

# A Survey of Regularization Methods of Solution of Volterra Integral Equations of the First Kind

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**Abstract** One of the most recent mathematical concepts, the regularization methods for solving integral equations, has brought in yet another dimension into the existing problems and has helped to usher in a new body of knowledge for further consideration. Indeed, new inputs have placed other numerical methods for solving integral equations in the fastest lane, since a number of methods were analysed with respect to accuracy, convergence and stability properties. Most of the work was done on the assumptions that the kernel  $k(x, s)$  and the right-hand side  $f(x)$  are known without error and that the approximating equation can be exact, whereas, as is often the case, these may not be true. Consequently, the effect can be observed in the study of regularization methods for solving Volterra type integral equations of the first kind. Classical regularization methods tend to destroy the non-anticipatory (or causal) nature of the original Volterra problem because such methods typically rely on the computation of the Volterra adjoint operator. In this paper, we highlight the general concept of the regularization method of solution for Volterra integral equation of the first kind without destroying the Volterra structure.

**Keywords** Volterra Integral Equation, Regularization method, Ill-posed

## 1. Introduction

The theory of integral equations is interesting, not only in itself but also in the absolute importance of its results in the analysis of numerical methods of solution of various equations. Besides the concept of existence and uniqueness with respect to differential equations, the theory is concerned, in particular, with questions of regularity and stability of their solutions.

In the last two and a half decades, there has been a tremendous increase in the applications of first kind integral equations of Volterra type. Among the areas where the applications of such class of equations abound in the natural sciences is the problem of restoration of incoming signals at the entrance of measuring instruments and observatory systems which, in view of their physical nature, allow some distortions of the observed and registered data [1].

An integral equation is a functional equation in which the unknown function appears under one or several integral signs. In an integral equation of Volterra type for instance, the integrals containing the unknown function are characterized by a variable upper limit of integration. To be

more precise, an integral equation of the form

$$\int_a^x k(x, s) y(s) ds = f(x) \quad (1)$$

is called a linear Volterra integral equation of the first kind, or that of the form

$$y(x) - \lambda \int_a^x k(x, s) y(s) ds = f(x) \quad (2)$$

is called a linear Volterra integral equation of the second kind. Here,  $x, s$  and  $a$  are real numbers,  $\lambda$  is a parameter,  $y(s)$  is an unknown function, while  $f(x)$  and  $k(x, s)$  are given functions which are square integrable on  $[a, b]$  and in the domain  $a \leq x \leq b, a \leq s \leq x$  respectively. The function  $f(x)$  is called the free term, while the function  $k(x, s)$  is called the kernel.

One distinguishing feature associated with this class of equations is the uncertainty surrounding their nature and this is reflected in the difficulties encountered when attempting to find their solutions using one or the other of the existing numerical methods [2]. On one hand, Volterra integral equations of the first kind appear to be special cases of Fredholm integral equations of the first kind and are consequently classified among the lists of ill-conditioned problems solvable by the classical regularization means. On the other hand, given a set of limitations (for example, given

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that their kernels and right-hand sides are “good” and “smooth”) Volterra integral equations of the first kind belong to the class of well-posed problems and hence can be solved through any direct method based on the discretization of the unknown solutions. Both approaches seem to be a little on the extreme. The use of discretization process transforms Volterra integral equations of the first kind into another problem (often systems of linear equations and may be solved by the well-known singular value decomposition) that can lead to solutions which may or may not deviate strongly from the (minimum norm) solution of the original equation [3]. A complete regularization (with its accompanying difficulties and unresolved questions) turns out to be a too complicated approach as it is observed that the methods for Volterra equations of the first kind are not nearly as ill-posed as methods for Fredholm integral equations of the first kind.

In this study, we investigate the regularization methods with a view to identifying the properties and types that can be used in solving Volterra type equations of the first kind, but without destroying the Volterra structure. The methods in view are perhaps the discrete regularization methods. The discrete approximation methods provide another approach to regularize the original problem. In this case, the regularization parameter is the discretization parameter (or step size) and coordination between this parameter and the amount of noise  $\delta$  in the problem is required to obtain good approximations in the presence of noise.

## 2. The Concept of a Regularizing Operator

We are already aware of the fact that the problem of solving Volterra integral equations of the first kind in some spaces, for example, in the triple  $(C, V, C)$  is, to a certain extent, ill-posed. Even in those spaces where the problems are well-conditioned, for example, in the space  $(C, V, C^{(1)})$ , we still encounter significant difficulties, leading to some slight instabilities in the results during actualization. Consequently, to improve on the stability of the solutions which implies the accuracy of the desired results, we go out for the most stable methods. Among such methods are the regularization methods.

Methods for construction of approximate solutions of ill-posed problems that are stable with respect to small perturbations of the initial data are referred to as regularization methods. The approach is based on what is called the fundamental concept of a regularizing operator [4]. In what follows, we examine the concept in detail.

Suppose the operator  $A$  in the equation

$$Ay = f, y \in Y, f \in F, \quad (3)$$

where  $Y$  and  $F$  are some metric spaces and  $A$  is a continuous operator mapping  $Y$  into  $F$ , is such that its inverse  $A^{-1}$  is also continuous in the set  $A \times Y$  and the set  $Y$  of possible solutions, is not compact.

If the right-hand member of the equation is an element

$f_\delta \in F$  that differs from the exact right-hand member  $f_T$  by no more than  $\delta$ , that is, if  $\rho_F(f_\delta, f_T) \leq \delta$ , it is obvious that the approximate solutions  $y_\delta$  of equation (3) cannot be defined as the exact solution of this equation with approximate right-hand member  $f = f_\delta$ , that is, according to the formula  $y_\delta = A^{-1}f_\delta$ .

