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A Dirichlet Boundary-Value Problem in the Quarter-Plane for the Modified Helmholtz Equation

Min-Hai Huang

College of Mathematics and Statistics, Zhaoqing University, Zhaoqing, Guang Dong, China

Abstract A Dirichlet boundary-value problem for the modified Helmholtz equation in the quarter-plane was discussed. By using the Fokas transform method, the solution in the form of integral representation was given.

Keywords Modified Helmholtz equation, Boundary-value problem, Fokas transform method

1. Introduction

The aim of this paper is to solve a Dirichlet boundary-value problem of the modified Helmholtz equation in a quarter-plane. The modified Helmholtz equation arises naturally in many physical applications [1], for example, in implicit marching schemes for the heat equation, in Debye-Huckel theory, in the linearization of the Poisson-Boltzmann equation, in diffusion of waves [2, 3, 4] and so on.

The mathematical formulation of our problem is as follow. Find a function $q(z,\overline{z})$ in the first quadrant Ω satisfying

$$\Delta q(z,\overline{z})-4\beta q(z,\overline{z})=0, \qquad z\in\Omega, \tag{1.1}$$

$$q=d_j(z), z\in\Gamma_j, \qquad j=1,2. \tag{1.2}$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial\overline{z}}$ is the usual Laplace operator, $z=x+iy$ and $\overline{z}=x-iy$, $\Gamma_j(j=1,2)$ is the borderline of the wedge domain Ω, $d_j(z)(j=1,2)$ is known function and $d_j\in L^1(R^+)\cap L^2(R^+)$, the $L^p(p=1,2)$ spaces are function spaces defined on R^+ using a natural generalization of the p-norm for finite-dimensional vector spaces.

Classical integral transform methods such as Fourier transform, Laplace and Mellin transform are important mathematical tools in investigating boundary-value problems of partial differential equations (see [5, 6]). For one concrete boundary value problem, appropriate integral transformation should be applied. Here, we shall use the Fokas transform method [7].

In the late 1990s Fokas introduced a novel flexible approach to solve the initial/boundary value problem for various two dimensional linear and integrable nonlinear PDE's [8]. In general, Fokas transform method involves three steps, namely:

(i) given a PDE, construct a closed differential form (this is related to the existence of a Lax pair);
(ii) given a domain, perform the spectral analysis associated with this differential form, yields an integral representation of the solution in terms of boundary value of the domain;
(iii) given appropriate boundary conditions, by analyzing the global relation, the unknown boundary values can be determined.

The main achievement of this method is that it yields explicit integral (as oppose to series) representations for a variety of boundary value problems. For some specific problems, the integral representation of the solutions have analytical and numerical advantage, and also facilitate to analysis the asymptotic behavior of the solution. Use of Fokas transform to construct the integral expression of solutions is valid for convex polygons, but fail for non-convex polygon. The readers are also referred to [7] for a systematic exposition of the method, its various applications and more references therein.

2. The Integral Representation of the Solution

LEMMA 1. ([7]) *Let* Ω *be the interior of the first quadrant of the complex z-plane. Assume that there exists a solution* $q(z,\overline{z})$ *of the modified Helmholtz equation (1.1) in* Ω, *and assume that this solution is sufficiently smooth on the boundary of* Ω. *Then* $q(z,\overline{z})$ *can be expressed as*

