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Oscillation Theorems for Linear Neutral Impulsive Differential Equations of the Second Order with Variable Coefficients and Constant Retarded Arguments

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Abstract This paper deals with the oscillations of a class of second order linear neutral impulsive ordinary differential equations with variable coefficients and constant retarded arguments. Here, we obtain sufficient conditions ensuring the oscillation of all solutions. Examples are provided to illustrate the abstract results.

Keywords Differential equations, Impulsive, Oscillations, Retarded arguments, Second order

1. Introduction

Since Sturm's famous memoir in the 17th century, it is observed that a great deal of interest has been focused on the behaviour of solutions of ordinary and delay differential equations in spite of the existence of extensive literature in these fields ([10], [12], [20], [22]). Delay differential equations has applications in the modeling of complex biological systems, population dynamics, neural network, etc ([19], [21], [23]). Stochastic functional differential equations with state-dependent delay, which have many important applications in mathematical models of real phenomena, is not left out in this seemingly unending quest for knowledge ([15], [16], [17], [18]). Still more interesting, the theory of impulsive differential equations has brought in yet another dimension to the whole scenario and has helped to usher in a new body of knowledge for further considerations. The effects of these new inputs can be observed in the study of oscillatory properties of impulsive differential equations with deviating arguments as well as the investigation of neutral impulsive differential equations which have recently captured the attention of many applied mathematicians as well as other scientists around the world.

In 1989 the paper of Gopalsamy and Zhang [11] was published, where the first investigation on oscillatory properties of impulsive differential equations was carried out. Since then, several authors including Butler [2], Lakshmikantham *et al.* [4], Travis [5], Wong [6] and Ladde *et al.* [12] have since studied oscillations of second-order ordinary differential equations. Lately, the pioneering efforts of Isaac and Lipcsey ([7], [8], [9], [13]) in identifying some of the essential oscillatory and non-oscillatory conditions of neutral impulsive differential equations of the first order is also worth commending. However, relatively less attention has been given to oscillations of second-order neutral delay differential equations with impulses.

This work therefore is concerned with the problem of oscillation of all solutions of a class of second order linear impulsive differential equations with variable coefficients and constant delays.

The theory of oscillations of neutral impulsive differential equations is gradually occupying a central place among the theories of oscillations of impulsive differential equations. This could be due to the fact that neutral impulsive differential equations play fundamental roles in the present drive to further develop information technology. Indeed, neutral differential equations appear in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits).

Impulsive differential equations are adequate mathematical models for description of evolution processes characterized by the combination of a continuous and jumps change of their state:

Now, let an evolution process evolve in a period of time J in an open set $\Omega \subset J \times R^n$ and let the function $f:\Omega \to R^n$ be at the least a continuous mapping fulfilling local Lipchitzian condition in $y \in R^n$, $\forall (t,y) \in \Omega$. Let the real numerical sequence $S=\{t_k\}_{k=1}^{\infty} \subset J$ be increasing without finite accumulation point such that $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim_{k\to+\infty} t_k = +\infty$,. The points t_k are called moments of impulse effect. Then the governing second order impulsive differential equation is of the form

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$$\begin{cases} y''(t)=f(t,y,y'),\ t\neq t_k \\ \Delta y'(t_k)=f_k(y,y'),\ t=t_k, \end{cases} \tag{1.1}$$

where $y'=\dfrac{dy}{dt}$, $y''=\dfrac{d^2y}{dt^2}$, $(t,y(t))\in\Omega$, $\Delta y^{(i)}(t_k)=y^{(i)}(t_k^+)-y^{(i)}(t_k^-)$, $i=0,1$ and $y(t_k^-)$, $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t=t_k$, respectively. For the sake of definiteness, we shall suppose that the functions $y(t)$ and $y'(t)$ are continuous from the left at the points t_k such that $y'(t_k^-)=y'(t_k)$, $y(t_k^-)=y(t_k)$.

For the description of the continuous change of such processes ordinary differential equations are used, while the moments and the magnitude of the change by jumps are given by the jump conditions. Now, in the case of unfixed moments of impulse effects, the impulse points may be time and state dependent, that is, $t_k=t_k(t,y(t))$. When the function t_k depends on the state of the system (1.1), then it is said to have impulses at variable times. This is reflected in the fact that different solutions will tend to undergo impulses at different times.

