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Applied Problems of Markov Processes

Roza Shakenova*, Alua Shakenova

Kazach National Research Technical University Named after K. I. Satpaev, Almaty, Kazakhstan

Abstract In this paper authors proposed that the human-being's state of health can be described as the function of three main factors: economics, ecology and politics of government. We obtained three models of the state of health from the worst to the best using Markov processes. We hope that our theoretical models can be applied in practice.

Keywords Limiting probability, Probability of state, Markov Processes

1. Introduction

Markov chains can be used in the description of different economical, ecological problems. The main problem is to find final (limit) probabilities of different states of a certain system. We may use graphs for the system with discrete states. The vertices of graph correspond to the states of the system on the picture #1. The arrows between graphs show the possibility of system's transfer from one state to another. The final states of human-beings' health are very important in medicine. Therefore, this paper considers the problem of determining final probabilities of human-being's state of health. Let's recall basic definitions.

Law of the random variable distribution is every relationship between the possible values of a random variable and their probabilities. The table below shows the law of distribution of a discrete random variable.

x_i	x_1	$x_2 \dots x_n$	
p_i	p_1	$p_2 \dots p_n$	$\sum_{i=1}^{n} p_i = 1$

The set of different states of one physical system with discrete states, where random process occurs, is finite or countable.

$$s_1, s_2, \dots, s_i, \dots \tag{1}$$

Random process is the sequence of events of the following type (1). We will consider this sequence («chain») of events. We will confine to the finite number of states. The system is able to transfer from one state S_i to another state S_j directly or through other states [2], [3]. Usually the graph of states describes the states visually, where the vertices of graph correspond to the states. If the probability of each state in the future for every time t_0 ($t > t_0$) of the system S with discrete states,, S_1, $S_2, \dots S_i, \dots$, depends on the state in the present ($t = t_0$), and does not depend on its behavior in the past (when $t < t_0$), then the random process is **Markov process**. Let's consider conditional probability of transferring of system S on the kth step in the state S_j, if it is known that it was in the state S_i on the (k-1)th step. Denote this probability by:

$$p_{ij}(k) = P\left\{S(k) = s_j \middle| S(k-1) = s_i\right\},$$
$$(i, j = 1, 2, \dots, n) \tag{2}$$

Probability $p_{ii}(k)$ is the probability that the system remains in the state s_i on kth step. Probabilities $p_{ij}(k)$ are transition probabilities of Markov chain on the kth step. Transition probabilities can be written in the form of square table (matrix) of size n. This is **Stochastic matrix**; the sum of all probabilities of one row is equal to 1 $\sum_{j=1}^{n} p_{ij} = 1$, because the system can be in one of mutually exclusive states.

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}. \tag{3}$$

* Corresponding author:
rshaken@mail.ru (Roza Shakenova)
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In order to find unconditional probabilities we need to know initial probability distribution, i.e. probabilities at the beginning of the process at time $t_0 = 0$: $p_1(0)$, $p_2(0)$,…, $p_n(0)$, with their sum equal to one.

Markov chain is called uniform, if transition probabilities do not depend on the step's number (3).

Let's consider only uniform Markov chains to simplify life.

2. The Formula for Total Probability

Let the event A can occur together with only one of following events H_1, H_2, …, H_n, which form the full group of pair wise mutually exclusive events, i.e. $H_i \cdot H_j = 0$, $i \neq j$ and $\sum_{i=1}^{n} H_i = \Omega$. Then the probability of A can be calculated using the formula of total probability.

$$P(A) = \sum_{i=1}^{n} P(H_i) \cdot P(A|H_i) \,. \tag{4}$$

Where H_1, H_2,..., H_n are called hypotheses, and the values $P(H_i)$ - probabilities of hypotheses.

Make hypothesis such that the system was in the state s_i at initial time with (k=0). The probability of this hypothesis is known and equal to $p_i(0) = P\{S(0) = s_i\}$. Assuming that this hypothesis takes place, the conditional probability of System S being in the state s_j on the first step is equal to transition probability

$$p_{ij} = P\left\{S(1) = s_j \middle| S(0) = s_i\right\}$$

Applying formula for total probability we obtain the following:

$$p_j(1) = \sum_{i=1}^{n} P\left\{S(1) = s_j \middle| S(0) = s_i\right\} \cdot P\left\{S(0) = s_i\right\} =$$

$$= \sum_{i=1}^{n} p_{ij}(0), \ (j = 1,2,...,n). \tag{5}$$

Now, find the distribution of probabilities on the second step, which depends on the distribution of probabilities on the first step and matrix of transition probabilities for Markov chain. Again make hypothesis such that the system was in the state s_i on the first step, The probability of this hypothesis is known and equal to $p_i(1) = P\{S(1) = s_i\}$, $(i = 1,2,...,n)$. Given this hypothesis, the conditional probability of the system being in the state s_j on the second step is equal to $p_{ij} = P\left\{S(2) = s_j \middle| S(1) = s_i\right\}$