* Corresponding author:
hmh9520@sina.com (Min-Hai Huang)
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$$q(z,\bar{z})=\frac{1}{4\pi i}\sum_{j=1}^{2}\int_{L_j}e^{i\beta(kz-\bar{z}/k)}\rho_j(k)\frac{dk}{k}, \tag{2.1}$$

where the spectral functions $\rho_1(k)$ *and* $\rho_2(k)$ *are defined as*

$$\rho_1(k)=-\int_0^{\infty}e^{\beta(k+1/k)y}\left[iq_x(0,y)+\beta\left(1/k-k\right)q(0,y)\right]dy,\quad \Re k\le 0, \tag{2.2}$$

$$\rho_2(k)=\int_0^{\infty}e^{-i\beta(k-1/k)x}[-iq_y(x,0)+i\beta(1/k+k)q(x,0)]dx,\quad \Im k\le 0, \tag{2.3}$$

and the rays $l_j(j=1,2)$ *are respectively the positive imaginary axis and the positive real axis in the complex* k *-plane. Furthermore, the following global relation is valid:*

$$\rho_1(k)+\rho_2(k)=0\ \text{ for } \pi\le\arg k\le 3\pi/2. \tag{2.4}$$

3. Determination of the Spectral Functions

In the section 2, the solutions of (**1.1**) has been given in Lemma 1 in the form of integral representation (**2.1**) with spectral functions $\rho_j(k)(j=1,2)$, which involve both the Dirichlet and Neumann boundary-values. For the concrete problem, some boundary-values are unknown. But, this can be achieved by help of the global relations and certain symmetrical properties. In this section, we'll give an derivation of the procedure.

Theorem 1. *Assume that* $q(x,y)$ *is a solution of the modified Helmholtz equation (***1.1***) in the first quadrant* Ω *, and satisfies the Dirichlet boundary condition (***1.2***), then it possesses the integral representation*

$$q(z,\bar{z})=\frac{1}{2\pi i}\int_{l_1}e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left[G_1(k)+G_2(-k)\right]\frac{dk}{k}+\frac{1}{2\pi i}\int_{l_2}e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left[G_2(k)+G_1(-k)\right]\frac{dk}{k}, \tag{3.1}$$

where, the rays $l_j(j=1,2)$ *are respectively the positive imaginary axis and the positive real axis in the complex* k *-plane, and*

$$G_1(k)=\beta(k-1/k)\int_0^{\infty}e^{\beta(k+1/k)s}d_1(is)ds,\quad Rek\le 0. \tag{3.2}$$

$$G_2(k)=i\beta(k+1/k)\int_0^{\infty}e^{-i\beta(k-1/k)s}d_2(s)ds,\quad Imk\le 0. \tag{3.3}$$

Proof: First of all, we introduce two auxiliary functions

$$N_1(k)=-\int_0^{\infty}e^{\beta(k+1/k)s}q_x(0,s)ds,\quad Rek\le 0. \tag{3.4}$$

$$N_2(k)=-\int_0^{\infty}e^{-i\beta(k-1/k)s}q_y(s,0)ds,\quad Imk\le 0. \tag{3.5}$$

Now, we have

$$\rho_1(k)=iN_1(k)+G_1(k);\rho_2(k)=iN_2(k)+G_2(k). \tag{3.6}$$

Then, it's easy to see, $G_1(k),N_1(k)$ and $G_2(k),N_2(k)$ are analytic functions in the left and lower half k -planes, respectively, and fulfill the readily verified symmetrical relations

$$\overline{G_1(\bar{k})}=G_1(k),\quad \overline{N_1(\bar{k})}=N_1(k),\quad Rek\le 0. \tag{3.7}$$

$$\overline{G_2(\bar{k})}=G_2(-k),\quad \overline{N_2(\bar{k})}=N_2(-k),\quad Imk\ge 0. \tag{3.8}$$

From the global relation (**2.4**), we have

$$iN_1(k)+G_1(k)+iN_2(k)+G_2(k)=0,\quad \pi\le\arg k\le\frac{3\pi}{2}. \tag{3.9}$$

