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Analytical Solutions of Fourth Order Critically Undamped Oscillatory Nonlinear Systems with Pairwise Equal Imaginary Eigenvalues

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Abstract In this article, we present asymptotic solutions of fourth order critically oscillatory undamped nonlinear systems in which the four eigenvalues are pairwise equal and imaginary. In this regard, the modified Krylov-Bogoliubov-Mitropolskii (KBM) method is used, which is considered to be a well-suited method for investigating the transient behaviour of oscillating systems, to obtain the solutions of fourth order critically oscillatory nonlinear systems. This paper suggests that the solutions obtained by the modified KBM method are quite well consonant with those obtained by the numerical method using *Mathematica.*

Keywords KBM method, Critically oscillatory system, Nonlinearity, Eigenvalues

1. Introduction

Generally, a significant approach, known as the expansion of the small parameter, is used to investigate nonlinear oscillatory systems, which is erected upon the perturbation theory. The perturbation theory can be defined as mathematical methods by which an approximate solution to a mathematical problem is figured out. This theory was used as a basis for one of the widely known methods called the Krylov-Bogoliubov-Mitropolskii (KBM) [1, 2] method which is applied to study nonlinear oscillatory and non-oscillatory differential systems with small nonlinearities. In the beginning, this method was developed by Krylov and Bogoliubov [1] to find out the periodic solutions of second order nonlinear differential systems with small nonlinearities. Shortly after that, Popov [3] extended it to damped oscillatory processes. Then, it was expanded further and justified by Bogoliubov and Mitropolskii [2]. However, due to the physical significance of the damped oscillatory systems, Popov's results were rediscovered by Mendelson [4]. Later, the method was extended by Murty and Deekshatulu [5] for over damped nonlinear systems. However, in 1971, Murty [6] propounded a unified KBM method for second order nonlinear systems which covered all the undamped, over-damped and damped oscillatory cases. Next, Osiniskii [7] developed it further and used it to solve third order nonlinear differential systems for the first time, imposing some restrictions on it. But this rendered the solution over-simplified. Therefore, Mulholland [8] lifted those restrictions and found the intended solutions of third order nonlinear systems. Mickens [9] modified the method for nonlinear oscillations with finite damping, and for harmonic oscillations without damping. Bojadziev [10] suggested damped oscillating processes in biological and biochemical systems. Bojadziev and Edwards [11] offered some methods for non-oscillatory and oscillatory processes. Later, Arya and Bojadziev [12] proposed time dependent oscillating systems with damping, slowly varying parameters, and delay. Bojadziv [13] subsequently examined the solutions of nonlinear systems by transforming the method to damped nonlinear oscillations for a 3-dimensional differential system. Then, Alam and Sattar [14] expounded time dependent third-order oscillating systems with damping. Later on, Akbar *et al.* [15] generalized the method which was less intricate than the method put forward by Murty *et al.* [16]. Akbar *et al.* [17] then expanded it for fourth order damped oscillatory systems in the case when the four eigenvalues were complex conjugate. Thereupon, the perturbation solution of fourth order critically damped oscillatory systems was expounded by Haque *et al.* [18] under the conditions that when two of the eigenvalues are real and equal and the other two are complex conjugate. Later, Rahman *et al.* [19] presented a technique for fourth order damped oscillatory nonlinear systems in the case when two of the eigenvalues are real and distinct and the other two are complex conjugate.

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This article provides the solutions of fourth order critically undamped oscillatory nonlinear systems where all the eigenvalues are pairwise equal and imaginary. This paper suggests that the obtained perturbation results are quite well consonant with the numerical results for different sets of initial conditions together with different sets of eigenvalues. In this study, all the results are computed using *Mathematica 9.0*.

2. Method

Let us consider a fourth order weakly nonlinear system

$$x^{(iv)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon\, f\left(x,\ \dot{x},\ \ddot{x},\ \dddot{x}\right) \tag{1}$$

where $x^{(iv)}$ denotes the fourth derivative and over dots indicate the first, second and third derivatives of x with respect to time t; k_1, k_2, k_3, k_4 stand for characteristic parameters; ε signifies a positive small parameter and $f\left(x,\ \dot{x},\ \ddot{x},\ \dddot{x}\right)$ is the given nonlinear function. Since the system (1) is of fourth order critically undamped oscillatory, so we consider the equation (1) has pairwise equal imaginary eigenvalues, and assume that the pairwise equal eigenvalues are $\pm i\lambda$.

