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On a Problem of Maximization in the Discrete Time Models of Economic Dynamics

S. I. Hamidov

Baku State University, Department of Mathematic Cybernetics, Baku, Azerbaijan

Abstract We consider the two-sector model of the economic dynamics. The problem of the distribution of labor between sectors is considered under the condition that the total consumption is maximized. As a production function is taken a function with constant elasticity of the substitution (CES). Potential opportunity of the sectors is analyzed.

Keywords Problem of maximization, Consumption, Production function

1. Introduction

Consider Neumann type two-sector model $(i = 1,2)$ [1] that is denoted as Z. Let's introduce the denotations:

$K_t^i -$ base funds of the $i-$th sector;

$L_t^i -$ number of the labor in the $i-$th sector;

$W_t^i -$ consumption fund of the $i-$th sector;

$\omega_t^i -$ specific consumption (wage rate) in the $i-$th sector;

$M_t^i -$ national income in the $i-$th sector;

$\eta_t^i -$ capital-labor force ratio in the $i-$th sector, $\eta_t^i = \dfrac{K_t^i}{L_t^1}$;

$\nu_i -$ ratio of fixed assets disposals in the $i-$th sector;

$F_i(K_t^i, L_t^i) -$ production function of the $i-$th sector. Sometimes instead of the function $F_i(K_t^i, L_t^i)$ we'll consider the function $f_i(\eta_t^i) = F_i(\eta_t^i, 1)$.

It is expected that the total labor force is $L_t = L_t^i + L_t^2 = 1.$ In paper [2] the dependence of the consumption volume on the labor force is investigated. In this paper the following problem is set: to distribute the labor between the sectors such that to maximize the total consumption. At the same time as a production function the function with constant elasticity of substitution (CES) is considered [1, 3, 4, 6, 8].

* Corresponding author:
sabir818@yahoo.com (S. I. Hamidov)
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2. Main Part

So, consider the problem of maximizing the total consumption

$$W_{t+1}^1(L_{t+1}^1) + W_{t+1}^2(L_{t+1}^2) \to \max \tag{1}$$

under the condition

$$L_{t+1}^1 + L_{t+1}^2 = 1. \tag{2}$$

Here $W_{t+1}^i = \omega_{t+1}^i L_{t+1}^i$ $(i = 1,2)$ is a consumption fund in the $i-$th sector, under the assumption that the *special consumption* ω_{t+1}^i $(i = 1,2)$ *is chosen by the formula*

$$\omega_{t+1}^i(\eta_{t+1}^i) = \frac{f_i(\eta_{t+1}^i) - \eta_{t+1}^i f_i'(\eta_{t+1}^i)}{\nu_i + f_i'(\eta_{t+1}^i)}, \tag{3}$$

where η_{t+1}^i is the unique root of the equation

$$\frac{\left[\nu_i \eta_t^i + f_i(\eta_t^i)\right] L_t^i}{L_{t+1}^i} = \frac{\nu_i \eta_{t+1}^i + f_i(\eta_{t+1}^i)}{\nu_i + f_i'(\eta_{t+1}^i)}. \tag{4}$$

Since L_{t+1}^i may be expressed as function of η_{t+1}^i:

$$L_{t+1}^i(\eta_{t+1}^i) = \frac{M_t^i}{Q_i(\eta_{t+1}^i)}, \tag{5}$$

where M_t^i and $Q_i\ (\eta_{t+1}^i)$ are defined by the equalities

$$M_t^i = \left[\nu_i \eta_t^i + f_i(\eta_t^i)\right] L_t^i, \quad Q_i(\eta_{t+1}^i) = \frac{\nu_i \eta + f_i(\eta)}{\nu_i + f_i'\ (\eta)},$$

then the following relations are valid

$$W_{t+1}^i(L_{t+1}^i) = M_t^i \frac{f_i(\eta_{t+1}^i) - \eta_{t+1}^i f_i'(\eta_{t+1}^i)}{\nu_i \eta_{t+1}^i + f_i(\eta_{t+1}^i)}, \tag{6}$$

where $\eta_{t+1}^i = \eta_{t+1}^i(L_{t+1}^i)$ is an inverse to function defined by the formula (5).

