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[0,1]truncated Fréchet-G Generator of Distributions

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Abstract In this paper, we introduce a new family of continuous distributions based on [0,1] truncated Fréchet distribution. [0,1] truncated Fréchet Generalized Gamma distribution is discussed as special cases. The cumulative distribution function, the rth moment, the mean, the variance, the skewness, the kurtosis, the mode, the median, the characteristic function, the reliability function and the hazard rate function are obtained for the distributions under consideration. It is well known that an item fails when a stress to which it is subjected exceeds the corresponding strength. In this sense, strength can be viewed as "resistance to failure". Good design practice is such that the strength is always greater than the expected stress. The safety factor can be defined in terms of strength and stress as strength/ stress. So, the [0,1] TFGG strength-stress model with different parameters will be derived here. The Shannon entropy and Relative entropy will be derived also.

Keywords [0,1] TFGG, Stress-strength model, Shannon entropy, Relative entropy

1. Introduction

Here, we proposed a distribution with the hope it will attract wider applicability in other fields. The generalization which is motivated by the work of Eugene et al. will be our guide. Eugene et al. (2002) [2] defined the beta G distribution from a quite arbitrary cumulative distribution function (cdf), G(x) by

$$F(x) = (1/\beta(a,b)) \int_0^{G(x)} w^{a-1} (1-w)^{b-1}\, dw \qquad (1)$$

where a > 0 and b > 0 are two additional parameters whose role is to introduce skewness and to vary tail weight and $\beta(a,b) = \int_0^1 w^{a-1} (1-w)^{b-1}\, dw$ is the beta function. The class of distributions (1) has an increased attention after the works by Eugene et al. (2002) [2] and Jones (2004) [5]. Application of $X = G^{-1}(V)$ to the random variable V following a beta distribution with parameters a and b, V ∼ B(a, b) say, yields X with cdf (1). Eugene et al. (2002) [2] defined the beta normal (BN) distribution by taking G(x) to be the cdf of the normal distribution and derived some of its first moments. General expressions for the moments of the BN distribution were derived (Gupta and Nadarajah, 2004 [4]). An extensive review of scientific literature on this subject is available in Abid and Hassan (2015) [1]. We can write (1) as,

$$F(x) = I_{G(x)}(a,b) \qquad (2)$$

Where, $I_y(a,b) = (1/B(a,b) \int_0^y w^{a-1} (1-w)^{b-1}\, dw$, denotes the incomplete beta function ratio, i.e., the cdf of the beta distribution with parameters a and b. For general a and b, we can express (2) in terms of the well-known hypergeometric function defined by,

$${}_2F_1(\alpha,\beta,\gamma;x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i\,(\beta)_i}{(\gamma)_i\, i!}\, x^i$$

Where $(\alpha)_i = \alpha(\alpha+1)\ldots(\alpha+i-1)$ denotes the ascending factorial. We obtain,

$$F(x) = \frac{G(x)^a}{a\,B(a,b)}\; {}_2F_1(a, 1-b, a+1; G(x))$$

The properties of the cdf, F(x) for any beta G distribution defined from a parent $G(x)$ in (1), could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of Gradshteyn and Ryzhik (2000) [3]. The probability density function (pdf) corresponding to (1) can be written in the form,

$$f(x) = \frac{1}{B(a,b)} G(x)^{a-1} \left(1 - G(x)\right)^{b-1} g(x) \qquad (3)$$

where $g(x) = \partial G(x)/\partial x$ is the pdf of the parent distribution.

Now, since the pdf and cdf of [0,1] truncated Fréchet distribution are respectively,

$$h(x) = \frac{ab}{e^{-a}}\, x^{-(b+1)}\, e^{-ax^{-b}} \qquad 0 < x < 1 \qquad (4)$$

$$H(x) = \frac{1}{e^{-a}}\, e^{-ax^{-b}} \qquad (5)$$

Graphs for some arbitrary parameters values of pdf and cdf are shown in figure (1) and figure (2) respectively,

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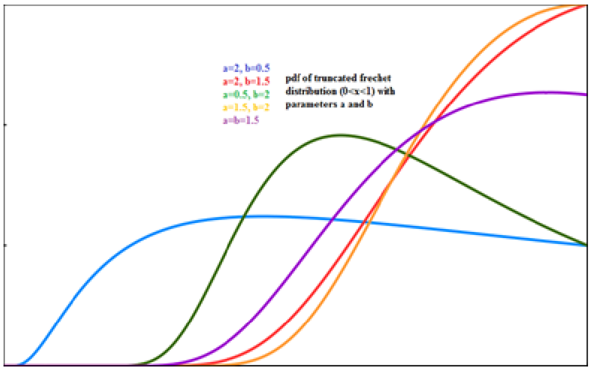

Figure (1) : pdf of (0,1)truncated Fréchet distribution

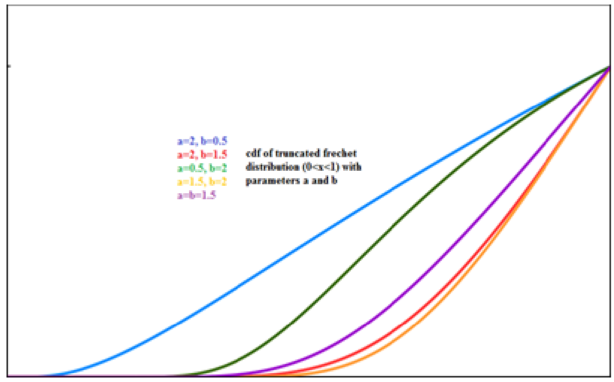

Figure (2) : cdf of (0,1)truncated Fréchet distribution

Now, Given two absolutely continuous cdfs, H and G, so that h and g are their corresponding pdfs. We suggest a new distribution F by composing H with G, so that $F(x) = H(G(x))$ is a CDF,

$$F(x) = \int_0^{G(x)} \frac{ab}{e^{-a}}\, t^{-(b+1)}\, e^{-at^{-b}}\, dt$$

$$= \frac{1}{e^{-a}}\, e^{-at^{-b}}\Big]_0^{G(x)} = \frac{1}{e^{-a}}\, e^{-aG(x)^{-b}} \qquad (6)$$

With pdf,

$$f(x) = \frac{\partial}{\partial x}\,\mathrm{F(x)} = \frac{\partial}{\partial x}\frac{e^{-aG(x)^{-b}}}{e^{-a}}$$

$$= \frac{ab}{e^{-a}}\, e^{-aG(x)^{-b}}\, \big(G(x)\big)^{-(b+1)}\, g(x) \qquad (7)$$

With $G(x)$ being a baseline distribution, we define in (6) and (7) above, a generalized class of distributions. We will name it the [0,1] truncated Fréchet -G distribution.

In the following section, we will assume that G is Generalized Gamma distribution.

2. [0,1] truncated Fréchet-generalized Gamma Distribution

Assume that $g(x) = \frac{\tau}{\alpha\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}}$ and $G(x) = \gamma\left[k, \left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)$ $(0 < x)$ are pdf and cdf of Generalized Gamma random variable [6, 8] respectively, then, by applying (6) and (7) above, we get the cdf pdf of [0,1] TFGG random variable as follows,

$$F(x) = \frac{1}{e^{-a}}\, e^{-a\left\{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)\right\}^{-b}} \quad , \; x \ge 0 \qquad (8)$$

$$f(x) = \frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)\right\}^{-(b+1)} e^{-a\left\{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)\right\}^{-b}} \;, \; x \ge 0 \qquad (9)$$

So, the reliability and hazard rate function are respectively

$$R(x) = 1 - F(x)$$

$$R(x) = 1 - \frac{e^{-a\left\{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)\right\}^{-b}}}{e^{-a}} = 1 - e^{-a\left[\left\{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)\right\}^{-b}-1\right]}$$

$$\lambda(x) = \frac{f(x)}{R(x)} = \frac{\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)\right\}^{-(b+1)} e^{-a\left\{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)\right\}^{-b}}}{1-e^{-a\left[\left\{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]/\Gamma(k)\right\}^{-b}-1\right]}}$$

The rth raw moment can be derived as follows,

$$E(x^r) = \int_0^{\infty} x^r \frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}} dx$$

Since, $e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^i \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-bi}$, then,

$$E(x^r) = \frac{ab\tau}{\alpha e^{-a}\Gamma(k)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^i \int_0^{\infty} x^r \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} dx$$

Since, $\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} = \left\{1-\left(1-\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)\right\}^{-(bi+b+1)}$

By using, $(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\Gamma(k)} z^j \quad |z|<1\ ,\ k>0$ and

$$(1-z)^b = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(b+1)}{\Gamma(b-m+1)} z^m \quad |z|<1, b>0 \tag{10}$$