Therefore, the characteristic parameters of equation (1) are defined by

k_1 = Sum of the eigenvalues $= 0$

k_2 = Sum of the eigenvalues taken two at a time $= 2\lambda^2$

k_3 = Sum of the eigenvalues taken three at a time $= 0$

k_4 = Product of the eigenvalues $= \lambda^4$

Thus, when $\varepsilon = 0$, the solution of the corresponding linear equation of (1) becomes

$$x\left(t,0\right) = (a_0 + c_0 t)\cos\lambda t + (b_0 + d_0 t)\sin\lambda t \tag{2}$$

where a_0, b_0, c_0, d_0 are constants of integration.

Nonetheless, if $\varepsilon \neq 0$, following Alam [20], an asymptotic solution of (1) becomes

$$\begin{aligned} x\left(t,\varepsilon\right) &= (a+ct)\cos\lambda t + (b+dt)\sin\lambda t \\ &\quad + \varepsilon\, u_1\left(a, b, c, d, t\right) + \cdots \end{aligned} \tag{3}$$

where a, b, c, d denote functions of t and they satisfy the following first order differential equations:

$$\begin{aligned} \dot{a}(t) &= \varepsilon A_1\left(a,\ b,\ c,\ d,\ t\right) + \cdots \\ \dot{b}(t) &= \varepsilon B_1\left(a,\ b,\ c,\ d,\ t\right) + \cdots \\ \dot{c}(t) &= \varepsilon C_1\left(a,\ b,\ c,\ d,\ t\right) + \cdots \\ \dot{d}(t) &= \varepsilon D_1\left(a,\ b,\ c,\ d,\ t\right) + \cdots \end{aligned} \tag{4}$$

Here, we only consider first few terms in the series expansion of (3) and (4). We evaluate the functions u_i and A_i, B_i, C_i, D_i for $i = 1, 2, 3, \cdots$ to the extent that a, b, c, d appearing in equation (3) and equation (4), satisfy the given differential equation (1). In order to determine these unknown functions, it should be noted that, in the KBM method generally the correction terms, u_i for $i = 1, 2, 3, \cdots$ exclude the terms, which are also known as 'secular terms', that render them larger. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, the solution is generally confined to a lower order, usually the first as suggested by Murty [6] owing to the rapidly growing algebraic complexity for the derivation of the formulae.

Now, differentiating the equation (3) four times with respect to *t*, substituting the value of *x* and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(iv)}$ in the original equation (1), using the relations presented in (4), and, finally, equating the coefficients of ε, we get

$$\begin{aligned} &\cos\lambda t\left(\frac{\partial^3 A_1}{\partial t^3} - 4\lambda^2\frac{\partial A_1}{\partial t} + 4\lambda\frac{\partial^2 B_1}{\partial t^2} + 4\frac{\partial^2 C_1}{\partial t^2} - 8\lambda^2 C_1 + 12\lambda\frac{\partial D_1}{\partial t}\right) + \sin\lambda t\left(-4\lambda\frac{\partial^2 A_1}{\partial t^2} + \frac{\partial^3 B_1}{\partial t^3} - 4\lambda^2\frac{\partial B_1}{\partial t} - 12\lambda\frac{\partial C_1}{\partial t}\right. \\ &\left. + 4\frac{\partial^2 D_1}{\partial t^2} - 8\lambda^2 D_1\right) + t\cos\lambda t\left(\frac{\partial^3 C_1}{\partial t^3} - 4\lambda^2\frac{\partial C_1}{\partial t} + 4\lambda\frac{\partial^2 D_1}{\partial t^2}\right) + t\sin\lambda t\left(-4\lambda\frac{\partial^2 C_1}{\partial t^2} + \frac{\partial^3 D_1}{\partial t^3} - 4\lambda^2\frac{\partial D_1}{\partial t}\right) + \left\{\frac{\partial^2}{\partial t^2} + \lambda^2\right\}^2 u_1 \\ &\qquad = -f^{(0)}(a,\, b,\, c,\, d,\, t) \end{aligned} \tag{5}$$

where $f^{(0)}\left(a,\ b,\ c,\ d,\ t\right) = f\left(x_0,\ \dot{x}_0,\ \ddot{x}_0,\ \dddot{x}_0\right)$ and $x_0 = (a+ct)\cos\lambda t + (b+dt)\sin\lambda t$

Now, expanding the function $f^{(0)}$ in the Taylor's series (see also Sattar [21], Alam [22], Alam and Sattar [23] for details) with respect to the origin in power of t.