The numerical parameter  $\delta$  characterizes the error in the right-hand member of equation (3). Therefore, it is natural to define  $y_\delta$  with the aid of an operator depending on a parameter having a value chosen in accordance with the error  $\delta$  in the initial data  $f_\delta$ . Specifically, as  $\delta \rightarrow 0$ , that is, as the right-hand member  $f_\delta$  of equation (3) approaches (in the metric of the  $F$ ) the exact value  $f_T$ , the approximate solution  $y_\delta$  must approach (in the metric of the space) the exact solution  $y_T$  that we are seeking for in the equation  $Ay = f_T$ .

Now, assuming that the elements  $y_T \in Y$  and  $f_T \in F$  are connected by  $Ay_T = f_T$ , we define the following concepts:

**Definition 2.1.** An operator  $R(f, \delta)$  is said to be a regularizing operator for the equation  $Ay = f$  in a neighbourhood of  $f = f_T$  if:

- i) there exists a positive number  $\delta_1$  such that the operator  $R(f, \delta)$  is defined for energy  $\delta$  in  $0 \leq \delta \leq \delta_1$  and every  $f_\delta \in F$  such that  $\rho_F(f_T, f_\delta) \leq \delta$ ;
- ii) for every  $\varepsilon > 0$ , there exists a  $\delta_0 = \delta_0(\varepsilon, f_T) \leq \delta$ , such that the inequality  $\rho_F(f_\delta, f_T) \leq \delta \leq \delta_0$ , implies the inequality  $\rho_Y(y_\delta, y_T) \leq \varepsilon$ , where  $y_\delta = R(f_\delta, \delta)$ .

**Definition 2.2.** An operator  $R(f, \alpha)$  depending on a parameter  $\alpha$  is called a regularizing operator for the equation  $Ay = f$  in a neighbourhood of  $f = f_T$  if:

- i) there exists a positive number  $\delta_1$  such that the operator  $R(f, \alpha)$  is defined for every  $\alpha > 0$  and every  $f \in F$  for which  $\rho_F(f, f_T) \leq \delta \leq \delta_1$ ;
- ii) there exists a function  $\alpha = \alpha(\delta)$  of  $\delta$  such that, for every  $\varepsilon > 0$ , there exists a number  $\delta(\varepsilon) \leq \delta_1$  such that the inclusion  $f_\delta \in F$  and the inequality  $\rho_F(f_T, f_\delta) \leq \delta(\varepsilon)$  imply  $\rho_Y(y_T, y_\alpha) \leq \varepsilon$ , where  $y_\alpha = R(f_\delta, \alpha(\delta))$ .

Again, there is no assumption of the uniqueness of the operator  $R(f_\delta, \alpha(\delta))$ . It is easy to point out here that the function  $\alpha = \alpha(\delta)$  also depends on  $f_\delta$ .

If  $\rho_T(f_T, f_\delta) \leq \delta$ , we know from studies by Tikhonov [4] that we can take for an approximate solution of equation (3) with approximately known right-hand member  $f_\delta$  the element  $y_\alpha = R(f_\delta, \alpha)$  obtained with the aid of the regularizing operator  $R(f, \alpha)$ , where  $\alpha = \alpha(\delta, f_\delta)$  in accordance with the error in the initial data  $f_\delta$ . This solution is called a regularized solution of equation (3). The numerical parameter  $\alpha$  is called the regularization parameter. Obviously, every regularizing operator defines a stable method of approximate construction of the solution of equations (3) provided the choice of  $\alpha$  is consistent with the accuracy  $\delta$  of the initial data ( $\alpha = \alpha(\delta)$ ). If we know that  $\rho_f(f_T, f_\delta) \leq \delta$ , we can, by definition of a regularizing operator, choose the value of the regularization parameter  $\alpha = \alpha(\delta)$  in such a way that, as  $\delta \rightarrow 0$ , the regularized solution  $y_\alpha = R(f_\delta, \alpha(\delta))$  approaches (in the metric of  $Y$ ) the desired exact solution  $y_T$ , that is,  $\rho_Y(y_T, y_{\alpha(\delta)}) \rightarrow 0$ . This justifies taking as an approximate solution of equation (1) the regularized solution. Thus, the problem of finding an approximate solution of equation (1) that is stable under small changes in the right-hand member reduces to:

- i) finding regularizing operators;
- ii) determining the regularization parameter  $\alpha$  from the supplementary information pertaining to the problem, for example, the size of the error in the right-hand member  $f_\delta$ .

This method of constructing approximate solutions is called the regularization method.

Out of all the operators  $R(f, \alpha)$  from  $F$  and  $Y$  that depend on the parameter  $\alpha$  and which are defined for every  $f \in F$  and every positive  $\alpha$ , we need to single out the operators that are continuous with respect to  $f$ . In doing this, we can give sufficient conditions for belonging to the set of regularizing operators of equation (3). This is an immediate consequence of the following theorem.

**Theorem 2.1.** Let  $A$  denote an operator from  $Y$  into  $F$  and let  $\bar{R}(f, \alpha)$  denote an operator from  $F$  into  $Y$  that is defined for every element  $f \in F$  and every positive  $\alpha$  that is continuous with respect to  $f$ . If  $\lim_{\alpha \rightarrow 0} \bar{R}(Ay, \alpha) = y$ , for every element  $y \in Y$ , then the operator  $\bar{R}(f, \alpha)$  is a regularizing operator for the equation  $Ay = f$ .