We get,

$\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} = \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \sum_{m=0}^{j} \frac{(-1)^m}{m!} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^m$ and then,

$$E(x^r) = \frac{b\tau}{\alpha e^{-a}\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^{\infty} x^r \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^m dx$$

$let\ y = \left(\frac{x}{\alpha}\right)^{\tau} \Rightarrow x = \alpha\, y^{\frac{1}{\tau}} \Rightarrow dx = \frac{\alpha}{\tau} y^{\frac{1}{\tau}-1} dy$, then,

$$E(x^r) = \frac{b\tau}{\alpha e^{-a}\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^{\infty} \alpha^r y^{\frac{r}{\tau}} \left(y^{\frac{1}{\tau}}\right)^{\tau k-1} e^{-y} \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^m \frac{\alpha}{\tau} y^{\frac{1}{\tau}-1} dy$$

$$= \frac{b\alpha^r e^a}{\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^{\infty} y^{\frac{r}{\tau}} y^{k-\frac{1}{\tau}} e^{-y} y^{\frac{1}{\tau}-1} \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^m dy$$

$$= \frac{b\alpha^r e^a}{\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^{\infty} y^{\frac{r}{\tau}+k-1} e^{-y} \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^m dy$$

By using [6],

$$I\left(k+\frac{r}{\tau},m\right) = \int_0^{\infty} y^{\frac{r}{\tau}+k-1} e^{-y} \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^m dy$$

$$= k^{-m}\Gamma\left(r/\tau + k(m+1)\right) F_A^{(m)}(r/\tau + k(m+1); k,\dots,k; k+1,\dots,k+1; -1,\dots-1)\Big) \tag{11}$$

Where, $F_A^{(m)}$ is the Lauricella function of type A, then,

$$E(x^r) = \frac{b\alpha^r e^a}{\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I\left(k+\frac{r}{\tau},m\right) \tag{12}$$

And then, the characteristic function is

$$Q_x(t) = E(e^{ixt}) = \left(\sum_{r=0}^{\infty} \frac{(ixt)^r}{r!}\right) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r)$$

$$Q_x(t) = \frac{be^a}{\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \frac{(it\alpha)^r}{r!} I\left(k+\frac{r}{\tau},m\right)$$

So, the mean μ and variance σ^2 of the of [0,1] TFGG random variable are,

$$\mu = E(x) = \frac{\alpha b e^a}{\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I\left(k+\frac{1}{\tau},m\right) \tag{13}$$

$$\sigma^2 = E(x^2) - (Ex)^2$$

$$\sigma^2 = \frac{\alpha^2 b e^a}{\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I\left(k+\frac{2}{\tau},m\right) - \left\{\frac{\alpha b e^a}{\Gamma(k)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I\left(k+\frac{1}{\tau},m\right)\right\}^2 \tag{14}$$

Since, $F(x) = \frac{e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}}{e^{-a}} = \frac{1}{2}$, then By solving the nonlinear equation

$\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]-\Gamma(k)\,\left(1+\frac{\ln(2)}{a}\right)^{\frac{-1}{b}}=0$, we obtain the median of X.

The skewness of [0,1] TFGG random variable will be, $sk=\frac{\mu_3}{\mu_2^{3/2}}=\frac{Ex^3-3\mu Ex^2+2\mu^3}{(\sigma^2)^{3/2}}$, so

$$sk=\frac{\left\{\begin{array}{c}\frac{\alpha^3 b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{3}{\tau},m\right)-3\\ \left\{\frac{\alpha b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}\\ \left\{\frac{\alpha^2 b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{2}{\tau},m\right)\right\}+2\\ \left\{\frac{\alpha b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}^3\end{array}\right\}}{\left\{\begin{array}{c}\frac{\alpha^2 b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{2}{\tau},m\right)-\\ \left\{\frac{\alpha b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}^2\end{array}\right\}^{3/2}} \tag{15}$$

Also, the kurtosis is, $kr=\frac{\mu_4}{\mu_2^2}-3=\frac{Ex^4-4\mu Ex^3+6\mu^2Ex^2-3\mu^4}{(\sigma^2)^2}-3$

$$=\frac{\left\{\begin{array}{c}\frac{\alpha^4 b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{4}{\tau},m\right)-4\\ \left\{\frac{\alpha b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}\\ \left\{\frac{\alpha^3 b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{3}{\tau},m\right)\right\}+6\\ \left\{\frac{\alpha b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}^2\\ \left\{\frac{\alpha^2 b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{2}{\tau},m\right)\right\}-3\\ \left\{\frac{\alpha b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}^4\end{array}\right\}}{\left\{\begin{array}{c}\frac{\alpha^2 b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{2}{\tau},m\right)-\\ \left\{\frac{\alpha b e^a}{\Gamma(k)}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}^2\end{array}\right\}^2}-3$$

$$=\frac{\left\{\begin{array}{c}\frac{(\Gamma(k))^3}{b^3e^{3a}}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{4}{\tau},m\right)-4\\ \frac{(\Gamma(k))^2}{b^2e^{2a}}\left\{\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}\\ \left\{\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{3}{\tau},m\right)\right\}+6\frac{\Gamma(k)}{be^a}\\ \left\{\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(\left(k+\frac{1}{\tau},m\right)\right)\right\}^2\\ \left\{\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{2}{\tau},m\right)\right\}-3\\ \left\{\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}^4\end{array}\right\}}{\left\{\begin{array}{c}\frac{\Gamma(k)}{be^a}\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{2}{\tau},m\right)\\ -\left\{\sum_{i,j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{1}{\tau},m\right)\right\}^2\end{array}\right\}^2}-3 \tag{16}$$

The quantile function x_q of [0,1] TFGG random variable can be obtained by solving the following nonlinear equation as,

$$\Rightarrow\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}-\left(1-\frac{\ln(q)}{a}\right)^{\frac{-1}{b}}=0\text{ , since }q=P\left(x<x_q\right)=F\left(x_q\right)=\frac{e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}}{e^{-a}}\text{ , }0<q<1\text{ , }x_q>0$$

So by using inverse transform method we can generate [0,1] TFGG random variable as follows,

$\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)} - \left(1 - \frac{\ln(u)}{a}\right)^{\frac{-1}{b}} = 0$, Where U is uniformly distributed random number in the unit interval [0,1].

2.1. Shannon and Relative Entropies

An entropy of a random variable X is a measure of variation of the uncertainty. The Shannon entropy of $[0,1]$ $\mathrm{TFGG}(a,b,\theta)$ random variable X can be found as follows,

$$H = -\int_0^{\infty} f(x)\ \ln\big(f(x)\big)\ dx$$

$$= -\int_0^{\infty} f(x) \ln\left[\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}\right] dx$$

$$= \int_0^{\infty} f(x)\left[-\ln\left(\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\right) - (\tau k-1)\ln\left(\frac{x}{\alpha}\right) + \left(\frac{x}{\alpha}\right)^{\tau} + (b+1)\ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\} + a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}\right] dx$$

$$= -\ln\left(\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\right) - (\tau k-1)E\left(\ln\left(\frac{x}{\alpha}\right)\right) + E\left(\left(\frac{x}{\alpha}\right)^{\tau}\right) + (b+1)E\left(\ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}\right) + aE\left(\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}\right)$$

Let, $I_1 = -(\tau k-1)E\left(\ln\left(\frac{x}{\alpha}\right)\right)$, $I_1 = -(\tau k-1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\int_0^{\infty}\ln\left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}} dx$

Since, $e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ a^i \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-bi}$, then

$$I_1 = -(\tau k-1)\frac{b\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ a^{i+1}\int_0^{\infty}\ln\left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} dx$$

By using equation (10) we get,

$\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^m}{m!}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}\left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^m$, and then,

$$I_1 = -(\tau k-1)\frac{b\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$
$$\int_0^{\infty}\ln\left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^m dx$$

by using expansion series incomplete Gamma function

$$\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)} = \frac{\left(\frac{x}{\alpha}\right)^{\tau k}}{\Gamma(k)}\sum_{d=0}^{\infty}\frac{\left(-\left(\frac{x}{\alpha}\right)^{\tau}\right)^d}{(k+d)d!} \tag{17}$$

We get,

$$I_1 = -(\tau k-1)\frac{b\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$
$$\int_0^{\infty}\ln\left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left[\frac{\left(\frac{x}{\alpha}\right)^{\tau k}}{\Gamma(k)}\sum_{d=0}^{\infty}\frac{\left(-\left(\frac{x}{\alpha}\right)^{\tau}\right)^d}{(k+d)d!}\right]^m dx$$