Now suppose that as a production function will be considered the function with a constant elasticity of substitution (CES)

$$F_i(K_t^i, L_t^i) = (A_i K_t^{i-\rho_i} + B_i L_t^{i-\rho_i})^{-\frac{1}{\rho_i}}, \qquad (7)$$

where $\rho_i > 0$.

It is proved that [2] in the case when F is a production function with a constant elasticity of substitution (CES) the consumption computed by the formula (6), reaches its maximum in some point $\overline{L}_{t+1}^i$, and $\overline{L}_{t+1}$ is a unique local extremum point, the function $W_{t+1}^i(L_{t+1}^i)$ has the only *inflection point* $\hat{L}_{t+1}$, *which changes the concavity to the convexity while* $\hat{L}_{t+1}^i > \overline{L}_{t+1}^i$.

Note that capital-labor ratio $\overline{\eta}_i$, in which the maximum of the consumption function of the defined by the formula (6) is reached is the same at all times. Therefore, the point $\overline{L}_{t+1}^i$ depends only on the national wealth M_t^i:

$$\overline{L}_t^i = \frac{M_t^i}{Q_i(\overline{\eta}^i)}, \qquad (8)$$

where $Q_i(\overline{\eta}^i) = const$.

Suppose that (3) has a unique solution $\tilde{L}_t^2$ $(\tilde{L}_t^{1'} = 1 - \tilde{L}_t^2)$ and $\overline{L}_t^i$ is a point in which the function $W_t^i(L_t^i)$ reaches its maximum. As above $\overline{\omega}^i$ and $\overline{\eta}^i$ are found from the conditions

$$\gamma_t^i(\overline{\omega}^i) = 1; \qquad \overline{\omega}^i = \omega_t^i(\overline{\eta}^i).$$

Here we give

Lemma 1. Let the following conditions be fulfilled

a) $\tilde{\eta}^2 > \overline{\eta}^2$

b) $\tilde{L}_\tau^2 \le \overline{L}_\tau^2$ for some moment τ.

Then

$\overline{L}_{\tau+1}^2 \le \overline{L}_\tau^2$

Proof. Let $\tilde{L}_\tau^2 \le \overline{L}_\tau^2$ and $\tilde{\eta}^2 > \overline{\eta}^2$. Since $Q(\eta)$ and $\omega(\eta)$ are increasing functions and

$$Q(\eta_\tau^2) = \frac{M_{\tau-1}^2}{\tilde{L}_\tau^2} \ge \frac{M_{\tau-1}^2}{\overline{L}_\tau^2} = Q(\tilde{\eta}^2),$$

then $\omega_\tau^2(\eta_\tau^2) \ge \omega^2(\tilde{\eta}^2) > \overline{\omega}^2(\overline{\eta}^2)$, and therefore $\gamma_\tau^2(\omega_\tau^2) < 1$. Since $M_\tau^2 = \gamma_\tau^2 M_{\tau-1}^2$ we have $M_\tau^2 < M_{\tau-1}^2$. Considering (8) we get $\overline{L}_{\tau+1}^2 \le \overline{L}_\tau^2$.

Lemma is proved.

Note. It is easy to check that the condition $\tilde{\eta} > \overline{\eta}$ is satisfied if

$$A \in \left[\frac{1}{\left((1-v)(1+\rho)^{1+\frac{1}{\rho}} \right)^{\rho}}, \frac{1}{(1-v)^{\rho}} \right].$$

Let’s show this. The points $\tilde{\eta}$ and $\overline{\eta}$ are solutions of the equations

$$vB\eta^\rho - vA\rho - A\rho y^{-\frac{1}{\rho}} = 0,$$

$$1 - v = Ay^{-\frac{1}{\rho}-1},$$

where $y = A + B\eta^\rho$. Then

$$(1-v)y = Ay^{-\frac{1}{\rho}} \quad \text{и} \quad \frac{1}{\rho}(vB\eta^\rho - vA\rho) = Ay^{-\frac{1}{\rho}}.$$

Let

$$\varphi_1(\eta) = \frac{v}{\rho}B\eta^\rho - vA,$$

$$\varphi_2(\eta) = (1-v)(A + B\eta^\rho),$$

$$\varphi_3(\eta) = A(A + B\eta^\rho)^{-\frac{1}{\rho}}.$$

Note that the functions φ_1 and φ_2 are increasing and φ_3 - decreasing and moreover

$$\varphi_1(0) = -vA, \quad \varphi_2(0) = (1-v)A, \quad \varphi_3(0) = A^{1-\frac{1}{\rho}}$$

and

$$\lim_{\eta\to+\infty}\varphi_1(\eta) = \lim_{\eta\to+\infty}\varphi_2(\eta) = +\infty, \quad \lim_{\eta\to+\infty}\varphi_3(\eta) = 0.$$