Taking complex conjugates for the formula (3.9), according to the symmetry relations (3.7)-(3.8), we obtain

$$-iN_1(k)+G_1(k)-iN_2(-k)+G_2(-k)=0,\frac{\pi}{2}\le\arg k\le\pi. \tag{3.10}$$

which is

$$iN_1(k)=G_1(k)-iN_2(-k)+G_2(-k),\frac{\pi}{2}\leq \arg k\leq \pi. \tag{3.11}$$

Formula (3.9) add (3.10) yields

$$2G_1(k)+iN_2(k)+G_2(k)-iN_2(-k)+G_2(-k)=0, \qquad \arg k=\pi. \tag{3.12}$$

Making transform $k\mapsto -k$, we have

$$iN_2(k)=2G_1(-k)+iN_2(-k)+G_2(-k)+G_2(k),\arg k=0. \tag{3.13}$$

Combining the formulas (3.13), (3.11), (3.6) and (2.1), one obtains

$$q(z,\bar{z})=\frac{1}{4\pi i}\sum_{j=1}^{2}\int l_j e^{i\beta(kz-\frac{\bar{z}}{k})}\rho_j(k)\frac{dk}{k}=\frac{1}{4\pi i}\sum_{j=1}^{2}\int l_j e^{i\beta(kz-\frac{\bar{z}}{k})}[iN_j(k)+G_j(k)]\rho_j(k)\frac{dk}{k}$$

$$=\frac{1}{4\pi i}\int l_1 e^{i\beta(kz-\frac{\bar{z}}{k})}[G_1(k)-iN_2(-k)+G_2(-k)+G_1(k)]\rho_j(k)\frac{dk}{k}$$

$$+\frac{1}{4\pi i}\int l_2 e^{i\beta(kz-\frac{\bar{z}}{k})}[2G_1(-k)+iN_2(-k)+G_2(-k)+G_2(k)+G_2(k)]\rho_j(k)\frac{dk}{k}. \tag{3.14}$$

By using Cauchy's integral theorem, it's easy deduced

$$\frac{-1}{4\pi i}\int_{l_1} e^{i\beta(kz-\frac{\bar{z}}{k})}iN_2(-k)\rho_j(k)\frac{dk}{k}+\frac{1}{4\pi i}\int_{l_2} e^{i\beta(kz-\frac{\bar{z}}{k})}iN_2(-k)\rho_j(k)\frac{dk}{k}=0,$$

$$\frac{1}{4\pi i}\int_{l_1} e^{i\beta(kz-\frac{\bar{z}}{k})}G_2(-k)\rho_j(k)\frac{dk}{k}=\frac{1}{4\pi i}\int_{l_2} e^{i\beta(kz-\frac{\bar{z}}{k})}G_2(-k)\rho_j(k)\frac{dk}{k} \tag{3.15}$$

It is readily seen that

$$q(z,\bar{z})=\frac{1}{4\pi i}\int_{l_1} e^{i\beta(kz-\frac{\bar{z}}{k})}[G_1(k)+G_2(-k)+G_1(k)]\rho_j(k)\frac{dk}{k}$$

$$+\frac{1}{4\pi i}\int_{l_2} e^{i\beta(kz-\frac{\bar{z}}{k})}[2G_1(-k)+G_2(-k)+G_2(k)+G_2(k)]\rho_j(k)\frac{dk}{k}$$

$$=\frac{1}{2\pi i}\int_{l_1} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left[G_1(k)+G_2(-k)\right]\frac{dk}{k}+\frac{1}{2\pi i}\int_{l_2} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left[G_2(k)+G_1(-k)\right]\frac{dk}{k}, \tag{3.16}$$

which justifies (**3.1**).

4. Conclusions

In this paper, we have used Fokas transform method to discuss a Dirichlet boundary-value problem for the modified Helmholtz equation in the quarter-plane. We deduced the formula of the spectral function in detail, and the solution in the form of integral representation was obtained. It was shown that the integral representations of the solution is closed, which is useful for further analysis of the solution (see [9]).

The Fokas' transform method is a powerful tool for solving problems of the initial/boundary value problem for various two dimensional linear and integrable nonlinear PDE's. This paper presents a simple application. To some extent, Fokas transform is an effective complement and promote of the Fourier transform and the inverse scattering transform.

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