$let\ \ y = \left(\frac{x}{\alpha}\right)^{\tau} \Rightarrow x = \alpha y^{\frac{1}{\tau}} \Rightarrow dx = \frac{\alpha}{\tau}\ y^{\frac{1}{\tau}-1}\ dy$, then,

$$I_1 = -(\tau k-1)\frac{b\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma(b(i+1)+1)}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$
$$\int_0^{\infty}\ln\left(y^{\frac{1}{\tau}}\right)\ y^{k-\frac{1}{\tau}}\ e^{-y}\left[\frac{y^k}{\Gamma(k)}\sum_{d=0}^{\infty}\frac{(-y)^d}{(k+d)d!}\right]^m\ \frac{\alpha}{\tau}\ y^{\frac{1}{\tau}-1}\ dy$$

$$= \frac{-(\tau k-1)}{\tau} \frac{b e^{a}}{(\Gamma(k))^{m+1}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} \frac{(-1)^{d_1+\cdots+d_m}}{(k+d_1)\dots(k+d_m)d_1!\dots d_m!} \int_0^{\infty} \ln(y)\, y^{k(m+1)+d_1+\cdots+d_m-1} e^{-y}\, dy$$

$$= \frac{-(\tau k-1)}{\tau} \frac{b e^{a}}{(\Gamma(k))^{m+1}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\sum_{d_1=0}^{\infty} \cdots \sum_{d_m=0}^{\infty} \frac{(-1)^{d_1+\cdots+d_m}}{(k+d_1)\dots(k+d_m)d_1!\dots d_m!} \Gamma(k(m+1)+d_1+\cdots+d_m)$$

$$\{\Psi(k(m+1)+d_1+\cdots+d_m)-\ln(1)\}$$

$$= -\frac{(\tau k-1)}{\tau} \frac{b e^{a}}{(\Gamma(k))^{m+1}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$k^{-m}\, \Gamma\big(k(m+1)\big)\, \Psi\big(k(m+1)\big)\, F_A^{(m)}(k(m+1); k, \dots k; k+1, \dots, k+1; -1, \dots, -1)$$

Now, since the Lauricella function of type A, $F_A^{(n)}(a; b_1, \dots, b_n; C_1, \dots, C_n; x_1, \dots, x_n)$ can be defined as $\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(C_1)_{m_1}\dots(C_n)_{m_n}} \frac{x_1^{m_1}\dots x_n^{m_n}}{m_1!\dots m_n!}$, where $(a)_i$ is the ascending factorial defined by $(a)_i = a(a+1)\dots(a+i-1)$, (with the convention that $(a)_0 = 1$).

And, $I_2 = E\left(\left(\frac{x}{\alpha}\right)^{\tau}\right)$

$$= \frac{ab\tau}{\alpha e^{-a}\Gamma(k)} \int_0^{\infty} \left(\frac{x}{\alpha}\right)^{\tau} \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}} dx$$

$$= \frac{ab\tau}{\alpha e^{-a}\Gamma(k)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^i \int_0^{\infty} \left(\frac{x}{\alpha}\right)^{\tau} \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} dx$$

by using equation (10) we get,

$$I_2 = \frac{ab\tau}{\alpha e^{-a}\Gamma(k)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^{\infty} \left(\frac{x}{\alpha}\right)^{\tau} \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^{m} dx$$

let $y = \left(\frac{x}{\alpha}\right)^{\tau} \Rightarrow x = \alpha y^{\frac{1}{\tau}} \Rightarrow dx = \frac{\alpha}{\tau} y^{\frac{1}{\tau}-1} dy$, then,

$$I_2 = \frac{ab\tau}{\alpha e^{-a}\Gamma(k)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^{\infty} y\, y^{k-\frac{1}{\tau}} e^{-y} \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^{m} \frac{\alpha}{\tau} y^{\frac{1}{\tau}-1} dy$$

$$= \frac{ab\, e^{a}}{\Gamma(k)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^{\infty} y^{k} e^{-y} \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^{m} dy$$

By using equation (11) we get,

$$= \frac{b e^{a}}{\Gamma(k)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I(k+1, m)$$

and,

$$I_3 = (b+1)E\left(\ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}\right)$$

$$= (b+1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)} \int_0^{\infty} \ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\} \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}} dx$$

$$= (b+1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^i \int_0^{\infty} \ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\} \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} dx$$

By using equation (10) we get,

$$I_3 = (b+1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{i+m}}{i!m!} a^{i} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\int_0^{\infty} \ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\} \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^{m} dx$$

By using expansion series of incomplete Gamma function, we get

$$\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]=\left(\frac{x}{\alpha}\right)^{\tau k}\Gamma(k)\,e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}$$

$$I_3=(b+1)\frac{b\tau e^{a}}{\alpha\Gamma(k)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^m}{m!}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\int_0^{\infty}\ln\left\{\left(\frac{x}{\alpha}\right)^{\tau k}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}\right\}\left(\frac{x}{\alpha}\right)^{\tau k-1}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left[\left(\frac{x}{\alpha}\right)^{\tau k}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{(k+d)!}\right]^{m}dx$$

$$=(b+1)\frac{b\tau e^{a}}{\alpha\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{1}{(k+d_1)!\ldots(k+d_m)!}\int_0^{\infty}\ln\left\{\left(\frac{x}{\alpha}\right)^{\tau k}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}\right\}\left(\frac{x}{\alpha}\right)^{\tau k-1}\left(\frac{x}{\alpha}\right)^{\tau km}$$

$$\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)}e^{-(m+1)\left(\frac{x}{\alpha}\right)^{\tau}}dx$$

Since, $\ln\left\{\left(\frac{x}{\alpha}\right)^{\tau k}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}\right\}=\tau k\ln\left(\frac{x}{\alpha}\right)-\left(\frac{x}{\alpha}\right)^{\tau}+\ln\left(\sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}\right)$

$$I_{31}=\tau k\int_0^{\infty}\ln\left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1}\left(\frac{x}{\alpha}\right)^{\tau km}\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)}e^{-(m+1)\left(\frac{x}{\alpha}\right)^{\tau}}dx$$

$$let\ \ y=\left(\frac{x}{\alpha}\right)^{\tau}\Rightarrow x=\alpha y^{\frac{1}{\tau}}\ \ \Rightarrow dx=\frac{\alpha}{\tau}\,y^{\frac{1}{\tau}-1}dy$$

$$I_{31}=\tau k\int_0^{\infty}\ln(y)^{\frac{1}{\tau}}\,y^{k-\frac{1}{\tau}}y^{km}\,y^{d_1+\cdots+d_m}e^{-(m+1)y}\frac{\alpha}{\tau}\,y^{\frac{1}{\tau}-1}dy$$

$$=\frac{\alpha k}{\tau}\int_0^{\infty}\ln(y)\,y^{k(m+1)+d_1+\cdots+d_m-1}e^{-(m+1)y}dy$$

$$=\frac{\alpha k}{\tau}(m+1)^{-(k(m+1)+d_1+\cdots+d_m)}\Gamma(k(m+1)+d_1+\cdots+d_m)$$

$$\{\Psi(k(m+1)+d_1+\cdots+d_m)-\ln(m+1)\}$$

$$I_{32}=-\int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau}\left(\frac{x}{\alpha}\right)^{\tau k-1}\left(\frac{x}{\alpha}\right)^{\tau km}\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)}e^{-(m+1)\left(\frac{x}{\alpha}\right)^{\tau}}dx$$

$$let\ \ y=\left(\frac{x}{\alpha}\right)^{\tau}\ \Rightarrow x=\alpha\,y^{\frac{1}{\tau}}\ \Rightarrow dx=\frac{\alpha}{\tau}y^{\frac{1}{\tau}-1}dy$$

$$I_{32}=-\frac{\alpha}{\tau}\int_0^{\infty}y^{k(m+1)+d_1+\cdots+d_m}\,e^{-(m+1)y}dy$$

$$=-\frac{\alpha}{\tau}\frac{\Gamma(k(m+1)+d_1+\cdots+d_m+1)}{(m+1)^{k(m+1)+d_1+\cdots d_m+1}}$$

$$I_{33}=\int_0^{\infty}\ln\left(\sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1}\left(\frac{x}{\alpha}\right)^{\tau km}\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)}e^{-(m+1)\left(\frac{x}{\alpha}\right)^{\tau}}dx$$

$$I_{33}=\int_0^{\infty}\ln\left(\sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}\right)\left(\frac{x}{\alpha}\right)^{\tau k(m+1)+\tau(d_1+\cdots+d_m)-1}e^{-(m+1)\left(\frac{x}{\alpha}\right)^{\tau}}dx$$