Using the formula for total probability we obtain:

$$p_j(2) = \sum_{i=1}^{n} p_i(1) \cdot p_{ij}, \ (j = 1,2,...,n).$$

Applying this method several times we obtain recurrent formula:

$$p_j(k) = \sum_{i=1}^{n} p_i(k-1) \cdot p_{ij},$$

$$(k = 1,2,...; j = 1,2,...,n) \tag{6}$$

In some conditions in Markov chain with the increase of the step k the stationary mode sets up, at which the system continues to wander over states, but the probabilities of these states do not depend on the number of step. These probabilities are denoted **final (limit) probabilities of Markov chains**. The equations for such probabilities can be written using mnemonic rule: Given stationary mode, total probability flux of the system remains constant: the flow into the state s is equal to the flow out of the state s.

$$\sum_{i=1}^{n} p_i \cdot p_{ij} = \sum_{i=1}^{n} p_j \cdot p_{ji} = p_j \cdot \sum_{i=1}^{n} p_{ji},$$

$$(i \neq j), \quad (j = 1,2,...,n). \tag{7}$$

This condition is balance condition for the state s_j. Add normalization condition $\sum_{i=1}^{n} p_i = 1$ to these n equations.

3. Markov Chains. The state of Health

Let the System C be human-being from certain ecological, economic sphere. It can be in the following states:

c_1 - healthy and works;

c_2 - ill, but the illness is not discovered;

c_3 - ill, the light treatment;

c_4 - does not work, under observation;

c_5 - ill, under serious medical treatment. Marked graph for the state of human-being is shown on the picture #1.

The problem: To construct the equation and find the final probabilities of human-being's state of health.

Solution: Let's consider c_5 on the graph. Two arrows are directed into this state; consequently, there are two terms for addition on the left side (7) for $j=5$ (state c_5). One arrow is directed out of this state, subsequently, there is only one term on the right side (7) for $j=5$ (state c_5). Hence, using balance condition (7), we obtain the first equation:

$$p_2 \cdot p_{25} + p_4 \cdot p_{45} = p_5 \cdot p_{51} \qquad (8)$$

Similarly, we write three more equations:

$$p_1 \cdot p_{12} = p_2(p_{23} + p_{25}),$$

$$p_2 \cdot p_{23} + p_4 \cdot p_{43} = p_3 \cdot p_{31}$$

$$p_1 \cdot p_{14} = p_4(p_{41} + p_{43} + p_{45})$$

The fifth equation is the normalization condition:

$p_1 + p_2 + p_3 + p_4 + p_5 = 1$. We rewrite the system of equations in the following way:

1) $p_5 = \dfrac{(p_2 \cdot p_{25} + p_4 \cdot p_{45})}{p_{51}}$,

2) $p_2 = \dfrac{p_1 \cdot p_{12}}{(p_{23} + p_{25})}$,

3) $p_3 = \dfrac{(p_2 \cdot p_{23} + p_4 \cdot p_{43})}{p_{31}}$,

4) $p_4 = \dfrac{p_1 \cdot p_{14}}{(p_{41} + p_{43} + p_{45})}$,

5) $p_1 + p_2 + p_3 + p_4 + p_5 = 1$.

Let's solve the system of equations. From 2) we find:

$p_2 = d_2 \cdot p_1$, where $d_2 = \dfrac{p_{12}}{(p_{23} + p_{25})}$. From 4) we find:

$p_4 = d_4 \cdot p_1$, where $d_4 = \dfrac{p_{14}}{(p_{41} + p_{43} + p_{45})}$. From 3) find:

$$p_3 = \frac{(d_2 \cdot p_{23} + d_4 \cdot p_{43}) \cdot p_1}{p_{31}} = d_3 \cdot p_1,$$

where $d_3 = \dfrac{(d_2 \cdot p_{23} + d_4 \cdot p_{43})}{p_{31}}$. From 1)

find: $p_5 = \dfrac{(d_2 \cdot p_{25} + d_4 \cdot p_{45}) \cdot p_1}{p_{51}} = d_5 \cdot p_1$,

where $d_5 = \dfrac{(d_2 \cdot p_{25} + d_4 \cdot p_{45})}{p_{51}}$.

Giving corresponding values of probabilities:

$p_{43} = 0.2$, $p_{14} = 0.1$, $p_{41} = 0.6$,
$p_{45} = 0.2$, $p_{31} = 0.5$, $p_{51} = 0.1$,
$p_{23} = 0.6$, $p_{12} = 0.2$, $p_{25} = 0.2$.