**Proof.** It will be sufficient to show that the operator  $\bar{R}(f, \alpha)$  possesses property (ii) of Definition 2.2.

Let  $y_T$  and  $f_T$  denote fixed elements of  $Y$  and  $F$  respectively such that  $Ay_T = f_T$ . Let  $\delta$  denote a fixed positive number. Then, for every element  $f_\delta \in F$  such that  $\rho_F(f_\delta, f_T) \leq \delta$ , we have

$$\rho_Y(\bar{R}(f_\delta, \alpha), y_T) \leq \rho_Y(\bar{R}(f_\delta, \alpha), \bar{R}(f_T, \alpha)) + \rho_Y(\bar{R}(f_T, \alpha), y_T). \quad (4)$$

Since the operator  $\bar{R}(f, \alpha)$  is continuous with respect to  $f$  at the “point”  $f_T$ , it follows that, for sufficiently small positive  $\delta \leq \delta_I$ , the inequality

$$\rho_F(f_\delta, f_T) \leq \delta \quad (5)$$

implies the inequality

$$\rho_Y(\bar{R}(f_\delta, \alpha), \bar{R}(f_T, \alpha)) \leq \omega(\delta), \quad (6)$$

where  $\omega(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

Since  $\lim_{\alpha \rightarrow 0} \bar{R}(Ay_T, \alpha) = \lim_{\alpha \rightarrow 0} \bar{R}(f_T, \alpha) = y_T$ , it holds that

$$\rho_Y(\bar{R}(f_T, \alpha), y_T) \leq \omega(\delta), \quad (7)$$

$\forall \delta > 0$  and  $\alpha_I = \alpha_I(\delta, y_T)$  with  $\alpha \leq \alpha_I$ .

It follows from inequalities (4), (6), and (7) that, for every  $\delta \leq \delta_I$  and  $\alpha \leq \alpha_I$ ,

$$\rho_Y(\bar{R}(f_\delta, \alpha), y_T) \leq 2\omega(\delta). \quad (8)$$

Since  $\omega(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ , there exist for every  $\varepsilon > 0$ , a  $\delta(\varepsilon)$  such that, for  $\delta \leq \delta(\varepsilon) \leq \delta_I$  and  $\alpha = \alpha_I(\delta, y_T)$ , inequalities (5) and (8) imply  $\rho_Y(\bar{R}(f_\delta, \alpha), y_T) \leq \varepsilon$ .

This completes the proof of Theorem 1.

## 3. Regularization Methods

### 3.1. Tikhonov's Regularization Method

Consider the Volterra integral equation of the first kind

$$Ay \equiv \int_a^x k(x, s)y(s)ds = f(x), \quad a \leq x \leq b, \quad (9)$$

where

$$k(x, s) \in L_2, f(x) \in L_2, y(s) \in W_2^{(1)}.$$

To ensure the stability of the solution of equation (9), we introduce the condition of minimum smoothing functional given as

$$\int_a^b [Ay - f(x)]^2 dx + \alpha \Omega[y] = \min_y, \quad (10)$$

where, as it is so – called, the stabilizing functional,  $\Omega[y]$ , is usually written in the form [3]

$$\Omega[y] = \int_a^b \left\{ y^2(x) + q[y'(s)]^2 \right\} ds. \quad (11)$$

What is more,  $\alpha > 0$  is called the regularization parameter,  $q \geq 0$  defines the order of the regularization (zero order if

$q=0$  and first order if  $q>0$ ). Expanding condition (10) using condition (11) and bearing in mind the Volterra structure of equation (9) yields the following equation of the second kind:

$$\alpha[y(t) - qy''(t)] + \int_a^b B(t,s)y(s)ds = F(t), \quad (12)$$

$a \leq t \leq b$ , where

$$B(t,s) = \begin{cases} R(t,s), & \text{if } a \leq s \leq t \\ R(s,t), & \text{if } t \leq s \leq b, \end{cases} \quad (13)$$

$$R(t,s) = \int_l^b k(x,t)K(x,s)dx, \quad (14)$$

and

$$F(t) = \int_t^b k(x,t)f(x)dx. \quad (15)$$

Expressions (13) and (14) may be written in another form as

$$B(t,s) = \int_{\max\{t,s\}}^b k(x,t)K(x,s)dx. \quad (16)$$

In place of equations (12) - (16), we may use the relation

$$\alpha[y_\alpha(t) - qy_\alpha''(t)] + \int_a^b R(t,s)y_\alpha(s)ds = F(t), \quad (17)$$

$a \leq t \leq b$ , where

$$R(t,s) = R(s,t) = \int_c^\infty \tilde{k}(x,t)\tilde{k}(x,s)dx, \quad (18)$$

$$F(t) = \int_c^\infty \tilde{k}(x,t)\tilde{f}(x)dx, \quad (19)$$

and

$$y'_\alpha(a) = y'_\alpha(b) = 0, \quad (20)$$

are valid for Fredholm first kind equation and setting in the later,  $c=a$ ,  $d=b$  and

$$k(x,s) = \begin{cases} k(x,s), & \text{for } x \geq s \\ 0, & \text{for } x \leq s. \end{cases} \quad (21)$$

In all the cases considered above, the original Volterra integral equations of the first kind are replaced with equations of the second kind (precisely, integro-differential equation of the second kind) of Fredholm type. Consequently, the Tikhonov's regularization method results in the loss of the Volterra structure but assists in the preliminary transformation of equation (10) (using, for example, the finite sum and the finite difference methods) into a system of linear algebraic equations with a full (positive definite) matrix (not triangular). This calls for a

reasonable loss of time when solving it using any computational device. In spite of all these shortcomings, the use of Tikhonov's regularization method in the solution of equation (9) still turns out to be one of the most effective of the regularization methods.