$=\eta(\tau,\alpha,k,m,d_1,\ldots,d_m)$ new function

$$I_3=(b+1)\frac{be^{a}}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{1}{(k+d_1)!\ldots(k+d_m)!}\left\{\begin{array}{r}k\frac{\Gamma(k(m+1)+d_1+\cdots+d_m)}{(m+1)^{(k(m+1)+d_1+\cdots+d_m)}}\{\Psi(k(m+1)+d_1+\cdots+d_m)-\ln(m+1)\}\\-\frac{\Gamma(k(m+1)+d_1+\cdots+d_m+1)}{(m+1)^{k(m+1)+d_1+\cdots+d_m+1}}+\frac{\tau}{\alpha}\eta(\tau,\alpha,k,m,d_1,\ldots,d_m)\end{array}\right\}$$

And, $I_4=\ aE\left(\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}\right)$

$$= \frac{a^2 b\tau}{\alpha e^{-a}\Gamma(k)} \int_0^\infty \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-b} \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-b}} dx$$

$$= \frac{b\tau e^a}{\alpha\Gamma(k)} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \int_0^\infty \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-(2b+1)} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-bi} dx$$

$$= \frac{b\tau e^a}{\alpha\Gamma(k)} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \int_0^\infty \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-(bi+2b+1)} dx$$

By using (10) we get,

$$I_4 = \frac{b\tau e^a}{\alpha\Gamma(k)} \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^m}{m!} \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^\infty \left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau} \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right)^m dx$$

$let\ y = \left(\frac{x}{\alpha}\right)^\tau \ \Rightarrow x = \alpha\, y^{\frac{1}{\tau}} \ \Rightarrow dx = \frac{\alpha}{\tau}\, y^{\frac{1}{\tau}-1}\, dy$, then,

$$I_4 = \frac{b\tau e^a}{\alpha\Gamma(k)} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^{i+2} \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^\infty y^{k-\frac{1}{\tau}} e^{-y} \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^m \frac{\alpha}{\tau} y^{\frac{1}{\tau}-1} dy$$

$$= \frac{be^a}{\Gamma(k)} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^{i+2} \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \int_0^\infty y^{k-1} e^{-y} \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^m dy$$

By using (11) we get,

$$= \frac{be^a}{\Gamma(k)} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^{i+2} \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I(k,m)$$

then

$$H = \ln\left(\frac{\alpha e^{-a}\Gamma(k)}{ab\tau}\right) - \frac{(\tau k-1)}{\tau} \frac{be^a}{(\Gamma(k))^{m+1}} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$k^{-m}\, \Gamma\big(k(m+1)\big)\, \Psi\big(k(m+1)\big) F_A^{(m)}(k(m+1); k,\dots,k; k+1,\dots,k+1; -1,\dots,-1) +$$

$$\frac{be^a}{\Gamma(k)} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I(k+1,m) + (b+1)\frac{be^a}{\Gamma(k)}$$

$$\sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \sum_{d_1=0}^\infty \cdots \sum_{d_m=0}^\infty \frac{1}{(k+d_1)!\dots(k+d_m)!}$$

$$\left\{\begin{matrix} k\,\frac{\Gamma(k(m+1)+d_1+\cdots+d_m)}{(m+1)^{(k(m+1)+d_1+\cdots+d_m)}}\{\Psi(k(m+1)+d_1+\cdots+d_m) - \ln(m+1)\} \\ -\frac{\Gamma(k(m+1)+d_1+\cdots+d_m+1)}{(m+1)^{k(m+1)+d_1+\cdots+d_m+1}} + \frac{\tau}{\alpha}\eta(\tau,\alpha,k,m,d_1,\dots,d_m) \end{matrix}\right\} +$$

$$\frac{be^a}{\Gamma(k)} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^{i+2} \frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I(k,m) \tag{18}$$

The relative entropy (or the Kullback–Leibler divergence) is a measure of the difference between two probability distributions F_1 and F_2. It is not symmetric in F_1 and F_2. In applications, F_1 typically represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution, while F_2 typically represents a theory, model, description, or approximation of F_1. Specifically, the Kullback–Leibler divergence of F_2 from F_1, denoted $D_{KL}(F_1 \parallel F_2)$, is a measure of the information gained when one revises ones beliefs from the prior probability distribution F_2 to the posterior probability distribution F_1. More exactly, it is the amount of information that is *lost* when F_2 is used to approximate F_1, defined operationally as the expected extra number of bits required to code samples from F_1 using a code optimized for F_2 rather than the code optimized for F_1.

The relative entropy $Dkl(F\|F^*)$ for a random variable [0,1] TFGG(a,b,α,τ,k) can be found as follows,

$$\frac{f(x)}{f^*(x)} = \frac{\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-b}}}{\frac{a_1 b_1 \tau_1}{\alpha_1 e^{-a_1}\Gamma(k_1)}\left(\frac{x}{\alpha_1}\right)^{\tau_1 k_1-1} e^{-\left(\frac{x}{\alpha_1}\right)^{\tau_1}} \left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-(b_1+1)} e^{-a_1\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}}}$$

$$Dkl=\int_0^\infty f(x)\ln\left(\frac{\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k-1}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}}{\frac{a_1b_1\tau_1}{\alpha_1e^{-a_1}\Gamma(k_1)}\left(\frac{x}{\alpha_1}\right)^{\tau_1k_1-1}e^{-\left(\frac{x}{\alpha_1}\right)^{\tau_1}}\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-(b_1+1)}e^{-a_1\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}}}\right)dx$$

$$=\int_0^\infty f(x)\left[\begin{array}{c}\ln\left(\frac{ab\tau\,\alpha_1e^{-a_1}\Gamma(k_1)}{a_1b_1\tau_1\alpha e^{-a}\Gamma(k)}\right)+(\tau k-1)\ln\left(\frac{x}{\alpha}\right)-\left(\frac{x}{\alpha}\right)^{\tau}-(b+1)\ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}-\\ a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}-(\tau_1k_1-1)\ln\left(\frac{x}{\alpha_1}\right)+\left(\frac{x}{\alpha_1}\right)^{\tau_1}+(b_1+1)\ln\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}\\ +a_1\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}\end{array}\right]dx$$

Let, $I_1=(\tau k-1)\int_0^\infty\ln\left(\frac{x}{\alpha}\right)f(x)\,dx$

$$I_1=(\tau k-1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\int_0^\infty\ln\left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}dx$$

$$=\frac{(\tau k-1)}{\tau}\frac{be^{a}}{(\Gamma(k))^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}k^{-m}$$
$$\Gamma\big(k(m+1)\big)\,\Psi\big(k(m+1)\big)\,F_A^{(m)}(k(m+1);k,\dots,k;k+1,\dots,k+1;-1,\dots,-1)$$

And, $I_2=-\int_0^\infty\left(\frac{x}{\alpha}\right)^{\tau}f(x)\,dx$

$$I_2=-\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\int_0^\infty\left(\frac{x}{\alpha}\right)^{\tau}\left(\frac{x}{\alpha}\right)^{\tau k-1}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}dx$$

$$=-\frac{be^{a}}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I(k+1,m)$$

And, $I_3=-(b+1)\int_0^\infty\ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}f(x)\,dx$

$$I_3=-(b+1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\int_0^\infty\ln\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}\left(\frac{x}{\alpha}\right)^{\tau k-1}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}dx$$

$$=-(b+1)\frac{be^{a}}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\sum_{d_1=0}^{\infty}\dots\sum_{d_m=0}^{\infty}\frac{1}{(k+d_1)!\dots(k+d_m)!}\left\{\begin{array}{c}k\frac{\Gamma(k(m+1)+d_1+\dots+d_m)}{(m+1)^{k(m+1)+d_1+\dots+d_m}}\{\Psi(k(m+1)+d_1+\dots+d_m)-\ln(m+1)\}\\-\frac{\Gamma(k(m+1)+d_1+\dots+d_m+1)}{(m+1)^{k(m+1)+d_1+\dots+d_m+1}}+\frac{\tau}{\alpha}\eta(\tau,\alpha,k,m,d_1,\dots,d_m)\end{array}\right\}$$

And, $I_4=-a\int_0^\infty\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}f(x)\,dx$

$$I_4=-\frac{a^2b\tau}{\alpha e^{-a}\Gamma(k)}\int_0^\infty\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}\left(\frac{x}{\alpha}\right)^{\tau k-1}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}dx$$

$$=-\frac{be^{a}}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I(k,m)$$

And, $I_5=-(\tau_1k_1-1)\int_0^\infty\ln\left(\frac{x}{\alpha_1}\right)f(x)\,dx$

$$I_5 = -(\tau_1 k_1 - 1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\int_0^\infty \ln\left(\frac{x}{\alpha_1}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-(b+1)} e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-b}} dx$$

$$= -(\tau_1 k_1 - 1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^\infty \frac{(-1)^i}{i!} a^i \int_0^\infty \ln\left(\frac{x}{\alpha_1}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} dx$$