In this paper, we shall restrict ourselves to the investigation of properties of the solutions of impulsive differential equations with fixed moments of impulse effect, that is, the moments of jump are previously fixed. Our equation under consideration is of the form

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]''+q(t)y(t-\sigma)=0,\ t\neq t_k \\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0,\ t=t_k, \end{cases} \tag{1.2}$$

where $t,t_k\geq 0$. The second order neutral delay impulsive differential equation (1.2) is a differential system comprising a second-order differential equation and its impulsive conditions in which the highest-order derivative of the unknown function appears in the differential equation both with and without delay.

Let $\rho=max\{\tau,\sigma\}$. We say that a real valued function $y(t)$ is the solution of equation (1.2) if there exists a number $t_0\in R$ such that $y(t)\in PC([t_0-\rho,\infty),R)$, the function $y(t)+p(t)y(t-\tau)$ is twice continuously differentiable for $t\geq t_0-\rho$, $t\neq t_k$, $k\in N$ and $y(t)$ satisfies equation (1.2) for all $t\geq t_0-\rho$.

Without further mentioning, we will assume throughout this paper that every solution $y(t)$ of equation (2.1) that is under consideration here, is continuous from the left and is nontrivial. That is, $y(t)$ is defined on some half-line $[T_y,\infty)$ and $sup\{|y(t)|:\ t\geq T\}>0$ for all $T\geq T_y$. Such a solution is called a regular solution of equation (2.1). We say that a real valued function $y(t)$ defined on an interval $[a,+\infty)$ fulfills some property *finally*, if there exists a number $T\geq a$ such that $y(t)$ has this property on the interval $[T,+\infty)$.

Definition 1.4 The solution $y(t)$ of an impulsive differential equation is said to be

i) finally positive (finally negative) if there exist $T\geq 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t\geq T$ [8];
ii) non-oscillatory, if it is either finally positive or finally negative; and
iii) oscillatory, if it is neither finally positive nor finally negative ([1], [9]).

In the sequel, all functional inequalities that we write are assumed to hold finally, that is, for all sufficiently large t.

2. Statement of the Problem

We are concerned with the oscillatory properties of the second order linear neutral delay impulsive differential equation with variable coefficients and constant deviating arguments of the form

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]''+q(t)y(t-\sigma)=0,\ t\geq t_0,\ t\in J\setminus S \\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0,\ t_k\geq t_0,\forall\ t_k\in S, \end{cases} \tag{2.1}$$

where $p(t),q(t)\in C([t_0,\infty),\mathbb{R})$ and τ and σ are non-negative real numbers. Our aim is to establish some sufficient conditions for every bounded solution of equation (2.1) to be oscillatory. Throughout this study, we shall assume the following:

C2.1: $q_k\geq 0\ \forall\ k\in\mathbb{N}$;

C2.2: $p(t)\in PC([t_0,\infty),\mathbb{R})$, $p_1\leq p(t)\leq p_2$ for $t\in[t_0,\infty)$, where $p_1,p_2\in\mathbb{R}$;

C2.3: $q(t)\in PC([t_0,\infty),\mathbb{R})$, $q(t)\geq q_1>0$ for $t\in[t_0,\infty)$.

Here, we demonstrate how well-known mathematical techniques and methods (due to studies by Bainov and Simeonov [1]), after suitable modifications, is extended in proving an oscillation theorem for impulsive delay differential equations. We shall restrict ourselves to the study of impulsive differential equations for which the impulse effects take place at fixed moments of time $\{t_k\}$.

Lemma 2.1 and Lemma 2.2, which are essential in carrying out our investigation are impulsive extensions of the work done by Grammatikopoulos *et al* [14] and Ladas and Stavroulakis [10], respectively, in their quest to find sufficient conditions for oscillation of all solutions of a type of neutral delay ordinary differential equations.

Lemma 2.1: Assume conditions C2.1—C2.3 satisfied and let $y(t)$ be a finally positive solution of equation (2.1). Set

$$z(t)=y(t)+p(t)y(t-\tau). \quad (2.2)$$

Then the following statements are true:

a) The functions $z(t)$ and $z'(t)$ are strictly monotone and either

$$\lim_{t\to\infty} z(t)=\lim_{t\to\infty} z'(t)=-\infty \quad (2.3)$$

or

$$\lim_{t\to\infty} z(t)=\lim_{t\to\infty} z'(t)=0;\ z(t)<0 \text{ and } z'(t)>0. \quad (2.4)$$

In particular, $z(t)$ is finally negative.

b) Assume that $p_1 \geq -1$, then condition (2.4) holds. In particular, $z(t)$ is bounded.

