and non-linear form.

4.1. Linear Homogeneous Goursat Problem

Considering a Goursat problem which is linear and homogeneous

$$\frac{\partial^2 u}{\partial x \partial t} = R(u)$$

$$u(x,0) = f(x), \quad u(0,t) = g(t)$$

$$f(0) = g(0) = u(0,0)$$

where, $R(u)$ is a function of first degree.

Example 1: Considering a Goursat problem which is linear and homogeneous-

$$\frac{\partial^2 u}{\partial x \partial t} = u \tag{13}$$

$$u(x,0) = e^x, \quad u(0,t) = e^t, \quad u(0,0) = 1 \tag{14}$$

Equation (13) is converted to this new one by double Elzaki transform

$$E_2 \left[\frac{\partial^2 u}{\partial x \partial t} \right] = E_2 [u]$$

$$\frac{1}{uv} T(u,v) - \frac{v}{u} T(u,0) - \frac{u}{v} T(0,v) + uv T(0,0) = T(u,v) \tag{15}$$

Operating single Elzaki transform, equation (14) reduces to

$$T(u,0) = \frac{u^2}{1-u}, \quad T(0,v) = \frac{v^2}{1-v}, \quad T(0,0) = 1$$

Using the above initial values, the equation (15) is reduced to-

$$T(u,v)(1-uv) = \frac{u^2 v^2}{1-u} + \frac{u^2 v^2}{1-v} - u^2 v^2$$

$$T(u,v)(1-uv) = \frac{u^2 v^2 (1-uv)}{(1-u)(1-v)}$$

$$T(u,v) = \frac{u^2 v^2}{(1-u)(1-v)}$$

The exact solution is found by using inverse double Elzaki transform

$$u(x,t) = e^{x+t}$$

which is verified by the results obtained by VIM and ADM.

4.2. Linear Non-Homogeneous Goursat Problem

Consider a Goursat problem which is linear and non-homogeneous-

$$\frac{\partial^2 u}{\partial x \partial t} = R(u) + h(x, t)$$

$$u(x, 0) = f(x), u(0, t) = g(t)$$

$$f(0) = g(0) = u(0, 0)$$

Where $R(u)$ is a function of first degree.

Example 2: Considering linear non-homogeneous Goursat problem-

$$\frac{\partial^2 u}{\partial x \partial t} = u + 4xt - x^2 t^2 \quad (16)$$

$$u(x, 0) = e^x, u(0, t) = e^t, u(0, 0) = 1 \quad (17)$$

Taking double Elzaki transform in equation (16)

$$\begin{aligned} E_2 \left[\frac{\partial^2 u}{\partial x \partial t} \right] &= E_2 [u] + 4E_2 [xt] - E_2 [x^2 t^2] \\ \frac{1}{uv} T(u, v) - \frac{v}{u} T(u, 0) - \frac{u}{v} T(0, v) + uv T(0, 0) \\ &= T(u, v) + 4u^3 v^3 - 4u^4 v^4 \end{aligned} \quad (18)$$

By single Elzaki transform equation (17) becomes

$$\begin{aligned} E(u(x, 0)) &= T(u, 0) = \frac{u^2}{1-u}, \\ E(u(0, t)) &= T(0, v) = \frac{v^2}{1-v}, T(0, 0) = 1 \end{aligned}$$

with the above conditions, the equation (18) is converted to

$$\begin{aligned} T(u, v)(1-uv) &= 4u^4 v^4 (1-uv) + \frac{u^2 v^2 (1-uv)}{(1-u)(1-v)} \\ T(u, v) &= 4u^4 v^4 + \frac{u^2 v^2}{(1-u)(1-v)} \end{aligned}$$

Applying inverse double Elzaki transform in the above equation

$$u(x, t) = x^2 t^2 + e^{x+t}$$

Which is the required exact solution. This result is completely similar with the result obtained by VIM and DTM.

4.3. Non-linear Goursat Problem

To illustrate the proposed DEDM, two non-linear homogeneous and non-homogeneous Goursat problem is considered.