By using equation (10) we get,

$$\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} = \sum_{j=0}^\infty \sum_{m=0}^j \frac{(-1)^m}{m!}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right)^m$$, then,

$$I_5 = -(\tau_1 k_1 - 1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^i \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\int_0^\infty \ln\left(\frac{x}{\alpha_1}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau}\left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^\tau\right]}{\Gamma(k)}\right)^m dx$$

By using equation (17) we get,

$$I_5 = -(\tau_1 k_1 - 1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^i \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\int_0^\infty \ln\left(\frac{x}{\alpha_1}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau}\left[\frac{1}{\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k}\sum_{d=0}^\infty \frac{\left(-\left(\frac{x}{\alpha}\right)^\tau\right)^d}{(k+d)d!}\right]^m dx$$

By applying of equation in section 0.314 of Gradshteyn and Ryzhik (2000) [3] for power series raised to power, we obtain for any m positive integer

$$\left[\sum_{d=0}^\infty a_d \left(\frac{x}{\alpha}\right)^{\tau d}\right]^m = \sum_{d=0}^\infty C_{m,d}\left(\frac{x}{\alpha}\right)^{\tau d}$$

Where the coefficients $C_{m,d}(for\ d = 1,2,\dots)$ satisfy the recurrence relation

$C_{m,d} = (da_0)^{-1}\sum_{p=1}^d (mp - d + p)\ a_p\ C_{m,d-p}\ , C_{m,0} = a_0^m\ \ and\ \ a_p = \frac{(-1)^p}{(k+p)p!}$.

$$I_5 = -(\tau_1 k_1 - 1)\frac{ab\tau}{\alpha e^{-a}(\Gamma(k))^{m+1}}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^i \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\sum_{d=0}^\infty C_{m,d}\int_0^\infty \ln\left(\frac{x}{\alpha_1}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^\tau}\left(\frac{x}{\alpha}\right)^{\tau km}\left(\frac{x}{\alpha}\right)^{\tau d} dx$$

$let\ \ y = \left(\frac{x}{\alpha}\right)^\tau \ \Rightarrow x = \alpha y^{\frac{1}{\tau}} \quad \Rightarrow dx = \frac{\alpha}{\tau}\ y^{\frac{1}{\tau}-1}\ dy$

$$I_5 = -(\tau_1 k_1 - 1)\frac{ab\tau}{\alpha e^{-a}(\Gamma(k))^{m+1}}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^i \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\sum_{d=0}^\infty C_{m,d}\int_0^\infty \ln\left(\frac{\alpha\, y^{\frac{1}{\tau}}}{\alpha_1}\right)\left(y^{\frac{1}{\tau}}\right)^{\tau k+\tau km+\tau d-1} e^{-y}\ \frac{\alpha}{\tau}\ y^{\frac{1}{\tau}-1}\ dy$$

$$= -(\tau_1 k_1 - 1)\frac{be^a}{(\Gamma(k))^{m+1}}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{m=0}^j \frac{(-1)^{i+m}}{i!m!} a^{i+1} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$

$$\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\sum_{d=0}^\infty C_{m,d}\int_0^\infty \left[\ln\left(\frac{\alpha}{\alpha_1}\right) + \frac{1}{\tau}\ln(y)\right] y^{k(m+1)+d-1} e^{-y}\ dy$$

$$I_{51} = \int_0^\infty \ln\left(\frac{\alpha}{\alpha_1}\right) y^{k(m+1)+d-1} e^{-y}\ dy = \ln\left(\frac{\alpha}{\alpha_1}\right)\Gamma(k(m+1)+d)$$

$$I_{52} = \int_0^\infty \frac{1}{\tau}\ln(y)\ y^{k(m+1)+d-1} e^{-y}\ dy$$

$$= \frac{1}{\tau}\int_0^\infty \ln(y)\ y^{k(m+1)+d-1} e^{-y}\ dy$$

Since $\int_0^\infty x^{s-1}\ e^{-mx}\ \ln(x)\, dx = m^{-s}\,\Gamma(s)\,\{\Psi(s) - \ln(m)\}$, then

$I_{52} = \frac{1}{\tau}\,\Gamma(k(m+1)+d)\,\Psi(k(m+1)+d)$, so

$$I_5 = -(\tau_1 k_1 - 1)\ \frac{b e^{a}}{(\Gamma(k))^{m+1}}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\sum_{d=0}^{\infty} C_{m,d}\ \left\{\ln\left(\frac{\alpha}{\alpha_1}\right)\ \Gamma(k(m+1)+d) + \frac{1}{\tau}\ \Gamma(k(m+1)+d)\Psi(k(m+1)+d)\right\}$$

And, $I_6 = \int_0^{\infty}\left(\frac{x}{\alpha_1}\right)^{\tau_1}\ f(x)\ dx$

$$I_6 = \frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\ \int_0^{\infty}\left(\frac{x}{\alpha_1}\right)^{\tau_1}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}\ e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}\ dx$$

$$= \frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\ a^{i}\int_0^{\infty}\frac{x^{\tau_1}}{\alpha_1^{\tau_1}}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-bi}\ dx$$

$$= \frac{ab\tau\, e^{a}}{\alpha\alpha_1^{\tau_1}\Gamma(k)}\ \sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\ a^{i}\ \int_0^{\infty} x^{\tau_1}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)}\ dx$$

By using equation (10) we get,

$$I_6 = \frac{ab\tau\, e^{a}}{\alpha\alpha_1^{\tau_1}\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}\ \int_0^{\infty} x^{\tau_1}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^{m}\ dx$$

$let\ \ y = \left(\frac{x}{\alpha}\right)^{\tau} \Rightarrow x = \alpha\, y^{\frac{1}{\tau}}\ \ \Rightarrow dx = \frac{\alpha}{\tau}\ y^{\frac{1}{\tau}-1}\ dy$, then,

$$I_6 = \frac{\tau b\, e^{a}}{\alpha\alpha_1^{\tau_1}\Gamma(k)}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}\int_0^{\infty}\alpha^{\tau_1}\ y^{\frac{\tau_1}{\tau}}y^{k-\frac{1}{\tau}}e^{-y}\ \frac{\alpha}{\tau}\ y^{\frac{1}{\tau}-1}\ \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^{m}\ dy$$

$$= \frac{(\alpha/\alpha_1)^{\tau_1}\ b e^{a}}{\Gamma(k)}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}\int_0^{\infty} y^{\frac{\tau_1}{\tau}+k-1}\ e^{-y}\ \left(\frac{\gamma[k,y]}{\Gamma(k)}\right)^{m}\ dy$$

By using (11) we get,

$$I_6 = \frac{(\alpha/\alpha_1)^{\tau_1}\ b e^{a}}{\Gamma(k)}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)} I\left(k+\frac{\tau_1}{\tau}, m\right)$$

Since, $I_7 = (b_1+1)\int_0^{\infty}\ln\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}\ f(x)\ dx$

$$I_7 = (b_1+1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\int_0^{\infty}\ln\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}\ e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}\ dx$$

$$= (b_1+1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\ \sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\ a^{i}\ \int_0^{\infty}\ln\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-bi}\ dx$$

$$= (b_1+1)\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}a^{i}\int_0^{\infty}\ln\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)}\ dx$$

By using equation (10) we get,

$$I_7 = (b_1+1)\frac{b\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\int_0^{\infty}\ln\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^{m}\ dx$$

By using expansion series of incomplete Gamma function

$$I_7 = (b_1+1)\frac{b\tau\, e^{a}}{\alpha\Gamma(k)}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\int_0^{\infty}\ln\left\{\left(\frac{x}{\alpha_1}\right)^{\tau_1 k_1}\ e^{-\left(\frac{x}{\alpha_1}\right)^{\tau_1}}\ \sum_{s=0}^{\infty}\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1 s}}{\Gamma(k_1+s+1)}\right\}\ \left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left[\left(\frac{x}{\alpha}\right)^{\tau k}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \sum_{d=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{\tau d}}{\Gamma(k+d+1)}\right]^{m}\ dx$$

$$= (b_1+1)\,\frac{b\tau e^{a}}{\alpha\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\,a^{i+1}\,\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\,\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{1}{(k+d_1)!\ldots(k+d_m)!}\int_0^{\infty}\ln\left\{\left(\frac{x}{\alpha_1}\right)^{\tau_1 k_1} e^{-\left(\frac{x}{\alpha_1}\right)^{\tau_1}}\sum_{s=0}^{\infty}\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1 s}}{\Gamma(k_1+s+1)}\right\}\left(\frac{x}{\alpha}\right)^{\tau k-1}$$

$$e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left(\frac{x}{\alpha}\right)^{\tau k m} e^{-m\left(\frac{x}{\alpha}\right)^{\tau}}\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)}dx$$