### 3.2. Tikhonov's Regularization Method for Equation of Convolution type (Sequential Tikhonov Regularization Method)

We consider the convolution type Volterra integral equation of the first kind

$$Ay \equiv \int_{-\infty}^x K(x-s)y(s)ds = f(x), \quad -\infty < x < \infty. \quad (22)$$

A complete discussion on this question has been provided by Savelova ([5], [6], [7], [8], [9], [10], [11]).

For the convolution type Volterra integral equation of the first kind

$$\int_0^x K(x-s)y(s)ds = f(x), \quad x \geq 0, \quad (23)$$

We first make use of the so-called techniques of extension [12]. We introduce the functions

$$f_T = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and

$$y_T(s) = \begin{cases} y(s), & s \geq 0 \\ 0, & s < 0 \end{cases}$$

and hence obtain from equation (23), a new statement

$$\int_{-\infty}^x K(x-s)y_T(s)ds = f_T(x), \quad -\infty < x < \infty.$$

It is only from this moment that the Tikhonov's regularization method may be applied.

### 3.3. Denisov's Regularization Method

Consider the equation

$$\int_a^x K(x,s)y(s)ds = f(x), \quad x \in [a,b]. \quad (24)$$

Let  $K(x,x)=I$  and the functions  $K_x^{(i)}(x,s), i=0,1,2$ , be continuous and the exact solution  $y(s) \in C'[a,b]$ . What is more, let it be given that  $y(a)=y_0 \neq 0$ . Suppose further, that in place of the right member  $f(x)$ , the approximate right member  $\tilde{f}(x)$  such that  $\|\tilde{f}(x) - f(x)\|_C \leq \delta$  is known. Then, in place of equation (24), we propose the solution of the  $\alpha$ -regularizing equation

$$\alpha y_\alpha(x) + \int_a^x K(x,s)y_\alpha(s)ds = \tilde{f}(x) + \alpha y_0. \quad (25)$$

Such a procedure is close to the Lavrentev's classical method. The asymptote of the solution of equation (25) is examined in the study by Blondel [13] when  $\alpha \rightarrow 0$  with the exact function  $f(x)$  (but without the term  $\alpha y_0$ ).

In the study by Denisov [14], it is proved that

$$\|y_\alpha(x) - y(x)\|_C \leq k \left( \frac{\delta}{\alpha} + \alpha \right), \quad (26)$$

where  $k = \text{constant}$ . When  $\delta \rightarrow 0$ , we select  $\alpha(\delta)$  such that  $\delta/\alpha \rightarrow 0$ . For instance, if  $\alpha(\delta)$  is chosen so that  $\alpha(\delta) = O(\delta^\rho)$ ,  $0 < \rho < 1$ , we observe that  $\|y_\alpha(x) - y(x)\|_C \rightarrow 0$ , which is the algorithm obtained from equation (24).

We introduce yet another value of  $\alpha$  (call it the quasi-optimal regularizer and denote it by  $\alpha_{q0}$ ), the minimizer of the right-hand side of equation (25). Hence, we obtain

$$\alpha_{q0} = O\left(\delta^{\frac{1}{2}}\right), \quad (27)$$

$$\|y_{\alpha_{q0}}(x) - y(x)\|_C = O\left(\delta^{\frac{1}{2}}\right). \quad (28)$$

An important feature of the method under consideration is the simple nature of equation (24). Furthermore, the Volterra structure of the original equation (as seen) is maintained. However, the method insists on an initial knowledge of the value of  $y(a) = y_0$  which, in a concrete situation, is not obtainable.

### 3.4. Apartsin's $(h, \alpha)$ – Regularization Method

Further modification of Denisov's  $\alpha$  – regularization method in conjunction with Apartsin–Bakushinskii's  $h$  – regularization method is called  $(h, \alpha)$  – regularization method. In the said  $(h, \alpha)$  – regularization method, it is clear from the onset that the procedure will involve the replacement of the integral of equation (24) with the corresponding approximate finite sum ( $h$ -regularization) on the grid and instead of battling with the identification of the preconceived appropriate values of  $h$ ,  $(h, \alpha)$  and  $\delta$ , a system of linear algebraic equations is put in place of equation (24) and is independent of  $\alpha y_0$ .

Now, we consider a system of linear algebraic equations with a triangular matrix of coefficients:

$$\alpha \tilde{y}_{i-\frac{1}{2}} + h \sum_{j=1}^{i-1} k_{i,j+\frac{1}{2}} \tilde{y}_{j+\frac{1}{2}} \approx \tilde{f}_i, \quad i = 2, 3, \dots, n, \quad (29)$$

where

$$y_{i-\frac{1}{2}} = \tilde{y}(s_{i-\frac{1}{2}}), \quad k_{i,j+\frac{1}{2}} = K(x_i, s_{j+\frac{1}{2}}), \quad \tilde{f}_i = \tilde{f}(x_i),$$

$$x_i = a + (i-1)h,$$

$$s_{j+\frac{1}{2}} = a + \left(j - \frac{1}{2}\right)h, \quad (n-1)h = b - a.$$

Notice that the system in equation (28) is obtained by replacing the integral in equation (24) with a finite sum defined by the mid-point rectangular formula having a step-size of  $h = \text{constant}$ .