As a result, we get

$$f^{(0)} = \sum_{q=0}^{\infty} t^q\left[E_q + \sum_{r=0}^{\infty}\left\{F_{q,r+1}\cos(r+1)\lambda t + G_{q,r+1}\sin(r+1)\lambda t\right\}\right] \tag{6}$$

Here, $E_q, F_{q,r+1}, G_{q,r+1}$ are functions of a, b, c, d and q, r are vary from 0 to ∞, but for a particular problem they have some fixed values. Thus, with the help of equation (6), equation (5) becomes

$$\cos\lambda t\left(\frac{\partial^3 A_1}{\partial t^3}-4\lambda^2\frac{\partial A_1}{\partial t}+4\lambda\frac{\partial^2 B_1}{\partial t^2}+4\frac{\partial^2 C_1}{\partial t^2}-8\lambda^2 C_1+12\lambda\frac{\partial D_1}{\partial t}\right)+\sin\lambda t\left(-4\lambda\frac{\partial^2 A_1}{\partial t^2}+\frac{\partial^3 B_1}{\partial t^3}-4\lambda^2\frac{\partial B_1}{\partial t}-12\lambda\frac{\partial C_1}{\partial t}\right.$$
$$\left.+4\frac{\partial^2 D_1}{\partial t^2}-8\lambda^2 D_1\right)+t\cos\lambda t\left(\frac{\partial^3 C_1}{\partial t^3}-4\lambda^2\frac{\partial C_1}{\partial t}+4\lambda\frac{\partial^2 D_1}{\partial t^2}\right)+t\sin\lambda t\left(-4\lambda\frac{\partial^2 C_1}{\partial t^2}+\frac{\partial^3 D_1}{\partial t^3}-4\lambda^2\frac{\partial D_1}{\partial t}\right)+\left\{\frac{\partial^2}{\partial t^2}+\lambda^2\right\}^2 u_1 \tag{7}$$
$$=-\sum_{q=0}^{\infty}t^q\left[E_q+\sum_{r=0}^{\infty}\left\{F_{q,r+1}\cos(r+1)\lambda t+G_{q,r+1}\sin(r+1)\lambda t\right\}\right]$$

It should be noted that, following the KBM method, Murty *et al.* [16], Sattar [21], Alam and Sattar [23] impose the condition that u_1 must not contains the fundamental terms of $f^{(0)}$. So, equation (7) can be separated for unknown functions A_1, B_1, C_1, D_1 and u_1 as follows:

$$\cos\lambda t\left(\frac{\partial^3 A_1}{\partial t^3}-4\lambda^2\frac{\partial A_1}{\partial t}+4\lambda\frac{\partial^2 B_1}{\partial t^2}+4\frac{\partial^2 C_1}{\partial t^2}-8\lambda^2 C_1+12\lambda\frac{\partial D_1}{\partial t}\right)+\sin\lambda t\left(-4\lambda\frac{\partial^2 A_1}{\partial t^2}+\frac{\partial^3 B_1}{\partial t^3}-4\lambda^2\frac{\partial B_1}{\partial t}\right.$$
$$\left.-12\lambda\frac{\partial C_1}{\partial t}+4\frac{\partial^2 D_1}{\partial t^2}-8\lambda^2 D_1\right)+t\cos\lambda t\left(\frac{\partial^3 C_1}{\partial t^3}-4\lambda^2\frac{\partial C_1}{\partial t}+4\lambda\frac{\partial^2 D_1}{\partial t^2}\right)+t\sin\lambda t\left(-4\lambda\frac{\partial^2 C_1}{\partial t^2}+\frac{\partial^3 D_1}{\partial t^3}-4\lambda^2\frac{\partial D_1}{\partial t}\right) \tag{8}$$
$$=-\sum_{q=0}^{1}t^q\left\{F_{q,1}\cos\lambda t+G_{q,1}\sin\lambda t\right\}$$

$$\left\{\frac{\partial^2}{\partial t^2}+\lambda^2\right\}^2 u_1=-\sum_{q=0}^{\infty}t^q\left[E_q+\sum_{r=1}^{\infty}\left\{F_{q,r+1}\cos(r+1)\lambda t+G_{q,r+1}\sin(r+1)\lambda t\right\}\right] \tag{9}$$