Proof: (a) From equation (2.1), we have that

$$z''(t)=-q(t)y(t-\sigma)\leq -q_1 y(t-\sigma)<0$$

and

$$\Delta z'(t_k)=-q_k y(t_k-\sigma)\leq -q_1 y(t_k-\sigma)<0 \quad (2.5)$$

which implies that $z'(t)$ is a strictly decreasing function of t and so $z(t)$ is a strictly monotone function. From the above observations it follows that either

$$\lim_{t\to\infty} z(t)=\lim_{t\to\infty} z'(t)=-\infty$$

or

$$\lim_{t\to\infty} z'(t)=\ell \text{ is finite.} \quad (2.6)$$

Let us assume that condition (2.6) holds. Integrating both sides of equation (2.5) from t_0 to t with t_0 sufficiently large, and letting $t\to\infty$, we obtain

$$\int_{t_0}^{\infty} q_1 y(s-\sigma)ds+\sum_{t_0\leq t_k<\infty} q_1 y(t_k-\sigma)\leq z'(t_0)-\ell,$$

which implies that $y(t)\in L_1[T,\infty)$ and so $z(t)\in L_1[T,\infty)$, where L_1 is the space of all Lebesgue integrable functions on $[T,\infty)$. Since $z(t)$ is monotone, it follows that

$$\lim_{t\to\infty} z(t)=0, \quad (2.7)$$

and therefore $\ell=0$. Finally, by equations (2.7) and (2.6) with $\ell=0$ and the decreasing nature of $z'(t)$, we conclude that $z(t)<0$ and $z'(t)>0$.

c) By contradiction, we assume condition (2.4) was false, then from condition (2.3), it would follow that

$$\lim_{t\to\infty} z(t)=-\infty. \quad (2.8)$$

Using the fact that $p_1\geq -1$ and $z(t)<0$, we obtain

$$y(t)<p(t)y(t-\tau)\leq -p_1 y(t-\tau)\leq y(t-\tau),$$

which implies that $y(t)$ is bounded, contradicting condition (2.8) and proving that condition (2.4) is fulfilled. This, therefore, completes the proof of Lemma 2.1.

We now present another lemma which will be useful in the discussion of the main results.

Lemma 2.2: Assume that r and μ are positive constants such that

$$r^{\frac{1}{2}}\frac{\mu}{2}>\frac{1}{e}.$$

Then the differential inequality

$$\begin{cases} x''(t)-rx(t-\mu)\leq 0,\ t\notin S \\ \Delta x'(t_k)-rx(t_k-\mu)\leq 0, \forall\, t_k\in S \end{cases}$$

has no finally negative bounded solution.

3. Main Results

The following theorems are the impulsive extensions of Theorem 3.1.2 and Theorem 3.1.3 of the monograph by Bainov and Mishev [3].

Theorem 3.1: Consider the neutral delay impulsive differential equation (2.1) and assume conditions C2.1—C2.3 satisfied. Furthermore, assume that $p(t)$ is not finally negative. Then every solution of equation (2.1) oscillates.

Proof: By contradiction, we assume that $y(t)$ is a finally positive solution of equation (2.1). Set

$$z(t)=y(t)+p(t)y(t-\tau).$$

We can see here that $z(t)\geq 0$ finally. However, by Lemma 2.1(a), $z(t)<0$ finally. This contradicts the statement of the theorem that $p(t)$ is not finally negative, and therefore, completes the proof of Theorem 3.1.

The following illustration will enhance clarity:

Example 3.1: Consider the equation

$$\begin{cases} \left[y(t)+\left(\frac{1}{2}-\sin t\right)y(t-2\pi)\right]''+\left(\frac{3}{2}-\sin t\right)y(t-4\pi)=0, t\geq 0,\ t\notin S \\ \Delta\left[y(t_k)+\left(\frac{1}{2}-\sin t_k\right)y(t_k-2\pi)\right]'+\left(\frac{3}{2}-\sin t_k\right)y(t_k-4\pi)=0, t_k\geq 0, \forall\, t_k\in S \end{cases} \quad (3.1)$$

It is easy to see that the assumptions of Theorem 3.1 are satisfied here. Therefore, every solution of equation (3.8) oscillates. For instance, $y(t)=\dfrac{\sin t}{\frac{3}{2}+\sin t}$ is an oscillating solution.