Let  $k(x,s) \in C_A^{(3)}$ ,  $f(x) \in C^{(3)}[a,b]$ ,  $f(a) = 0$  and  $\min_{a \leq x \leq b} |K(x,x)| \neq 0$ . Then, for a regularized framework  $\{\tilde{y}_{i-\frac{1}{2}}\}$ ,  $i = 2, 3, \dots, n$  satisfying equation (29), the following estimate in respect of the error of the approximate solution is valid:

$$\max_{2 \leq i \leq n} |\varepsilon_{i-\frac{1}{2}}| = \max_{2 \leq i \leq n} |\tilde{y}_{i-\frac{1}{2}} - y_{i-\frac{1}{2}}| = \begin{cases} \left( \frac{2\delta + c_1 h^3 + \alpha h L_1}{\alpha + hk} \right) c_2, & y_0 = 0, \\ \left( \frac{2\delta + c_1 h^3 + \alpha L_1}{\alpha + hk} \right) c_3, & y_0 \neq 0, \end{cases} \quad (30)$$

$$\text{where} \quad \min_{2 \leq i \leq n} |K_{i,i+\frac{1}{2}}| \geq k > 0, \quad L_1 = \max_{a \leq s \leq b} |y'(s)|,$$

$$c_1, c_2, c_3 \in R, \quad L = \max_{a \leq s \leq b} |y(s)|, \quad \|\tilde{f}(x) - f(x)\|_C \leq \delta.$$

Let us analyze the estimate in equation (29). The first component on the right-hand side

$$\frac{2\delta}{\alpha + hk} c_2 \quad \text{or} \quad \frac{2\delta}{\alpha + hk} c_3, \quad (31)$$

characterizes the influence on the exact regularized framework by the right-hand side error. The second component

$$\frac{c_1 h^3}{\alpha + hk} c_2 \quad \text{or} \quad \frac{c_1 h^3}{\alpha + hk} c_3,$$

is connected with the approximation of the quadrature integral, while the third component

$$\frac{\alpha h L_1}{\alpha + hk} c_2 \quad \text{or} \quad \frac{\alpha L}{\alpha + hk} c_3,$$

reflects the contribution of the total error of distortion of the desired equation (28) due to the additional  $\alpha$ -component.

It is seen that the error estimate in equation (29) unites the  $h$ -regularization estimate. This can be observed in equation (23) with the Denisov's  $\alpha$ -regularization estimate in equation (25).

We call the quasi-optimal regularizer (QOR) of the parameters  $\alpha$  and  $h$ , the minimizer of the right-hand side of equation (29). This implies

$$\alpha_{QOR} = \begin{cases} 0 \left( \delta^{\frac{2}{3}} \right), & y_0 = 0 \\ 0(\delta), & y_0 \neq 0, \end{cases} \quad (32)$$

$$h_{QOR} = 0 \left( \delta^{\frac{1}{3}} \right) \quad (33)$$

and

$$\max_{2 \leq i \leq n} |\varepsilon_{i-\frac{1}{2}}| = \max_{2 \leq i \leq n} |y_{i-\frac{1}{2}} - y_{i-\frac{1}{2}}| \leq 0 \left( \delta^{\frac{2}{3}} \right). \quad (34)$$

Since, in practice, it is not clear which of the cases results in  $y_0 = 0$  or  $y_0 \neq 0$ , it is, therefore, good to select one of the relationships in the equation (32) for the asymptote  $\alpha_{QOR}(\delta)$  but maintaining the condition in equation (32).

Indeed, if we set  $\alpha_{QOR} = 0 \left( \delta^{\frac{2}{3}} \right)$ , then

$$\max_{2 \leq i \leq n} |\varepsilon_{i-\frac{1}{2}}| \leq \begin{cases} 0 \left( \delta^{\frac{2}{3}} \right), & \text{if } y_0 = 0 \\ 0 \left( \delta^{\frac{2}{3}} \right), & \text{if } y_0 \neq 0. \end{cases} \quad (35)$$

If we set  $\alpha_{QOR} = 0(\delta)$ , then

$$\max_{2 \leq i \leq n} |\varepsilon_{i-\frac{1}{2}}| \leq \begin{cases} 0(\delta), & \text{if } y_0 = 0 \\ 0 \left( \delta^{\frac{2}{3}} \right), & \text{if } y_0 \neq 0. \end{cases} \quad (36)$$

In all cases, however,  $\max_{2 \leq i \leq n} |\varepsilon_{i-\frac{1}{2}}| \rightarrow 0$  as  $\delta \rightarrow 0$ , irrespective of whether  $y_0 = 0$  or  $y_0 \neq 0$ , that is, the algorithm obtained from solving the system of linear algebraic equation (28) and also equation (22) taking advantage of asymptotes (32) and (33) are regularized. Besides, unlike Denisov's, the method does not use the  $\alpha y_0$  component, and the requirement of  $y_0 = 0$  or  $y_0 \neq 0$  is compensated by changing the dependence of the quasi-optimal asymptote  $\alpha_{QOR}$  and  $\delta$ . Finally, we note that as long as

$$\alpha = \begin{cases} 0(h^2), & \text{if } y_0 = 0 \\ 0(h_3), & \text{if } y_0 \neq 0, \end{cases}$$

then as  $\delta \rightarrow 0$ ,  $\alpha$  decreases faster than  $h$  (the carrier of the basic stability load) and so, parameters  $h$  and  $\alpha$  share homogeneous characterizations. As it now stands, the step size  $h$  should be regarded as the basic parameter while  $\alpha$  plays the role of the auxiliary. This is reflected by the fact that the presence or absence of  $\alpha$  does not influence the state of the asymptote  $h(\delta)$ . This being so, as soon as  $h=0$ ,  $\alpha = 0(\delta^{\frac{1}{2}})$ , the  $\alpha$ -smoothing equation quickly requires

the presence of the component  $\alpha y_0$ ; whereas, when  $h \neq 0$ ,  $\alpha = 0(\delta^{\frac{2}{3}})$  or  $\alpha = 0(\delta)$ , the asymptote  $h(\delta)$  is such that it replaces the function of  $\alpha(\delta)$  and continues to support the method of regularization without the introduction of the component  $\alpha y_0$  in the desired equation. However, the use of the parameter  $\alpha$  in place of  $h$  increases the effectiveness of the method since, for a small  $h$  (not to mention when  $h \rightarrow 0$ ), the error of the solution is notably compressed.