Now, contrasting the coefficients of $\cos\lambda t, \sin\lambda t, t\cos\lambda t$ and $t\sin\lambda t$ from both sides of equation (8), we get

$$\frac{\partial^3 A_1}{\partial t^3}-4\lambda^2\frac{\partial A_1}{\partial t}+4\lambda\frac{\partial^2 B_1}{\partial t^2}+4\frac{\partial^2 C_1}{\partial t^2}-8\lambda^2 C_1+12\lambda\frac{\partial D_1}{\partial t}=-F_{0,1} \tag{10}$$

$$-4\lambda\frac{\partial^2 A_1}{\partial t^2}+\frac{\partial^3 B_1}{\partial t^3}-4\lambda^2\frac{\partial B_1}{\partial t}-12\lambda\frac{\partial C_1}{\partial t}+4\frac{\partial^2 D_1}{\partial t^2}-8\lambda^2 D_1=-G_{0,1} \tag{11}$$

$$\frac{\partial^3 C_1}{\partial t^3}-4\lambda^2\frac{\partial C_1}{\partial t}+4\lambda\frac{\partial^2 D_1}{\partial t^2}=-F_{1,1} \tag{12}$$

$$-4\lambda\frac{\partial^2 C_1}{\partial t^2}+\frac{\partial^3 D_1}{\partial t^3}-4\lambda^2\frac{\partial D_1}{\partial t}=-G_{1,1} \tag{13}$$

Now, for determining the unknown functions $A_1, B_1, C_1, D_1,$ we have four equations (10) to (13). Therefore, in order to obtain the unknown functions $A_1, B_1, C_1, D_1,$ we are compelled to use some widely known operator method. From equations (12) and (13) we can determine the unknown functions C_1 and $D_1,$ then substitute them into the equations (10) and (11) we can find out the other two unknown functions A_1 and B_1.

Since $\dot{a}, \dot{b}, \dot{c}, \dot{d}$ are proportional to the small parameter $\varepsilon,$ they become slowly varying functions of time t and, for first approximate solution, we may consider them as constants which are in the right hand side. It should be noted that Murty and Deekshatulu [5], and Murty *et al.* [16] first made this assumption. Therefore, the solution of the equation (4) becomes

$$\begin{aligned} a&=a_0+\varepsilon\int_0^t A_1(a,b,c,d,t)\,dt\\ b&=b_0+\varepsilon\int_0^t B_1(a,b,c,d,t)\,dt\\ c&=c_0+\varepsilon\int_0^t C_1(a,b,c,d,t)\,dt\\ d&=d_0+\varepsilon\int_0^t D_1(a,b,c,d,t)\,dt \end{aligned} \tag{14}$$

It should be pointed out that equation (9) is an inhomogeneous linear ordinary differential equation. Thus, it can be solved by the widely known operator method.

Now, substituting the values of a, b, c, d and u_1 in the equation (3), we obtain the complete solution of (1). Thus, the determination of the first approximate solution is complete.

3. Example

An example has been worked out here using the aforementioned method. Let us consider the Duffing type equation of fourth order nonlinear differential system

$$x^{(iv)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon\, x^3 \tag{15}$$

Comparing (15) with (1), we get

$$f(x, \dot{x}, \ddot{x}, \dddot{x}) = x^3$$

Thus

$$\begin{aligned} f^{(0)} &= \frac{3}{4}\left\{\left(a^3 + ab^2\right)\cos\lambda t + \left(a^2 b + b^3\right)\sin\lambda t\right\} + \frac{3}{4}t\left\{\left(3a^2 c + b^2 c + 2abd\right)\cos\lambda t\right. \\ &\left. + \left(2abc + a^2 d + 3b^3\right)\sin\lambda t\right\} + \frac{3}{4}t^2\left\{\left(3ac^2 + 2bcd + ad^2\right)\cos\lambda t + \left(bc^2 + 2acd + 3bd^2\right)\sin\lambda t\right\} \\ &+ \frac{3}{4}t^3\left\{\left(c^3 + cd^2\right)\cos\lambda t + \left(c^2 d + d^3\right)\sin\lambda t\right\} + \frac{1}{4}\left\{\left(a^3 - 3ab^2\right)\cos 3\lambda t + \left(3a^2 b - b^3\right)\sin 3\lambda t\right\} \\ &+ \frac{3}{4}t\left\{\left(a^2 c - b^2 c - 2abd\right)\cos 3\lambda t + \left(2abc + a^2 d - b^2 d\right)\sin 3\lambda t\right\} + \frac{3}{4}t^2\left\{\left(ac^2 - 2bcd - ad^2\right)\cos 3\lambda t\right. \\ &\left. + \left(bc^2 - bd^2 + 2acd\right)\sin 3\lambda t\right\} + \frac{1}{4}t^3\left\{\left(c^3 - 3cd^2\right)\cos 3\lambda t + \left(3c^2 d - d^3\right)\sin 3\lambda t\right\} \end{aligned} \tag{16}$$