Example 3: Considering the non-linear non-homogeneous Goursat problem

$$\frac{\partial^2 u}{\partial x \partial t} = -u^3 + x^3 + 3x^2 t + 3xt^2 + t^3 \quad (19)$$

$$u(x, 0) = x, u(0, t) = t, u(0, 0) = 0 \quad (20)$$

Directly use the double Elzaki transform in equation (19) implies that

$$\begin{aligned} \frac{1}{uv} T(u, v) - \frac{v}{u} T(u, 0) - \frac{u}{v} T(0, v) + uv T(0, 0) \\ = -E_2 [u^3] + 6u^5 v^2 + 6u^4 v^3 + 6u^3 v^4 + 6u^2 v^5 \end{aligned}$$

Applying single Elzaki transform in equation (20), conditions reduce to

$$\begin{aligned} E(u(x, 0)) &= T(u, 0) = u^3, \\ E(u(0, t)) &= T(0, v) = v^3, T(0, 0) = 0 \end{aligned}$$

The transformed equation reduces with these conditions as

$$\begin{aligned} T(u, v) &= u^3 v^2 + u^2 v^3 + 6u^6 v^3 \\ &+ 6u^5 v^4 + 6u^4 v^5 + 6u^3 v^6 - uv E_2 [u^3] \end{aligned}$$

Again apply inverse double Elzaki transform in the above equation on both sides,

$$\begin{aligned} u(x, t) &= x + t + \frac{x^4 t}{4} + \frac{x^3 t^2}{2} \\ &+ \frac{x^2 t^3}{2} + \frac{xt^4}{4} - E_2^{-1} [uv E_2 [u^3]] \end{aligned} \quad (21)$$

In DEDM, solution is represented in infinite series form as below,

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \quad (22)$$

And the non-linear term can be written as follows,

$$Nu(x, t) = u^3 = \sum_{n=0}^{\infty} A_n(u) \quad (23)$$

where $A_n(u)$ are Adomian polynomials of components $u_0(x, y), u_1(x, y), \dots, u_n(x, y), n \geq 0$ which is defined by the formula (11).

Some Adomian polynomials of A_n are found as

$$\begin{aligned} A_0 &= u_0^3 \\ A_1 &= 3u_0^2 u_1 \\ &\dots \dots \dots \end{aligned}$$

Then the equation (21) is transformed to

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, t) &= x + t + \frac{x^4 t}{4} + \frac{x^3 t^2}{2} \\ &+ \frac{x^2 t^3}{2} + \frac{xt^4}{4} - E_2^{-1} [uv E_2 [\sum_{n=0}^{\infty} A_n]] \end{aligned}$$

The following recursive relation is found by comparing the above equation on both sides,

$$\left. \begin{aligned} u_0(x, y) &= x + t \\ u_{n+1} &= \frac{x^4 t}{4} + \frac{x^3 t^2}{2} + \frac{x^2 t^3}{2} \\ &+ \frac{xt^4}{4} - E_2^{-1} [uv E_2 [A_n]], n \geq 0 \end{aligned} \right\}$$

The following few components of $u_n(x, y)$, $n > 0$ are determined from the above relation

$$\begin{aligned} u_1 &= \frac{x^4 t}{4} + \frac{x^3 t^2}{2} + \frac{x^2 t^3}{2} + \frac{xt^4}{4} - E_2^{-1}[uvE_2[A_0]] \\ &= \frac{x^4 t}{4} + \frac{x^3 t^2}{2} + \frac{x^2 t^3}{2} + \frac{xt^4}{4} - E_2^{-1}[uvE_2[(x+t)^3]] \\ &= \frac{x^4 t}{4} + \frac{x^3 t^2}{2} + \frac{x^2 t^3}{2} + \frac{xt^4}{4} \\ &\quad - E_2^{-1}[6u^6 v^3 + 6u^5 v^4 + 6u^4 v^5 + 6u^3 v^6] \\ &= \frac{x^4 t}{4} + \frac{x^3 t^2}{2} + \frac{x^2 t^3}{2} + \frac{xt^4}{4} \\ &\quad - \frac{x^4 t}{4} - \frac{x^3 t^2}{2} - \frac{x^2 t^3}{2} - \frac{xt^4}{4} \\ &= 0 \end{aligned}$$

Consequently, $u_n(x, y) = 0$, $n \geq 1$

Substituting all values of $u_n(x, y)$, $n \geq 0$ in equation (22), finally the series solution of the given equation is

$$u(x, t) = x + t$$

This results are in complete compliance with the result given by VIM and DTM.