It follows from the last that the equation $\varphi_2(\eta) = \varphi_3(\eta)$ has a unique solution $\overline{\eta} > 0$ if and only if, when $\varphi_2(0) < \varphi_3(0)$, that is equivalent to the inequality

$$A < \frac{1}{(1-v)^\rho}.$$

Then $\varphi_1(\eta) = 0$ if and only if when $\eta^\rho = \frac{A\rho}{B}$. Due to the properties of φ_1 and φ_3 we have $\overline{\eta} > \left(\frac{A^\rho}{B}\right)^{\frac{1}{\rho}}$ (Fig.1).

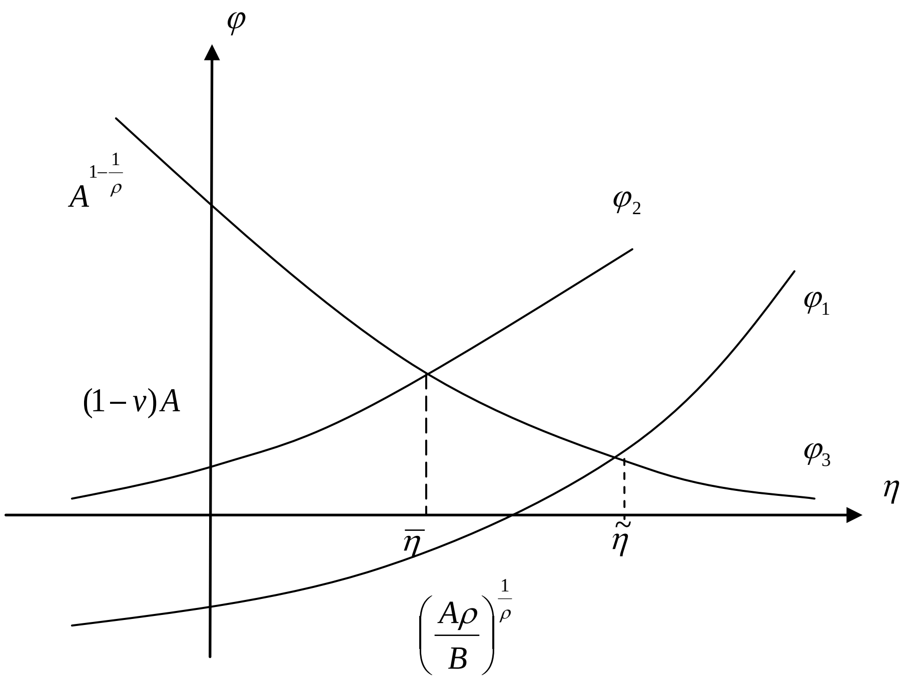

Figure 1

Therefore the inequality $\tilde{\eta} > \bar{\eta}$ is satisfied if

$$\varphi_2\left(\left(\frac{A\rho}{B}\right)^{\frac{1}{\rho}}\right) \geq \varphi_3\left(\left(\frac{A\rho}{B}\right)^{\frac{1}{\rho}}\right) \tag{9}$$

Let's calculate $\varphi_2\left(\left(\frac{A\rho}{B}\right)^{\frac{1}{\rho}}\right)$ and $\varphi_3\left(\left(\frac{A\rho}{B}\right)^{\frac{1}{\rho}}\right)$:

$$\varphi_2\left(\left(\frac{A\rho}{B}\right)^{\frac{1}{\rho}}\right) = (1-v)(1+\rho)A,$$

$$\varphi_3\left(\left(\frac{A\rho}{B}\right)^{\frac{1}{\rho}}\right) = A^{1-\frac{1}{\rho}}(1+\rho)^{-\frac{1}{\rho}}.$$