For equation (15), the equations (9) to (13) respectively become

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} + \lambda^2\right)^2 u_1 &= -\frac{3}{4}\Big[t^2\left\{\left(3ac^2 + 2bcd + ad^2\right)\cos\lambda t + \left(bc^2 + 2acd + 3bd^2\right)\sin\lambda t\right\} + t^3\left\{\left(c^3 + cd^2\right)\cos\lambda t\right. \\ &\left. + \left(c^2 d + d^3\right)\sin\lambda t\right\} + \frac{1}{3}\left\{\left(a^3 - 3ab^2\right)\cos 3\lambda t + \left(3a^2 b - b^3\right)\sin 3\lambda t + t\left\{\left(a^2 c - b^2 c - 2abd\right)\cos 3\lambda t\right.\right. \\ &+ \left(2abc + a^2 d - b^2 d\right)\sin 3\lambda t + t^2\left\{\left(ac^2 - 2bcd - ad^2\right)\cos 3\lambda t + \left(bc^2 - bd^2 + 2acd\right)\sin 3\lambda t \right. \\ &+ \frac{1}{3}\left\{t^3\left(c^3 - 3cd^2\right)\cos 3\lambda t + \left(3c^2 d - d^3\right)\sin 3\lambda t\right\}\Big] \end{aligned} \tag{17}$$

$$\frac{\partial^3 A_1}{\partial t^3} - 4\lambda^2\frac{\partial A_1}{\partial t} + 4\lambda\frac{\partial^2 B_1}{\partial t^2} + 4\frac{\partial^2 C_1}{\partial t^2} - 8\lambda^2 C_1 + 12\lambda\frac{\partial D_1}{\partial t} = -\frac{3}{4}\left(a^3 + ab^2\right) \tag{18}$$

$$-4\lambda\frac{\partial^2 A_1}{\partial t^2} + \frac{\partial^3 B_1}{\partial t^3} - 4\lambda^2\frac{\partial B_1}{\partial t} - 12\lambda\frac{\partial C_1}{\partial t} + 4\frac{\partial^2 D_1}{\partial t^2} - 8\lambda^2 D_1 = -\frac{3}{4}\left(a^2 b + b^3\right) \tag{19}$$

$$\frac{\partial^3 C_1}{\partial t^3} - 4\lambda^2\frac{\partial C_1}{\partial t} + 4\lambda\frac{\partial^2 D_1}{\partial t^2} = -\frac{3}{4}\left(3a^2 c + b^2 c + 2abd\right) \tag{20}$$

$$-4\lambda\frac{\partial^2 C_1}{\partial t^2} + \frac{\partial^3 D_1}{\partial t^3} - 4\lambda^2\frac{\partial D_1}{\partial t} = -\frac{3}{4}\left(2abc + a^2 d + 3b^2 d\right) \tag{21}$$

Solving equations (20) and (21), we achieve

$$C_1 = \frac{3}{8\lambda^3}\left\{(a^2 c + 2a^2 d + 6b^2 d) + (6a^2 c + 2a^2 d + abd)\lambda t\right\} \tag{22}$$

$$D_1 = \frac{3}{16\lambda^3}\left\{(2abc + a^2 d + 3b^2 d)\lambda t - (3a^2 c + b^2 c + 2abd)\right\} \tag{23}$$

Inputting the values of C_1 and D_1 from equations (22) and (23) into the equations (18) and (19), and solving them, we get

$$A_1=\frac{3}{32\lambda^4}\Big\{(3a^2c+b^2c+2abd)+(2a^2b+2b^3)\lambda+(2a^3\lambda+2ab^2\lambda-4abc-2a^2d-6b^2d)\lambda t \\ -(6a^2c+2b^2c+4abd)\lambda^2t^2\Big\} \tag{24}$$

$$B_1=\frac{3}{16\lambda^4}\Big\{(abc+2a^2d+6b^2d)-(a^3+ab^2)\lambda+(a^2c+b^2c+2abd+a^2b\lambda+b^3\lambda)\lambda t \\ -(2abc+a^2d+3b^2d)\lambda^2t^2\Big\} \tag{25}$$