Since, $\ln\left\{\left(\frac{x}{\alpha_1}\right)^{\tau_1 k_1} e^{-\left(\frac{x}{\alpha_1}\right)^{\tau_1}}\sum_{s=0}^{\infty}\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1 s}}{\Gamma(k_1+s+1)}\right\} = \tau_1 k_1\ln\left(\frac{x}{\alpha_1}\right) - \left(\frac{x}{\alpha_1}\right)^{\tau_1} + \ln\left(\sum_{s=0}^{\infty}\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1 s}}{\Gamma(k_1+s+1)}\right)$, then,

$$I_7 = (b_1+1)\frac{b\tau e^{a}}{\alpha\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\,a^{i+1}\,\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\,\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$

$$\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{1}{(k+d_1)!\ldots(k+d_m)!}\int_0^{\infty}\left\{\begin{array}{c}\tau_1 k_1\ln\left(\frac{x}{\alpha_1}\right)-\left(\frac{x}{\alpha_1}\right)^{\tau_1}+\\ \ln\left(\sum_{s=0}^{\infty}\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1 s}}{\Gamma(k_1+s+1)}\right)\end{array}\right\}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left(\frac{x}{\alpha}\right)^{\tau k m}$$

$$e^{-m\left(\frac{x}{\alpha}\right)^{\tau}}\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)}dx$$

$$I_{71} = \tau_1 k_1\int_0^{\infty}\ln\left(\frac{x}{\alpha_1}\right)\left(\frac{x}{\alpha}\right)^{\tau k-1}\left(\frac{x}{\alpha}\right)^{\tau k m}\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)} e^{-(m+1)\left(\frac{x}{\alpha}\right)^{\tau}}dx$$

$let\ y = \frac{x}{\alpha_1} \Longrightarrow x = \alpha_1\, y \Longrightarrow dx = \alpha_1\, dy$

$$I_{71} = \tau_1 k_1\left(\frac{\alpha_1}{\alpha}\right)^{\tau k+\tau k m+\tau(d_1+\cdots+d_m)-1}\int_0^{\infty}\ln(y)\; y^{\tau k+\tau k m+\tau(d_1+\cdots+d_m)-1}$$

$$e^{-(m+1)\left(\frac{\alpha_1}{\alpha}\right)^{\tau} y^{\tau}}\,\alpha_1\, dy$$

$let\ t = y^{\tau} \Longrightarrow y = t^{\frac{1}{\tau}} \Longrightarrow dy = \frac{1}{\tau}\, t^{\frac{1}{\tau}-1}\, dt$

$$I_{71} = \tau_1 k_1\alpha_1\left(\frac{\alpha_1}{\alpha}\right)^{\tau k+\tau k m+\tau(d_1+\cdots+d_m)-1}\int_0^{\infty}\frac{1}{\tau}\ln(t)\; t^{k+km+d_1+\cdots+d_m-\frac{1}{\tau}}$$

$$e^{-(m+1)\left(\frac{\alpha_1}{\alpha}\right)^{\tau} t}\,\frac{1}{\tau}\, t^{\frac{1}{\tau}-1}\, dt$$

$$= \frac{\alpha_1\tau_1 k_1}{\tau^2}\left(\frac{\alpha_1}{\alpha}\right)^{\tau k+\tau k m+\tau(d_1+\cdots+d_m)-1}\int_0^{\infty}\ln(t)\; t^{k+km+d_1+\cdots+d_m-1} e^{-(m+1)\left(\frac{\alpha_1}{\alpha}\right)^{\tau} t}dt$$

$$= \frac{\alpha_1\tau_1 k_1}{\tau^2}\left(\frac{\alpha_1}{\alpha}\right)^{\tau k+\tau k m+\tau(d_1+\cdots+d_m)-1}\left(\frac{\alpha_1}{\alpha}\right)^{-(\tau k m+\tau k+\tau(d_1+\cdots+d_m))}(m+1)^{-(k+km+d_1+\cdots+d_m)}$$

$$\Gamma(k+km+d_1+\cdots+d_m)\left\{\Psi(k+km+d_1+\cdots+d_m)-\ln\left((m+1)\left(\frac{\alpha_1}{\alpha}\right)^{\tau}\right)\right\}$$

$$= \frac{\tau_1 k_1\alpha}{\tau^2}\frac{\Gamma(k+km+d_1+\cdots+d_m)}{(m+1)^{k+km+d_1+\cdots+d_m}}\left\{\Psi(k+km+d_1+\cdots+d_m)-\ln\left((m+1)\left(\frac{\alpha_1}{\alpha}\right)^{\tau}\right)\right\}$$

And, $I_{72} = -\int_0^{\infty}\left(\frac{x}{\alpha_1}\right)^{\tau_1}\left(\frac{x}{\alpha}\right)^{\tau k-1}\left(\frac{x}{\alpha}\right)^{\tau k m}\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)} e^{-(m+1)\left(\frac{x}{\alpha}\right)^{\tau}}dx$

$let\ y = \left(\frac{x}{\alpha}\right)^{\tau} \Longrightarrow x = \alpha y^{\frac{1}{\tau}} \Longrightarrow dx = \frac{\alpha}{\tau}\, y^{\frac{1}{\tau}-1}\, dy$

$$I_{72} = -\left(\frac{\alpha}{\alpha_1}\right)^{\tau_1}\frac{\alpha}{\tau}\int_0^{\infty} y^{\frac{\tau_1}{\tau}+k(m+1)+d_1+\cdots+d_m-1} e^{-(m+1)y}dy$$

$$= -\left(\frac{\alpha}{\alpha_1}\right)^{\tau_1}\frac{\alpha}{\tau}\frac{\Gamma\left(\frac{\tau_1}{\tau}+k(m+1)+d_1+\cdots+d_m\right)}{(m+1)^{\frac{\tau_1}{\tau}+k(m+1)+d_1+\cdots+d_m}}$$

And, $I_{73} = \int_0^{\infty}\ln\left(\sum_{s=0}^{\infty}\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1 s}}{\Gamma(k_1+s+1)}\right)\left(\frac{x}{\alpha}\right)^{\tau k+\tau k m+\tau(d_1+\cdots+d_m)-1} e^{-(m+1)\left(\frac{x}{\alpha}\right)^{\tau}}dx$

$$= \eta^{*}(\tau,\alpha,k,m,\tau_1,\alpha_1,k_1,d_1,\ldots,d_m)$$

$$I_7 = (b_1+1)\ \frac{be^a}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{1}{(k+d_1)!\ldots(k+d_m)!}$$
$$\left\{\begin{matrix}\frac{\tau_1 k_1}{\tau}\frac{\Gamma(k+km+d_1+\cdots+d_m)}{(m+1)^{k(m+1)+d_1+\cdots+d_m}}\left\{\Psi(k+km+d_1+\cdots+d_m)-\ln\left((m+1)\left(\frac{\alpha_1}{\alpha}\right)^{\tau}\right)\right\}- \\ \left(\frac{\alpha}{\alpha_1}\right)^{\tau_1}\frac{\Gamma\left(\frac{\tau_1}{\tau}+k(m+1)+d_1+\cdots+d_m\right)}{(m+1)^{\frac{\tau_1}{\tau}+k(m+1)+d_1+\cdots d_m}}+\frac{\tau}{\alpha}\eta^*(\tau,\alpha,k,m,\tau_1,\alpha_1,k_1,d_1,\ldots,d_m)\end{matrix}\right\}$$

Since, $I_8 = a_1\int_0^{\infty}\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}\ f(x)\,dx$

$$I_8 = \frac{a_1ab\tau}{\alpha e^{-a}\Gamma(k)}\int_0^{\infty}\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}\left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}\ e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}\ dx$$

$$= \frac{a_1b\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ a^{i+1}\ \int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)}\ dx$$

$$= \frac{a_1b\tau}{\alpha e^{-a}\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}\int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}\ \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^{m}\ dx$$

Since, $\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1} = \left\{1-\left(1-\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right)\right\}^{-b_1}$