Example 4: Consider the non-linear homogeneous Goursat problem

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-2u} \quad (24)$$

$$u(x, 0) = 0, u(0, t) = 0, u(0, 0) = 0 \quad (25)$$

Applying double Elzaki transform the equation (24) becomes-

$$\frac{1}{uv}T(u, v) - \frac{v}{u}T(u, 0) - \frac{u}{v}T(0, v) + uvT(0, 0) = E_2[e^{-2u}]$$

Using the single Elzaki transform the conditions in equation (25) reduces to

$$E(u(x, 0)) = T(u, 0) = 0,$$

$$E(u(0, t)) = T(0, v) = 0, T(0, 0) = 0$$

with these new conditions the above equation gives

$$T(u, v) = uvE_2[u^3]$$

Operating inverse double Elzaki transform, find that

$$u(x, t) = E_2^{-1}[uvE_2[u^3]] \quad (26)$$

The infinite series solution in DEDM is presented by

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \quad (27)$$

and the non-linear term can be written as-

$$Nu(x, t) = e^{-2u} = \sum_{n=0}^{\infty} A_n(u) \quad (28)$$

where $A_n(u)$ are Adomian polynomials of components $u_0(x, y), u_1(x, y), \dots, u_n(x, y)$, $n \geq 0$ which is defined by the formula (11). Adomian polynomials A_n are

$$\begin{aligned} A_0 &= e^{-2u_0} \\ A_1 &= -2e^{-2u_0}u_1 \\ A_2 &= -2e^{-2u_0}u_2 + 2u_1^2e^{-2u_0} \\ &\dots \dots \dots \end{aligned}$$

Then the equation (26) becomes

$$\sum_{n=0}^{\infty} u_n(x, t) = E_2^{-1}[uvE_2[\sum_{n=0}^{\infty} A_n]]$$

The recursive relationships are obtained by comparing both sides of the above equation

$$\left. \begin{aligned} u_0(x, y) &= 0 \\ u_{n+1} &= E_2^{-1}[uvE_2[A_n]], n \geq 0 \end{aligned} \right\}$$

The following new components of $u_n(x, y)$, $n > 0$ have been found from the above relationship.

$$\begin{aligned} u_1(x, t) &= E_2^{-1}[uvE_2[A_0]] \\ &= E_2^{-1}[uvE_2[e^0]] \\ &= E_2^{-1}[u^3v^3] \\ &= xt \\ u_2(x, t) &= E_2^{-1}[uvE_2[-2e^{-2u_0}u_1]] \\ &= E_2^{-1}[uvE_2[-2xt]] \\ &= -\frac{x^2t^2}{2} \end{aligned}$$

Like-wise

$$\begin{aligned} u_3(x, t) &= E_2^{-1}[uvE_2[A_2]] = \frac{x^3t^3}{3} \\ u_4(x, t) &= -\frac{x^4t^4}{4} \text{ and so on.} \end{aligned}$$

Putting all values of $u_n(x, y)$, $n \geq 0$ in equation (27), finally the series solution of the given equation is

$$\begin{aligned} u(x, t) &= xt - \frac{x^2t^2}{2} + \frac{x^3t^3}{3} - \frac{x^4t^4}{4} + \dots \dots \dots \\ &= \ln(1 + xt) \end{aligned}$$

This finding is in complete compliance with the result obtained by VIM and DTM.