The solution of the equation (17) becomes

$$u_1=u_{11}\cos\lambda t+u_{12}\sin\lambda t+u_{13}\cos 3\lambda t+u_{14}\sin 3\lambda t \tag{26}$$

where
$$u_{11}=\frac{3}{128\lambda^7}(9c^2d+9d^3+15ac^2\lambda+5bcd\lambda+5ad^2\lambda)+\frac{3t}{128\lambda^6}(15c^3+15cd^2-bc^2\lambda-acd\lambda-3bd^2\lambda) \\ -\frac{9t^2}{64\lambda^5}(2c^2d+2d^3+3ac^2\lambda+2bcd\lambda+9ad^2\lambda)+\frac{t^3}{64\lambda^4}(bc^2\lambda+8acd\lambda+12bd^2\lambda-9c^3-9cd^2) \\ +\frac{t^4}{64\lambda^3}(3c^2d+3d^3+3ac^2+2bcd\lambda+ad^2\lambda)+\frac{3t^3}{320\lambda^2}(c^3+cd^2)$$

$$u_{12}=\frac{3}{128\lambda^7}(5bc^2\lambda+10acd\lambda+15bd^2-9c^3-9cd^2)+\frac{3t}{128\lambda^6}(15c^2d+15d^3+24ac^2\lambda+16bcd\lambda+8ad^2\lambda) \\ +\frac{9t^2}{64\lambda^5}(2c^3+2cd^2-bc^2\lambda-2acd\lambda-3bd^2\lambda)-\frac{t^3}{64\lambda^4}(9c^2d+9d^3+3ac^2\lambda+bcd\lambda+ad^2\lambda) \\ +\frac{t^4}{64\lambda^3}(bc^2\lambda+2acd\lambda+3bd^2\lambda-3c^3-3cd^2)+\frac{3t^5}{320\lambda^2}(c^2d+d^3)$$

$$u_{13}=\frac{1}{2048\lambda^7}(162c^2d-54d^3+69ac^2-138bcd\lambda-69ad^2\lambda-72abc\lambda^2-36a^2d\lambda^2+36b^2d\lambda^2-8a^3\lambda^3 \\ +24ab^2\lambda^3)+\frac{3t}{2048\lambda^6}(23c^3-69cd^2-24bc^2\lambda-48acd\lambda+24bd^2\lambda-8a^2c\lambda^2+8b^2c\lambda^2+16abd\lambda^2) \\ +\frac{3t^2}{512\lambda^5}(2ad^2\lambda+4bcd\lambda+3d^3-2ac^2\lambda-9c^2d\lambda)+\frac{t^3}{256\lambda^4}(3cd^2-c^3)$$

$$u_{14}=\frac{1}{2048\lambda^7}(162cd^2-54c^3+69bc^2\lambda+138acd\lambda-69bd^2\lambda+36a^2c\lambda^2-36b^2c\lambda^2-72abd\lambda^2 \\ -24a^2b\lambda^3+8b^3\lambda^3)+\frac{3t}{2048\lambda^6}(69c^2d-69d^3+24ac^2\lambda-48bcd\lambda-24ad^2\lambda-12abc\lambda^2-8a^2d\lambda^2 \\ +8b^2d\lambda^2)+\frac{3t^2}{512\lambda^5}(3c^3-9cd^2-2bc^2\lambda-4acd\lambda+2bd^2\lambda)+\frac{t^3}{256\lambda^4}(d^3-3c^2d)$$

Substituting the values of A_1, B_1, C_1, D_1 from equations (24), (25), (22) and (23) respectively into equation (4), we get

$$\dot a=\frac{3\varepsilon}{32\lambda^4}\Big\{(3a^2c+b^2c+2abd)+(2a^2b+2b^3)\lambda+(2a^3\lambda+2ab^2\lambda-4abc-2a^2d-6b^2d)\lambda t \\ -(6a^2c+2b^2c+4abd)\lambda^2t^2\Big\}$$
$$\dot b=\frac{3\varepsilon}{16\lambda^4}\Big\{(abc+2a^2d+6b^2d)-(a^3+ab^2)\lambda+(a^2c+b^2c+2abd+a^2b\lambda+b^3\lambda)\lambda t \\ -(2abc+a^2d+3b^2d)\lambda^2t^2\Big\}$$
$$\dot c=\frac{3\varepsilon}{8\lambda^3}\Big\{(a^2c+2a^2d+6b^2d)+(6a^2c+2a^2d+abd)\lambda t\Big\}$$
$$\dot d=\frac{3}{16\lambda^3}\Big\{(2abc+a^2d+3b^2d)\lambda t-(3a^2c+b^2c+2abd)\Big\} \tag{27}$$