### 3.5. Sergeev's Regularization Method

Consider Volterra integral equation of the first kind

$$\int_0^x K(x, s) y(s) ds = f(x), \quad x \in [0, b]. \quad (37)$$

Let

$$k(x, s) \in C^{(n)}([0, b], [0, b]), \quad n \geq 1,$$

$$\begin{aligned} \frac{\partial^i k(x, s)}{\partial x^i} \Big|_{s=x} &= 0, \quad i = 1, 2, \dots, n-2, \\ \frac{\partial^{n-1} k(x, s)}{\partial x^{n-1}} \Big|_{s=x} &= 1. \end{aligned} \quad (38)$$

If  $\frac{\partial^{n-1} k(x, s)}{\partial x^{n-1}} \Big|_{s=x} \neq 1$  and  $\min_{x \in [0, b]} \Big|_{s=x} \neq 0$ , we can divide both sides of equation (37) by  $\frac{\partial^{n-1} k(x, s)}{\partial x^{n-1}} \Big|_{s=x}$  to obtain an equation with property (38).

Now, let the function  $\frac{\partial^n k(x, s)}{\partial x^n}$  be continuous with respect to  $x$  and  $s$ . Let  $\left| \frac{\partial^n k(x, s)}{\partial x^n} \right| \leq k_n$ . Then after the  $n$ th differentiation of equation (37), we obtain a second kind equation

$$y(x) + \int_0^x \frac{\partial^n k(x, s)}{\partial x^n} y(s) ds = f^{(n)}(x), \quad x \in [0, b]. \quad (39)$$

If  $f(x) \in C_0^{(n)}[a, b] = \{ \xi(x) : \xi \in C^{(n)}[0, b], \xi^{(i)}(0) = 0, i = 0, 1, \dots, n-1 \}$ ,

Then equation (39) has a unique solution  $y(x) \in C[0, b]$ , (by virtue of the given exact data). But, previously, it was assumed that  $y(x) \in C^{(m)}[0, b], m > 0$ . However, if  $f(x)$  together with the error is known, then the determination of the derivative  $f^{(n)}(x)$  becomes an ill-posed problem and, hence, requires the use of any of the regularization methods. In the studies by Lavrent'ev *et al* and Sergeev ([15], [16]), a regularization method which also transforms the equation into a second kind equation but does not require the calculation of the said derivative of  $f(x)$  without losing the Volterra structure is proposed.

Suppose the values

$y^{(k)}(x)|_{x=0} = y_k$ ,  $k = 0, 1, \dots, m-1$ , are known. We introduce the function

$$W_\alpha(x) = e^{-\frac{x}{\alpha}} \sum_{k=0}^{n+1} \frac{1}{\alpha^k} C_{m+k-1}^{m-1} \frac{x^{m+k-1}}{(m+k-1)!}. \quad (40)$$

For this equation, let

$$W_\alpha^{(j)}(0) = \frac{\partial^j W_\alpha(x)}{\partial x^j} \Big|_{x=0} = \begin{cases} 0, & 0 \leq j \leq m+n, \quad j \neq m-1, \\ 1, & j = m-1. \end{cases}$$

Let it be assumed that in place of the exact right-hand member  $f(x)$ , the approximate right-hand member  $\tilde{f}(x)$  is known such that  $\|\tilde{f}(x) - f(x)\|_C \leq \delta$ .

Therefore, in place of the exact equation (37) or formally, its equivalent equation (39), we propose the solution of the following approximate Volterra equation of the second kind:

$$y_\alpha(x) - \int_0^x L(x,s) y_\alpha(s) ds = U(x), \quad x \in [0, b], \quad (41)$$

where

$$L(x,s) = \int_0^x W_\alpha^{(M)}(x-t) \frac{\partial^n k(t,s)}{\partial t^n} dt,$$

$$U(x) = f_0(x) - \int_0^t W_\alpha^{(n+m)}(x-t) \tilde{f}(t) dt$$

and

$$f_0(x) = \sum_{k=0}^{m-1} y_k W_\alpha^{(m-k-1)}(x).$$

The estimate

$$\|y_\alpha - y\|_C \leq \left( \alpha^m \varphi_m + 2^{n+m} \frac{\delta}{\delta^m} \right) A e^{Bb}, \quad (42)$$

where

$$\varphi_m = \max_{x \in [a,b]} |y^{(m)}(x)|, \quad A = C_{m+n+1}^m, \quad B = 2^m K_n A,$$

is valid. The minimum value of the right-hand side of equation (42) is obtained if

$$\alpha(\delta) = 2 \left( \frac{n}{m \varphi_m} \right)^{\frac{1}{n+m}} \delta^{\frac{1}{n+m}}. \quad (43)$$

For such value of  $\alpha(\delta)$ ,

$$\|y_\alpha - y_c\|_C \leq 2^m A e^{Bb} \left( 1 + \frac{m}{n} \right) \varphi_m^{\frac{n}{n+m}} \left( \frac{n}{m} \right)^{\frac{m}{n+m}} \delta^{\frac{m}{n+m}}. \quad (44)$$

It is clear that  $\|y_\alpha - y\|_C \rightarrow 0$  as  $\delta \rightarrow 0$ , and what is more,

$$\|y_\alpha - y\|_C = O \left( \delta^{\frac{n}{n+m}} \right), \quad (45)$$

that is, the algorithm obtained from equation (41) is regularizing.