Calculating the following values:

$$d_2 = \frac{0.2}{(0.6 + 0.2)} = \frac{0.2}{0.8} = 0.25 = \frac{1}{4},$$

$$d_4 = \frac{0.1}{(0.6 + 0.2 + 0.2)} = \frac{0.1}{1} = 0.1 = \frac{1}{10}.$$

$$d_3 = \frac{(0.6 \cdot 0.25 + 0.1 \cdot 0.2)}{0.5} = 0.34,$$

$$d_5 = \frac{(0.25 \cdot 0.2 + 0.1 \cdot 0.2)}{0.1} = \frac{0.07}{0.1} = 0.7 \cdot$$

According to the equality5) we have:

$$p_1 + 0.25 \cdot p_1 + 0.34 \cdot p_1 + 0.1 \cdot p_1 + 0.7 \cdot p_1 = 1$$

$$p_1(1 + 0.25 + 0.34 + 0.1 + 0.7) = 1,$$

so $p_1 = \dfrac{1}{2.39} = \dfrac{100}{239}$.

$$p_2 = d_2 \cdot p_1 = \frac{100}{239} \cdot \frac{1}{4} = \frac{25}{239},$$

$$p_3 = d_3 \cdot p_1 = \frac{34}{100} \cdot \frac{100}{239} = \frac{34}{239}.$$

$$p_4 = d_4 \cdot p_1 = \frac{1}{10} \cdot \frac{100}{239} = \frac{10}{239},$$

$p_5 = d_5 \cdot p_1 = \dfrac{7}{10} \cdot \dfrac{100}{239} = \dfrac{70}{239}$. Normalization condition $p_1 + p_2 + p_3 + p_4 + p_5 = 1$

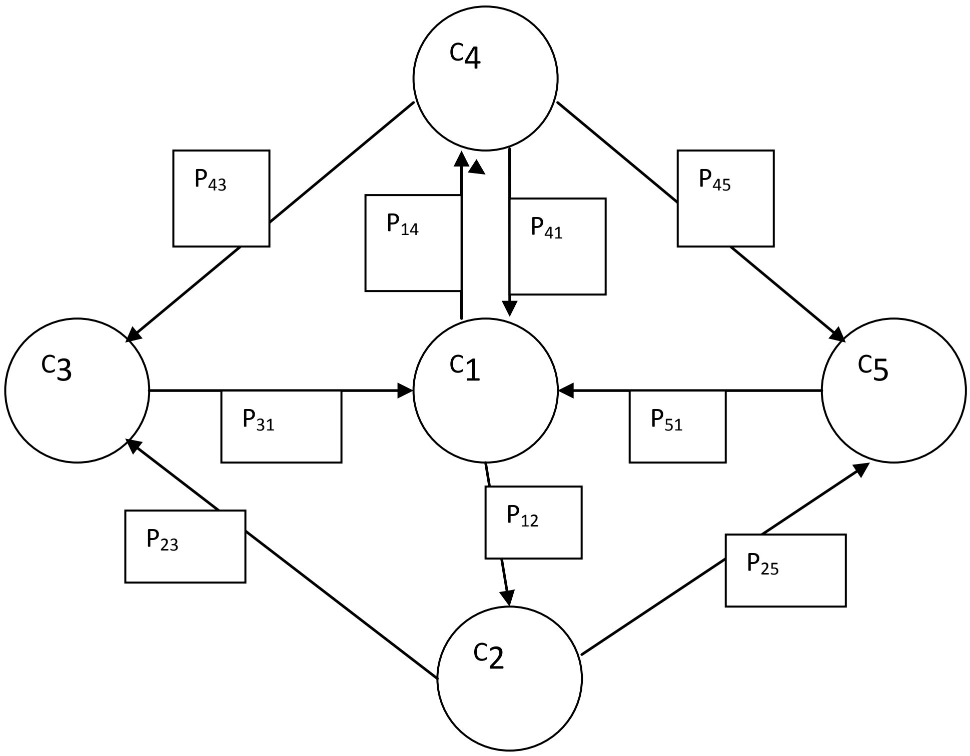
$$\frac{100}{239} + \frac{25}{239} + \frac{34}{239} + \frac{10}{239} + \frac{70}{239} = \frac{239}{239} = 1$$

works. We did not need the probabilities p_{11}, p_{22}, p_{33}, p_{44}, p_{55}.

4. Conclusions

In summary, we have found final probabilities of human-being's state of health, considering the system C- a human-being from a certain ecological, economical sphere. Looking at initial stages of disease in medicine it is possible to make prognosis about the final probabilities of sick human-being's state of health. Doctors together with researchers could invent such project for human-being's health recovery.

Appendix



Picture #1.

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