Finally, we discover that all of the equations of (27) have no exact solutions. Nonetheless, since $\dot a, \dot b, \dot c, \dot d$ are proportional to the small parameter ε, they are slowly varying functions of time t. It is, therefore, possible to replace a, b, c, d by their respective values obtained in linear case (*i.e.*, the values of a, b, c, d obtained when $\varepsilon=0$) in the right hand side of equation (27). This type of substitution was first introduced by Murty *et al*. [16], and Murty and Deekshatulu [5] to solve similar type of nonlinear equations. Thus, the solution of (27) becomes

$$\begin{aligned}
a &= a_0 + \frac{3\varepsilon}{32\lambda^4}\Big\{(3a_0{}^2c_0 + b_0{}^2c_0 + 2a_0b_0d_0)t + (2a_0{}^2b_0 + 2b_0{}^3)\lambda t + (a_0{}^3\lambda + a_0b_0{}^2\lambda \\
&\quad - 2a_0b_0c_0 - a_0{}^2d_0 - 3b_0{}^2d_0)\lambda t^2 - (2a_0{}^2c_0 + \frac{2}{3}b_0{}^2c_0 + \frac{4}{3}a_0b_0d_0)\lambda^2t^3\Big\} \\
b &= b_0 + \frac{3\varepsilon}{48\lambda^4}\Big\{3(a_0b_0c_0 + 2a_0{}^2d_0 + 6b_0{}^2d_0)t - 3(a_0{}^3 + a_0b_0{}^2)\lambda t + \frac{3}{2}(a_0{}^2c_0 + b_0{}^2c_0 \\
&\quad + 2a_0b_0d_0 + a_0{}^2b_0\lambda + b_0{}^3\lambda)\lambda t^2 - (2a_0b_0c_0 + a_0{}^2d_0 + 3b_0{}^2d_0)\lambda^2t^3\Big\} \\
c &= c_0 + \frac{3\varepsilon}{16\lambda^3}\Big\{(2a_0{}^2c_0 + 4a_0{}^2d_0 + 12b_0{}^2d_0) + (6a_0{}^2c_0 + 2a_0{}^2d_0 + a_0b_0d_0)\lambda t\Big\} \\
d &= d_0 + \frac{3\varepsilon}{32\lambda^3}\Big\{(2a_0b_0c_0 + a_0{}^2d_0 + 3b_0{}^2d_0)\lambda t^2 - (6a_0{}^2c_0 + 2b_0{}^2c_0 + 4a_0b_0d_0)t\Big\}
\end{aligned} \tag{28}$$

Consequently, we obtain the first approximate solution of the equation (15) as

$$x(t,\varepsilon) = \{(a+ct)\cos\lambda t + (b+dt)\sin\lambda t\} + \varepsilon u_1 \tag{29}$$

where a, b, c, d are given by the equation (28) and u_1 is given by (26).

4. Results and Discussion

In order to test the accuracy of our analytical solution obtained by certain perturbation method, we contrast the perturbation results to the numerical results obtained using *Mathematica 9.0* for the different sets of initial conditions. Here, $x(t, \varepsilon)$ is computed by equation (29), where a, b, c, d are calculated from equation (28); and (26) is used to obtain u_1 when $\varepsilon = 0.1$, together with the different sets of initial conditions. We get the results from (29) for different values of t and figure out the corresponding numerical solution using *Mathematica 9.0*. Figure 1 to Figure 4 represent all the results, which show the perturbation results and numerical results coincide satisfactorily.

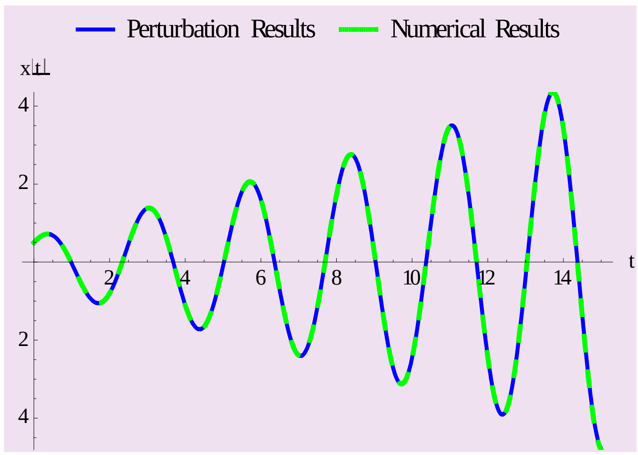

Figure 2. Comparison between perturbation and numerical results for $\lambda = 3\pi/4$ and $\varepsilon = 0.1$ with the initial conditions $a_0 = 0.50$, $b_0 = 0.40$, $c_0 = 0.15$, $d_0 = 0.20$