In what appears to be the simplest case, where  $n = m = 1$ , it is expected to solve the equation

$$\alpha y_\alpha(x) + \int_0^x p(x,s) y_\alpha(s) ds = Q(x), \quad x \in [0, b], \quad (46)$$

where

$$p(x,s) = \frac{1}{2} \int_0^x e^{-\frac{x-t}{\alpha}} \left( \frac{x-t}{\alpha} \right)^2 K'_t(t,s) dt$$

and

$$Q(t) = y_0 e^{-\frac{x}{\alpha}} \left( \alpha + x + \frac{x^2}{2\alpha} \right) -$$

$$\int_0^x e^{-\frac{(x-t)}{\alpha}} \left( \frac{x-t}{\alpha^2} \right) \left( \frac{x-t}{2\alpha} - 1 \right) \tilde{f}(t) dt.$$

Consequently, we have that

$$\|y_\alpha - y\|_C \leq 3 \left( \alpha \varphi_1 + 4 \frac{\delta}{\alpha} \right) e^{6k_1 b}. \quad (47)$$

The minimum value of the right-hand side of equation (47) is attained when

$$\alpha(\delta) = 2 \varphi_1^{\frac{1}{2}} \delta^{\frac{1}{2}}. \quad (48)$$

For such  $\alpha(\delta)$ ,

$$\|y_\alpha - y\|_C \leq 12 e^{6k_1 b} \varphi_1^{\frac{1}{2}} \delta^{\frac{1}{2}}. \quad (49)$$

Here,

$$y_0 = y(0), \quad \varphi_1 = \max_{x \in [0,b]} |y'(x)|, \quad k_1 = \max_{x,s \in [0,b]} \left| \frac{\partial k(x,s)}{\partial x} \right|.$$

### 3.6. Magnicki's Regularization Method

This is closely related to Sergeev's method but has a wider range of applications. We consider the equation

$$\int_0^x K(x,s) y(s) ds = f(x), \quad 0 \leq x \leq 1. \quad (50)$$

Let

$$\left. \begin{aligned}
& k(x, s) \in C^{(n+m)}([0, I]; [0, I]), n \geq 1, m \geq 1, \\
& \left. \frac{\partial^i k(x, s)}{\partial x^i} \right|_{s=x} = 0, i = 0, 1, \dots, n-2, \\
& \left. \frac{\partial^{n-1} k(x, s)}{\partial x^{n-1}} \right|_{s=x} = 1, \\
& k_{n+i}(x, s) = \frac{\partial^{n+i} k(x, s)}{\partial x^{n+i}}, \\
& \quad i = 0, 1, \dots, m, \text{ continuous in } x \text{ and } s \\
& k_{n+i,j}(x) = \frac{\partial^j k_{n+i}(x, x)}{\partial x^j}, \\
& \quad j = 0, 1, \dots, m-1, i = 0, 1, \dots, m-1 \\
& \text{continuous in } x, \\
& |k_{n+i}(x, t)| \leq k_{n+i}, |k_{n+i,j}(x)| \leq k_{n+i,j}.
\end{aligned} \right\} \quad (51)$$

Furthermore, let

$$\begin{aligned}
f(x) \in C_0^{(n)}[0, I] &= \{ \xi(x) : \xi(x) \in C^{(n)}[0, I]; \xi^{(i)}(0) = 0; \\
& i = 0, 1, 2, \dots, n-1 \}.
\end{aligned}$$

Then, differentiating equation (50)  $n$  times, we obtain the equation

$$y(x) + \int_0^x \frac{\partial^n k(x, s)}{\partial x^n} y(s) ds = f^{(n)}(x), \quad x \in [0, I], \quad (52)$$

which has a unique solution  $y(x) \in C[0, I]$ . Again, similar to Sergeev's method, we assume that  $y(x) \in C^{(m)}[0, I]$ , and that  $y_k = y^{(k)}(0)$ ,  $k = 0, 1, \dots, m-1$  are known.

Therefore, in place of equation (50) or (52), we propose the solution of a Volterra integral equation of the second kind

$$\begin{aligned}
& y_\alpha(x) + \int_0^x \frac{\partial^n k(x, s)}{\partial x^n} y_\alpha(s) ds = \\
& = y_0 + u_0(x) - \frac{I}{\omega^{(n-1)}(0) \alpha^{n+1}} \int_0^x \omega^{(n+m)} \left( \frac{x-t}{\alpha} \right) \tilde{f}(t) dt, \\
& x \in [0, I], \quad (53)
\end{aligned}$$

Subject to the conditions:

$$u_0(x) = \sum_{i=0}^{m-1} \frac{\alpha^i}{\omega^{(m-1)}(0)} \omega^{(m-i-1)} \left( \frac{x}{\alpha} \right) u^{(i)}(0),$$

$$u_0(x) = 0 \text{ for } m = 1,$$

$$u_0(x) = \int_0^x \frac{\partial^n K(x, s)}{\partial x^n} y(s) ds,$$

$\omega(\xi)$  a class of functions defined by the following conditions:

$$\int_0^\infty |\omega^{(i)}(\xi)| d\xi \leq C; \quad i = 0, 1, \dots, m+n,$$

$$\omega^{(m-1)}(0) \neq 0, \quad \omega^{(i)}(0) = 0, \quad i = 0, 1, \dots, m-2, m, \dots, n+1,$$

and there exist  $\rho > n+m$  such that  $\omega^{(\rho)} \neq 0$ .