This illustration shows that if the hypothesis of $p(t)$ not being finally negative in Theorem 3.1 is violated, the result may be wrong.

Example 3.2: Consider the equation

$$\begin{cases} \left[y(t)+(t-1)^{-\frac{1}{2}}y(t-1)\right]''+\frac{1}{4}t^{-\frac{3}{2}}(t-2)^{-\frac{1}{2}}y(t-2)=0,\ t\geq 2, t\notin S \\ \Delta\left[y(t_k)+(t_k-1)^{-\frac{1}{2}}y(t_k-1)\right]'+\frac{1}{4}t_k^{-\frac{3}{2}}(t_k-2)^{-\frac{1}{2}}y(t_k-2)=0, t_k\geq 2, \forall\, t_k\in S. \end{cases}$$

All assumptions of Theorem 3.2, except (ii) are satisfied. Note, however, that $y(t)=t^{\frac{1}{2}}$ is a non-oscillatory solution.

Theorem 3.2: Assume that conditions C2.1—C2.3 are satisfied with

$$-1\leq p_1\leq p_2<0\,. \tag{3.2}$$

Suppose also that there exists a positive constant r such that

$$\begin{cases} \dfrac{q(t)}{p(t+\tau-\sigma)}\leq -r \\ \dfrac{q_k}{p(t_k+\tau-\sigma)}\leq -r \end{cases} \tag{3.3}$$

and

$$r^{\frac{1}{2}}\frac{\sigma-\tau}{2}>\frac{1}{e}\,. \tag{3.4}$$

Then every solution of equation (2.1) is oscillatory.

Proof: By contradiction, we assume that $y(t)$ is a finally positive solution of equation (2.1). Set

$$z(t)=y(t)+p(t)y(t-\tau)\,.$$

Then a direct substitution shows that $z(t)$ is a twice piece-wise continuously differentiable solution of the neutral delay impulsive differential equation

$$\begin{cases} z''(t)+R(t)z''(t-\tau)+q(t)z(t-\sigma)=0,\ t\geq t_0,\ t\notin S \\ \Delta z'(t_k)+R(t_k)\Delta z'(t_k-\tau)+q_k z(t_k-\sigma)=0,\ t_k\geq t_0,\ t_k\in S, \end{cases} \tag{3.5}$$

where

$$R(t_k)=p(t_k-\sigma)\frac{q_k}{q(t_k-\tau)}\,.$$

From equation (2.1) we have that

$$z''(t), \Delta z'(t_k)<0\,, \tag{3.6}$$

and in view of equation (3.2), Lemma 2.1(b) implies that $z(t)$ is a finally negative bounded function. Using condition (3.6), equation (3.5) yields

$$\begin{cases} R(t)z''(t-\tau)+q(t)z(t-\sigma)>0,\ t\notin S \\ R(t_k)\Delta z'(t_k-\tau)+q_k z(t_k-\sigma)>0, \forall\, t_k\in S. \end{cases}$$

Hence, in view of equation (3.3), we obtain

$$\begin{cases} z''(t)-r\,z(t-(\sigma-\tau))<0,\ t\notin S \\ \Delta z'(t_k)-r\,z(t_k-(\sigma-\tau))<0, \forall\, t_k\in S. \end{cases}$$

But due to condition (3.4), Lemma 2.2 implies that it is impossible for this inequality to have a finally negative bounded solution, which is a contradiction. This completes the proof of Theorem 3.2.

The following illustration shows that if the condition $-1\leq p_1$ of Theorem 3.2 is violated, the result may not be true.

Example 3.3: Consider the neutral delay impulsive differential equation

$$\begin{cases} \left[y(t)-\left(e^2+e^{-t}\right)y(t-1)\right]''+e^2(e-1)y(t-2)=0,\ t\ge 0, t\notin S \\ \Delta\left[y(t_k)-\left(e^2+e^{-t}\right)y(t_k-1)\right]''+e^2(e-1)y(t_k-2)=0,\ t_k\ge 0, t_k\in S. \end{cases}$$

We observe that all conditions of Theorem 3.2, except for $-1\le p_1$ are satisfied. Note that $y(t)=e^t$ is a non-oscillating solution of this equation.

4. Conclusions

By appropriate imposition of impulse controls, all solutions of a certain class of second order neutral impulsive differential equations are observed to be oscillatory. In this paper, we generalized and proved the results of oscillations of second order neutral differential equations with constant coefficients obtained by Bainov and Mishev [3] for impulsive differential equations.

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