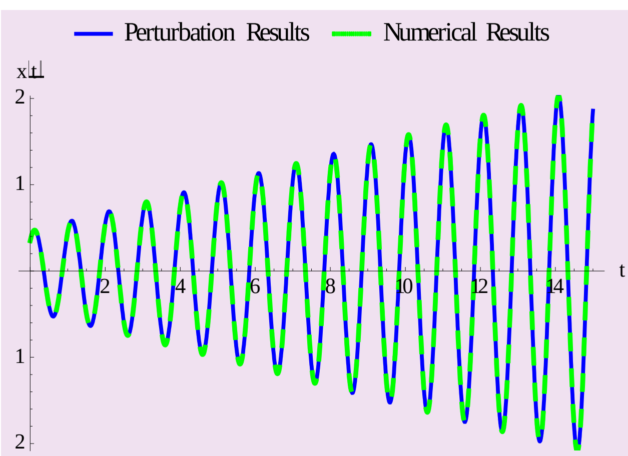

Figure 1. Comparison between perturbation and numerical results for $\lambda = 2\pi$ and $\varepsilon = 0.1$ with the initial conditions $a_0 = 0.35$, $b_0 = 0.40$, $c_0 = 0.05$, $d_0 = 0.25$

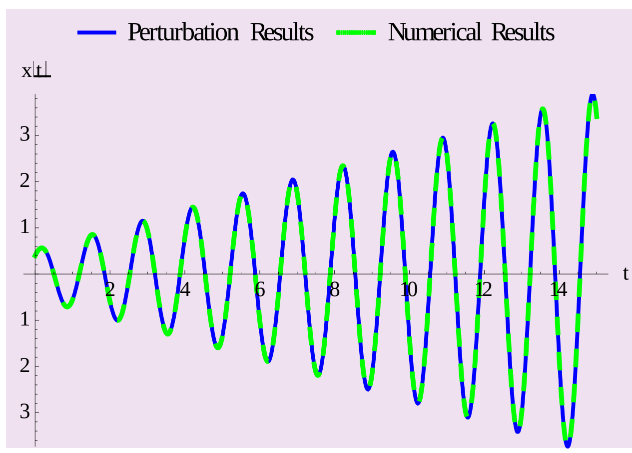

Figure 3. Comparison between perturbation and numerical results for $\lambda = 3\pi/2$ and $\varepsilon = 0.1$ with the initial conditions $a_0 = 0.40$, $b_0 = 0.35$, $c_0 = 0.10$, $d_0 = 0.20$

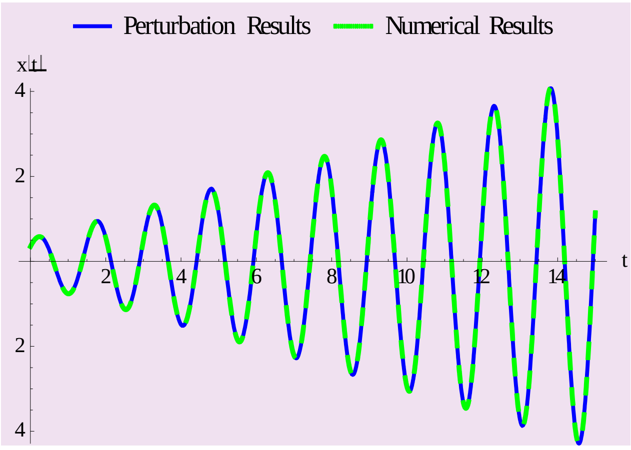

Figure 4. Comparison between perturbation and numerical results for $\lambda = 4\pi/3$ and $\varepsilon = 0.1$ with the initial conditions $a_0 = 0.35$, $b_0 = 0.40$, $c_0 = 0.05$, $d_0 = 0.25$

5. Conclusions

In conclusion, it can be said that we have modified the Krylov-Bogoliubov-Mitropolskii (KBM) method in this paper, which is regarded as one of the most widely used methods to study the transient behaviour of oscillating systems. Subsequently, we have successfully applied the modified KBM method to the fourth order critically nonlinear oscillatory systems. In relation to the fourth order critically nonlinear oscillatory systems, we have obtained the solutions in such circumstances where in the eigenvalues are pairwise equal and imaginary. It should be noted here that much error generally occurs in the KBM method in the case of rapid changes of x with respect to time t. Nevertheless, all the above figures reveal that, with respect to the different sets of initial conditions, the perturbation solutions of the modified KBM method correspond completely to the numerical solutions. It is, therefore, suggested that the modified KBM method gives highly accurate results which can be applied for different types of nonlinear differential systems.

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