The following estimate is valid

$$\|y_\alpha - y\|_C \leq \left| \alpha^m C_0 (A + \phi_m) + \frac{\delta}{\alpha^n} C_{n+m} \right| \frac{e^{k_n}}{\omega^{(m-1)}(0)}, \quad (54)$$

where

$$\|U^{(m)}(k)\| \leq A \|y^{(m)}(x)\| \leq \phi_m.$$

The minimum value of the right-hand side is attained

$$\text{when } \alpha(\delta) = \left[ \frac{C_{n+1}}{C_0(A + \phi_m)} \frac{n}{m} \right]^{\frac{1}{m+n}} \frac{1}{\delta^{\frac{1}{m+n}}}.$$

For such values of  $\alpha(\delta)$ ,

$$\begin{aligned}
\|y_\alpha - y\|_C &\leq C_{n+m}^{\frac{1}{m+n}} e^{K_n} \frac{C_0^{\frac{1}{m+n}}}{\omega^{(m-1)}(0)} (A + \phi_m)^{\frac{1}{m+n}} \times \\
&\times \left( \frac{n}{m} \right)^{\frac{1}{m+n}} \left( 1 + \frac{m}{n} \right) \delta^{\frac{1}{m+n}}.
\end{aligned}$$

As  $\delta \rightarrow 0$ ,  $\|y_\alpha - y\|_C \rightarrow 0$ , that is, the algorithm obtained from equation (53) is regularizing.

For  $m = n = 1$ ,

$$k_I(x, s) = \frac{\partial}{\partial s} k(x, s) \text{ and } k_{II}(x, s) = \frac{\partial^2 k(x, s)}{\partial x \partial s} \quad \text{are}$$

continuous in  $x$  and  $s$ ,

$$\left| \frac{\partial k(x, s)}{\partial s} \right| \leq k_I, \quad \left| \frac{\partial^2 k(x, s)}{\partial x \partial s} \right| \leq k_{II},$$

we propose a special equation (which is not necessarily a consequence of equation (53)):

$$\alpha y_\alpha(x) + \int_0^x G(x, s) y_\alpha(s) ds = F(x), \quad x \in [0, I],$$

where

$$\begin{aligned}
G(x, s) &= \\
&= \frac{I}{\omega(0)} \int_s^x \left[ \omega^{(l)} \left( \frac{x-t}{\alpha} \right) k_I(t, t) + \int_0^x \omega^{(l)} \left( \frac{x-t}{\alpha} \right) k_{II}(\tau, t) d\tau \right] dt,
\end{aligned}$$

$$F(x) = \alpha y_0 - \frac{I}{\omega(0) \alpha} \int_0^x \omega^{(2)} \left( \frac{x-t}{\alpha} \right) \tilde{f}(t) dt.$$



The following estimate (for an optimal value of  $\alpha(\delta)$ ) is valid:

$$\|y_\alpha - y\|_C \leq \frac{2C_2^2 C_0^{\frac{1}{2}}}{\omega(0)} \exp \left[ \frac{C_1(k_I + k_{II})}{\omega(0)} \right] \varphi_I^{\frac{1}{2}} \delta^{\frac{1}{2}}.$$

Now, with  $m = n = I$  conditions similar to those of equation (51), it becomes necessary to solve the following consequence of equation (51):

$$\begin{aligned} y_\alpha(x) + \int_0^x \frac{\partial k(x, s)}{\partial x} y_\alpha(s) ds \\ = y_0 - \frac{I}{\omega(0)\alpha^2} \int_0^x \omega^2 \left( \frac{x-t}{\alpha} \right) \tilde{f}(t) dt, \\ x \in [0, I]. \end{aligned}$$

We obtain the following estimates

$$\begin{aligned} \alpha(\delta) &= \left[ \frac{C_2}{C_0(A + \varphi_I)} \right]^{\frac{1}{2}} \delta^{\frac{1}{2}}, \\ \|y_\alpha - y\|_C &\leq 2C_2^{\frac{1}{2}} e^{K_I} \frac{C_0^{\frac{1}{2}}}{\omega(0)} (A + \varphi_I)^{\frac{1}{2}} \delta^{\frac{1}{2}}. \end{aligned}$$

## 4. Conclusions

Although this study primarily examined regularization methods for first kind Volterra integral equations that maintain the Volterra structure, we never lost sight of the distinguishing features associated with this class of equations, which reflect in the uncertainty and difficulties surrounding their nature when attempting to find their solutions using one or other existing numerical methods. On the other hand, an attempt to solve Volterra integral equations of the first kind using analytic techniques may not be fruitful after all. This is so, because Volterra integral equations of the first kind, in a sense appears to be situated mid-way between Volterra integral equations of the second kind and Fredholm integral equation of the first kind. Precisely, if a Volterra integral equation of the second kind is well-posed and can be effectively solved by any classical means, then a Fredholm integral equation of the first kind is ill-posed, given any preconceived functional space and solvable by special approximating methods. Again, Volterra integral equation of the first kind may be well-posed or ill-posed depending on the choice of the solution space and can only be solved effectively by regularization methods.

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