5. Conclusions

To find the approximate series solutions we successfully

implemented the DEDM of linear and non-linear Goursat problems. The mentioned process eliminates the computational difficulties that occur in other existing methods and is directly implemented without the use of linearization, transformation, discretization or limiting assumptions. By applying DEDM, many examples of linear and non-linear Goursat problems were evaluated and the findings have shown noticeable and more satisfactory results. The improvement of this methodology would lead to the possibility of increasing the order and accuracy of the system further.

REFERENCES

- [1] Elzaki, T. M. (2011). Application of new transform “Elzaki transform” to partial differential equations. *Global Journal of pure and applied Mathematics*, 7(1), 65-70.
- [2] Hassaballa, A. A., & Salih, Y. A. (2015). On Double Elzaki Transform and Double Laplace Transform. *IOSR Journal of Mathematics (IOSR-JM)*, 11(1), 35-41.
- [3] Elzaki, T. M., Hilal, E. M., Arabia, J. S., & Arabia, J. S. (2012). Solution of Telegraph Equation by Modified of Double Sumudu Transform” Elzaki Transform. *Mathematical Theory and Modeling*, 2(4), 95-103.
- [4] Idrees, M. I., Ahmed, Z., Awais, M., & Perveen, Z. (2018). On the convergence of double Elzaki transform. *Int Journal of Advanced and Applied Sciences*, 5(6), 19-24.
- [5] Hassan, M. A. (2020). Double Elzaki Transform Decomposition Method for Solving Non-Linear Partial Differential Equations. *Journal of Applied Mathematics and Physics*, 8(8), 1463-1471.
- [6] MAHGOB, M. (2015). Solution of partial integro-differential equations by double Elzaki transform method. *Mathematical Theory and Modeling*, 5(5), 61-65.
- [7] Wazwaz, A. M. (2007). The variational iteration method for a reliable treatment of the linear and the nonlinear Goursat problem. *Applied Mathematics and Computation*, 193(2), 455-462.
- [8] Taghvafard, H., & Erjaee, G. H. (2010). Two-dimensional differential transform method for solving linear and non-linear Goursat problem. *International Journal for Engineering and Mathematical Sciences*, 6(2), 103-106.
- [9] Wazwaz, A. M. (1995). The decomposition method for approximate solution of the Goursat problem. *Applied Mathematics and Computation*, 69(2-3), 299-311.
- [10] Day, J. T. (1966). A Runge-Kutta method for the numerical solution of the Goursat problem in hyperbolic partial differential equations. *The Computer Journal*, 9(1), 81-83.
- [11] Ahmad J, Mushtaq M. (2015). Exact Solution of Linear and Non-linear Day, J. T. (1966). A Runge-Kutta method for the numerical solution of the Goursat problem in hyperbolic partial differential equations. *The Computer Journal*, 9(1), 81-83. Goursat Problems. *Univers J Comput Math*. 3 (1): 14-17.
- [12] Noor, M. A., & Mohyud-Din, S. T. (2008). Modified variational iteration method for Goursat and Laplace problems. *World Applied Sciences Journal*, 4(4), 487-498.
- [13] Pandey, P. K. (2014). A Finite Difference Method for Numerical Solution of Goursat Problem of Partial Differential Equation. *Open Access Libr J*, 1, 1-6.
- [14] Usman M, Zubair T, Ali U, Mohyud-din ST. (2012). On Goursat Problems. *Int J Mod Math Sci*.3 (3): 63-76.
- [15] Al-Fayadh, A., & Faraj, D. S. (2019). Laplace Substitution–Variational Iteration Method for Solving Goursat Problems Involving Mixed Partial Derivatives. *American Journal of Mathematical and Computer Modelling*, 4(1), 16-20.
- [16] Hossain, M. B. & Datta, M. (2018). Solutions of Linear Partial Differential Equations with Mixed Partial Derivatives by Elzaki Substitution Method. *American Journal of Computational and Applied Mathematics* 8:59-64.
- [17] Datta, M., Habiba, U., & Hossain, M. B. (2020). Elzaki Substitution Method for Solving Nonlinear Partial Differential Equations with Mixed Partial Derivatives Using Adomain Polynomial. *International Journal*, 8(1), 6-12.