By using equation (10) we get,

$$\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1} = \sum_{l=0}^{\infty}\sum_{q=0}^{l}\frac{(-1)^q}{q!}\ \frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\ \frac{\Gamma(l+1)}{\Gamma(l-q+1)}\left(\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right)^{q}$$

$$I_8 = \frac{a_1b\tau}{\alpha e^{-a}\Gamma(k)}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+m+q}}{i!m!q!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$
$$\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\ \frac{\Gamma(l+1)}{\Gamma(l-q+1)}\ \int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left(\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right)^{q}\ \left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^{m}\ dx$$

By using eq. expansion of incomplete gamma (17) we get,

$$I_8 = \frac{a_1b\tau}{\alpha e^{-a}\Gamma(k)}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+m+q}}{i!m!q!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$
$$\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\ \frac{\Gamma(l+1)}{\Gamma(l-q+1)}\ \int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ \left[\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1k_1}}{\Gamma(k_1)}\ \sum_{s=0}^{\infty}\frac{\left(-\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right)^{s}}{(k_1+s)s!}\right]^{q}\left[\frac{\left(\frac{x}{\alpha}\right)^{\tau k}}{\Gamma(k)}\ \sum_{d=0}^{\infty}\frac{\left(-\left(\frac{x}{\alpha}\right)^{\tau}\right)^{d}}{(k+d)d!}\right]^{m}\ dx$$

$$= \frac{a_1b\tau e^{a}}{\alpha\left(\Gamma(k)\right)^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+m+q}}{i!m!q!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\ \frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$
$$\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}\ \frac{\Gamma(l+1)}{\Gamma(l-q+1)}\ \sum_{s_1=0}^{\infty}\cdots\sum_{s_q=0}^{\infty}\frac{(-1)^{s_1+\cdots+s_q}\ \alpha_1^{-\left(\tau_1(s_1+\cdots+s_q)\right)}\ \alpha_1^{-\tau_1k_1q}}{(k_1+s_1)\ldots(k_1+s_q)s_1!\ldots s_q!}\ \frac{1}{\left(\Gamma(k_1)\right)^{q}}$$
$$\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{(-1)^{d_1+\cdots+d_m}}{(k+d_1)\ldots(k+d_m)d_1!\ldots d_m!}\ \int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1}\ e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\ x^{\tau_1k_1q}\ x^{\tau_1(s_1+\cdots+s_q)}$$
$$\left(\frac{x}{\alpha}\right)^{\tau km}\ \left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)}\ dx$$

$let\ \ y = \left(\frac{x}{\alpha}\right)^{\tau} \Rightarrow x = \alpha y^{\frac{1}{\tau}} \Rightarrow dx = \frac{\alpha}{\tau}\ y^{\frac{1}{\tau}-1}\ dy$

Since, $\int_0^{\infty} y^{k-\frac{1}{\tau}}\ e^{-y}\ \alpha^{\tau_1k_1q}y^{\frac{\tau_1k_1q}{\tau}}\ \alpha^{\tau_1(s_1+\cdots+s_q)}\ y^{\frac{\tau_1(s_1+\cdots+s_q)}{\tau}}\ y^{km}\ y^{d_1+\cdots+d_m}\ \frac{\alpha}{\tau}\ y^{\frac{1}{\tau}-1}\ dy$

$$= \alpha^{\tau_1k_1q+\tau_1(s_1+\cdots+s_q)}\ \frac{\alpha}{\tau}\ \int_0^{\infty} y^{\frac{\tau_1k_1q}{\tau}+k(m+1)+\frac{\tau_1}{\tau}(s_1+\cdots+s_q)+d_1+\cdots+d_m-1}\ e^{-y}\ dy$$

$= \alpha^{\tau_1 k_1 q+\tau_1(s_1+\cdots+s_q)} \frac{\alpha}{\tau} \Gamma\left(\frac{\tau_1 k_1 q}{\tau}+k(m+1)+\frac{\tau_1}{\tau}(s_1+\cdots+s_q)+d_1+\cdots+d_m\right)$, then,

$$I_8=\frac{a_1 b e^{a}}{(\Gamma(k))^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+m+q}}{i!m!q!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}$$
$$\frac{\Gamma(l+1)}{\Gamma(l-q+1)}\sum_{s_1=0}^{\infty}\cdots\sum_{s_q=0}^{\infty}\frac{(-1)^{s_1+\cdots+s_q}\alpha_1^{-(\tau_1 k_1 q+\tau_1(s_1+\cdots+s_q))}}{(k_1+s_1)\ldots(k_1+s_q)s_1!\ldots s_q!}\frac{1}{(\Gamma(k_1))^{q}}$$
$$\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{(-1)^{d_1+\cdots+d_m}\alpha^{\tau_1 k_1 q+\tau_1(s_1+\cdots+s_q)}}{(k+d_1)\ldots(k+d_m)d_1!\ldots d_m!}\Gamma\left(\frac{\tau_1 k_1 q}{\tau}+k(m+1)+\frac{\tau_1}{\tau}(s_1+\cdots+s_q)+d_1+\cdots+d_m\right)$$

So

$$DKL(F\|F^*)=\ln\left(\frac{ab\tau\,\alpha_1 e^{-a_1}\Gamma(k_1)}{a_1 b_1\tau_1\alpha e^{-a}\Gamma(k)}\right)+\frac{(\tau k-1)}{\tau}\frac{be^{a}}{(\Gamma(k))^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}$$
$$\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}k^{-m}\Gamma(k(m+1))\Psi(k(m+1))$$
$$F_A^{(m)}(k(m+1);k,\ldots,k;k+1,\ldots,k+1;-1,\ldots,-1)-\frac{be^{a}}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}$$
$$I(k+1,m)\quad-(b+1)\frac{be^{a}}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{1}{(k+d_1)!\ldots(k+d_m)!}$$
$$\left\{\begin{matrix}k\frac{\Gamma(k(m+1)+d_1+\cdots+d_m)}{(m+1)^{(k(m+1)+d_1+\cdots+d_m)}}\{\Psi(k(m+1)+d_1+\cdots+d_m)-\ln(m+1)\}\\-\frac{\Gamma(k(m+1)+d_1+\cdots+d_m+1)}{(m+1)^{k(m+1)+d_1+\cdots+d_m+1}}+\frac{\tau}{\alpha}\eta(\tau,\alpha,k,m,d_1,\ldots,d_m)\end{matrix}\right\}-$$
$$\frac{be^{a}}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\Gamma([b(i+2)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I(k,m)-(\tau_1 k_1-1)\frac{be^{a}}{(\Gamma(k))^{m+1}}$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\sum_{d=0}^{\infty}C_{m,d}$$
$$\left\{\ln\left(\frac{\alpha}{\alpha_1}\right)\Gamma(k(m+1)+d)+\frac{1}{\tau}\Gamma(k(m+1)+d)\Psi(k(m+1)+d)\right\}+\frac{(\alpha/\alpha_1)^{\tau_1}be^{a}}{\Gamma(k)}$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}I\left(k+\frac{\tau_1}{\tau},m\right)$$
$$+(b_1+1)\frac{be^{a}}{\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{i+m}}{i!m!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{1}{(k+d_1)!\ldots(k+d_m)!}$$
$$\left\{\begin{matrix}\frac{\tau_1 k_1}{\tau}\frac{\Gamma(k+km+d_1+\cdots+d_m)}{(m+1)^{k(m+1)+d_1+\cdots+d_m}}\left\{\Psi(k+km+d_1+\cdots+d_m)-\ln\left((m+1)\left(\frac{\alpha_1}{\alpha}\right)^{\tau}\right)\right\}-\\\left(\frac{\alpha}{\alpha_1}\right)^{\tau_1}\frac{\Gamma\left(\frac{\tau_1}{\tau}+k(m+1)+d_1+\cdots+d_m\right)}{(m+1)^{\frac{\tau_1}{\tau}+k(m+1)+d_1+\cdots+d_m}}+\frac{\tau}{\alpha}\eta^*(\tau,\alpha,k,m,\tau_1,\alpha_1,k_1,d_1,\ldots,d_m)\end{matrix}\right\}+$$
$$\frac{a_1 b e^{a}}{(\Gamma(k))^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+m+q}}{i!m!q!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\frac{\Gamma(b_1+l)}{l!\Gamma(b_1)}$$
$$\frac{\Gamma(l+1)}{\Gamma(l-q+1)}\sum_{s_1=0}^{\infty}\cdots\sum_{s_q=0}^{\infty}\frac{(-1)^{s_1+\cdots+s_q}\alpha_1^{-(\tau_1 k_1 q+\tau_1(s_1+\cdots+s_q))}}{(k_1+s_1)\ldots(k_1+s_q)s_1!\ldots s_q!}\frac{1}{(\Gamma(k_1))^{q}}\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{(-1)^{d_1+\cdots+d_m}\alpha^{\tau_1 k_1 q+\tau_1(s_1+\cdots+s_q)}}{(k+d_1)\ldots(k+d_m)d_1!\ldots d_m!}$$
$$\Gamma\left(\frac{\tau_1 k_1 q}{\tau}+k(m+1)+\frac{\tau_1}{\tau}(s_1+\cdots+s_q)+d_1+\cdots+d_m\right)\qquad(19)$$