As follows from (9)

$$(1-v)(1+\rho)^{1+\frac{1}{\rho}}A^{\frac{1}{\rho}} \geq 1.$$

From this we obtain

$$A \geq \frac{1}{\left[(1-v)(1+\rho)^{1+\frac{1}{\rho}}\right]^{\rho}}.$$

Consequently, for the fulfillment of the condition a) of Lemma 1 it is enough to take A_2 from the interval

$$\left[\frac{1}{\left((1-v_2)(1+\rho_2)^{1+\frac{1}{\rho_2}}\right)^{\rho_2}},\ \frac{1}{(1-v_2)^{\rho_2}}\right].$$

Let $\pi_t = 1-\bar{L}_t^2$ and $\left[v_1\eta_t^1 + f_1(\eta_t^1)\right]\pi_t = R_t$.
It is valid

Theorem 1. Let the conditions

a) $\eta_{\tau-1}^1 < \bar{\eta}, \quad \tilde{\eta}^2 > \bar{\eta}^2,$

b) $\tilde{L}_{\tau-1}^2 \geq \bar{L}_\tau^2$ and $M_{\tau-1}^1 \geq Q_1(\tilde{\eta})$

be satisfied. Then $\tilde{L}_t^2 \leq \bar{L}_\tau^2$ for all $t > \tau,$ and moreover $\bar{L}_t^2$ decreases.

Proof. First we show that $M_\tau^1 \geq Q_1(\tilde{\eta}).$ Actually, from the condition $M'_{\tau-1} \geq Q_1(\tilde{\eta})$ and (8) follows that $\bar{L}'_\tau \geq 1;$ Since the total number of the labor is equal to unit, the function $W'_\tau(L')$ is increasing and concave on the interval $[0,1]$. Considering that $\bar{L}_\tau^2 \leq 1,$ we obtain from the last that $W_\tau^2(L^2)$ increases on the interval $[0,\pi_\tau]$ and decreases on $[\pi_\tau,1]$. Therefore the solution $\tilde{L}_\tau^1$ of problem (1) lies on the interval $[\pi_\tau,1]$, i.e. $\tilde{L}_\tau^1 \geq \pi_\tau$. It gives

$$M_\tau^1 = \left[v_1\eta_\tau^1 + f_1(\eta_\tau^1)\right]\tilde{L}_\tau^1 \geq R_\tau.$$

Suppose that $\eta_\tau^1 \geq \eta_{\tau-1}^1$. Then from the second condition of b) follows that $R_\tau \geq M_{\tau-1}^1$. This means that $M_\tau^1 \geq Q_1(\tilde{\eta})$. If $\eta_\tau^1 < \eta_{\tau-1}^1$, then $\eta_\tau^1 < \overline{\eta}$ and then $\gamma_\tau^1 > 1$.

From $M_{\tau-1}^1 \geq Q_1(\tilde{\eta})$, we get $M_\tau^1 \geq Q_1(\tilde{\eta})$.

Thus $M_\tau^1 \geq Q_1(\tilde{\eta})$. Besides in the proof was shown that $\tilde{L}_\tau^1 \geq \pi_\tau$ and so considering $\tilde{L}_\tau^1 = 1 - \tilde{L}_\tau^2$, $\pi_\tau = 1 - \overline{L}_\tau^2$ we get $\tilde{L}_\tau^2 \leq \overline{L}_\tau^2$.

Now using Lemma 1 we obtain

$$\overline{L}_{\tau+1}^2 \leq \overline{L}_\tau^2.$$

From this considering $M_\tau^1 \geq Q_1(\tilde{\eta})$ after the similar considerations may be shown that $\tilde{L}_{\tau+1}^2 \leq \overline{L}_{\tau+1}^2$ and $M_{\tau+1}^1 \geq Q_1(\tilde{\eta})$.

Continuing this process we arrive at the proof of the theorem.

3. Consequence

If the conditions of Theorem 1 are satisfied then $M_t^2 \to 0$, and total consumption

$$\omega_t^1 \tilde{L}_t^1 + \omega_t^2 \tilde{L}_t^2$$

tends to $\overline{\omega}^1$.

Assume that the potential of the second sector is higher than the first sector. Then Theorem 1 is an example of the fact that under certain assumptions in the problem of maximization of the total consumption in contrast to the problems with the same wage rates we do not observe the replacement of the labor force to the second, more "better" production [2]. Moreover, in fact, the labor force in the second sector is upper bounded by the decreasing sequence. Also limit the total consumption is less than the similar limit in the problem with the same wages.

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