2.2. Stress-Strength Reliability

Let y and x be the stress and strength random variable, independent of each other, follow respectively [0,1] TFGG(a,b,α,τ,k) and [0,1] TFGG$(a_1,b_1,\alpha_1,\tau_1,k_1)$, then,

$$R=P(Y<X)=\int_0^{\infty}f_X(x)\,F_Y(x)\,dx$$

$$R=\int_0^{\infty}\frac{ab\tau}{\alpha e^{-a}\Gamma(k)}\left(\frac{x}{\alpha}\right)^{\tau k-1}e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(b+1)}e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}}\frac{e^{-a_1\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}}}{e^{-a_1}}dx$$

Since, $e^{-a\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-b}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\, a^i \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-bi}$ and

$e^{-a_1\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(a_1)^n \left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1 n}$, then,

$$R = \frac{b\tau e^{a+a_1}}{\alpha\Gamma(k)}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{i+n}}{i!n!}\, a^{i+1}\,(a_1)^n \int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}} \left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} \left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1 n} dx$$

By using equation (10) we get,

$$\left\{\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right\}^{-(bi+b+1)} = \sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^m}{m!}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^m$$

$$\left\{\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right\}^{-b_1 n} = \sum_{l=0}^{\infty}\sum_{q=0}^{l}\frac{(-1)^q}{q!}\frac{\Gamma(b_1 n+l)}{l!\Gamma(b_1 n)}\frac{\Gamma(l+1)}{\Gamma(l-q+1)}\left(\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right)^q$$, then,

$$R = \frac{b\tau e^{a+a_1}}{\alpha\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+n+m+q}}{i!n!m!q!}\, a^{i+1}\,(a_1)^n\, \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$
$$\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\frac{\Gamma(b_1 n+l)}{l!\Gamma(b_1 n)}\frac{\Gamma(l+1)}{\Gamma(l-q+1)}\int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left(\frac{\gamma\left[k,\left(\frac{x}{\alpha}\right)^{\tau}\right]}{\Gamma(k)}\right)^m\left(\frac{\gamma\left[k_1,\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right]}{\Gamma(k_1)}\right)^q dx$$

By using expansion of incomplete gamma function (17) we get,

$$R = \frac{b\tau e^{a+a_1}}{\alpha\Gamma(k)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+n+m+q}}{i!n!m!q!}\, a^{i+1}\,(a_1)^n\, \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$
$$\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\frac{\Gamma(b_1 n+l)}{l!\Gamma(b_1 n)}\frac{\Gamma(l+1)}{\Gamma(l-q+1)}\int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1} e^{-\left(\frac{x}{\alpha}\right)^{\tau}}\left[\frac{\left(\frac{x}{\alpha}\right)^{\tau k}}{\Gamma(k)}\sum_{d=0}^{\infty}\frac{\left(-\left(\frac{x}{\alpha}\right)^{\tau}\right)^d}{(k+d)d!}\right]^m\left[\frac{\left(\frac{x}{\alpha_1}\right)^{\tau_1 k_1}}{\Gamma(k_1)}\sum_{s=0}^{\infty}\frac{\left(-\left(\frac{x}{\alpha_1}\right)^{\tau_1}\right)^s}{(k_1+s)s!}\right]^q dx$$

$$= \frac{b\tau e^{a+a_1}}{\alpha\left(\Gamma(k)\right)^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+n+m+q}}{i!n!m!q!}\, a^{i+1}\,(a_1)^n\, \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$
$$\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\frac{\Gamma(b_1 n+l)}{l!\Gamma(b_1 n)}\frac{\Gamma(l+1)}{\Gamma(l-q+1)}\frac{1}{\left(\Gamma(k_1)\right)^q}\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{(-1)^{d_1+\cdots+d_m}}{(k+d_1)\ldots(k+d_m)d_1!\ldots d_m!}$$
$$\sum_{s_1=0}^{\infty}\cdots\sum_{s_q=0}^{\infty}\frac{(-1)^{s_1+\cdots+s_q}}{(k_1+s_1)\ldots(k_1+s_q)s_1!\ldots s_q!}\int_0^{\infty}\left(\frac{x}{\alpha}\right)^{\tau k-1}\left(\frac{x}{\alpha}\right)^{\tau k m}\left(\frac{x}{\alpha}\right)^{\tau(d_1+\cdots+d_m)}\alpha_1^{-\tau_1 k_1 q}$$
$$x^{\tau_1 k_1 q}\,\alpha_1^{-\tau_1(s_1+\cdots+s_q)}\,x^{\tau_1(s_1+\cdots+s_q)}\,e^{-\left(\frac{x}{\alpha}\right)^{\tau}}dx$$

$let\ y = \left(\frac{x}{\alpha}\right)^{\tau} \Rightarrow x = \alpha y^{\frac{1}{\tau}} \quad \Rightarrow dx = \frac{\alpha}{\tau}\, y^{\frac{1}{\tau}-1}\, dy$, then,

$$R = \frac{be^{a+a_1}}{\left(\Gamma(k)\right)^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+n+m+q}}{i!n!m!q!}\, a^{i+1}\,(a_1)^n\, \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$
$$\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\frac{\Gamma(b_1 n+l)}{l!\Gamma(b_1 n)}\frac{\Gamma(l+1)}{\Gamma(l-q+1)}\frac{1}{\left(\Gamma(k_1)\right)^q}\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{(-1)^{d_1+\cdots+d_m}}{(k+d_1)\ldots(k+d_m)d_1!\ldots d_m!}$$
$$\sum_{s_1=0}^{\infty}\cdots\sum_{s_q=0}^{\infty}\frac{(-1)^{s_1+\cdots+s_q}\,(\alpha/\alpha_1)^{\tau_1 k_1 q+\tau_1(s_1+\cdots+s_q)}}{(k_1+s_1)\ldots(k_1+s_q)s_1!\ldots s_q!}\int_0^{\infty} y^{\frac{\tau_1 k_1 q}{\tau}+k(m+1)+\frac{\tau_1}{\tau}(s_1+\cdots+s_q)+d_1+\cdots+d_m-1}\, e^{-y}\, dx$$

$$= \frac{be^{a+a_1}}{\left(\Gamma(k)\right)^{m+1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{j}\sum_{q=0}^{l}\frac{(-1)^{i+n+m+q}}{i!n!m!q!}\, a^{i+1}\,(a_1)^n\, \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}$$
$$\frac{\Gamma(j+1)}{\Gamma(j-m+1)}\frac{\Gamma(b_1 n+l)}{l!\Gamma(b_1 n)}\frac{\Gamma(l+1)}{\Gamma(l-q+1)}\frac{1}{\left(\Gamma(k_1)\right)^q}\sum_{d_1=0}^{\infty}\cdots\sum_{d_m=0}^{\infty}\frac{(-1)^{d_1+\cdots+d_m}}{(k+d_1)\ldots(k+d_m)d_1!\ldots d_m!}$$
$$\sum_{s_1=0}^{\infty}\cdots\sum_{s_q=0}^{\infty}\frac{(-1)^{s_1+\cdots+s_q}\,(\alpha/\alpha_1)^{\tau_1 k_1 q+\tau_1(s_1+\cdots+s_q)}}{(k_1+s_1)\ldots(k_1+s_q)s_1!\ldots s_q!}\Gamma\left(\frac{\tau_1 k_1 q}{\tau}+k(m+1)+\frac{\tau_1}{\tau}\left(s_1+\cdots+s_q\right)+d_1+\cdots+d_m\right) \tag{20}$$

3. Summary and Conclusions

In statistical analysis a lot of distributions are used to represent set(s) of data. Recently, new distributions are derived to extend some of well-known families of distributions, such that the new distributions are more flexible than the others to model real data. The composing of some distributions with each other's in some way has been in the foreword of data modeling.

In this paper, we presented a new family of continuous distributions based on [0,1] truncated Fréchet distribution. [0,1] truncated Fréchet Generalized Gamma ([0,1]TFGG) distribution is discussed as special case. Properties of [0,1] TFGG is derived. We provide form for characteristic function, rth raw moment, mean, variance, skewness, kurtosis, mode, median, reliability function, hazard rate function, Shannon entropy function and Relative entropy function. This paper deals also with the determination of stress-strength R=p[y<x] when x (strength) and y (stress) are two independent [0,1] TFGG distribution with different parameters.

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