

Oscillatory Motion of a Viscous Fluid in a Thin-Walled Elastic Tube with Induced Magnetic Field: A Proposed Therapy for Cancer and Hypertension Treatment

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Abstract The motion of a fluid in an elastic tube when subjected to a pressure-gradient which is a periodic function of time is modeled and analyzed. This was done by considering the equations of the motion of the fluid and those of the motion of the tube. We obtained the average velocity across the tube and also a better approximation for the rate of flow was obtained.

Keywords Pressure-Gradient, Periodic function, Oscillatory motion, Thin-Walled elastic tube, Magnetic field

1. Introduction

The problem of determining the motion of a liquid in an elastic tube when subjected to a pressure-gradient which is a periodic function of time, arises in connection with the flow of blood in the large arteries [1]. Performance modeling and analysis of blood flow in elastic arteries was investigated by G.C. Sharma et al. They estimated the effect of magnetic field on rheological model of blood. One of these models, is the Power-Low model and the other is generalized Maxwell model [2]. Thurston [8] attempted to study all of the rheological properties of blood with a model including non-Newtonian Viscosity, viscoelasticity and thixotropy.

J. C. Misra and G. C. Shit studied the effect of magnetic field on blood flow through an artery, using a numerical model. Blood consisting of a suspension of red blood cells containing hemoglobin which contains iron oxide, it is quite apparent that blood is electrically conducting and exhibits magnetohydrodynamic flow characteristics [3]. P.N. Habu et al [12] and [13] investigated the effect of Magnetic Field in an Oscillatory flow and particle suspension in a fluid through an elastic tube:- An application to blood flow in arteries and also Oscillatory flow and particle suspension in a fluid through an elastic tube, respectively.

Erica M. Cherry et al [4] studied Magnetic Drug Targeting Applications (MDTA) is a promising proposed technique for treating localized disease such as cancer [4]. In the present study, we investigated the Oscillatory motion of a viscous fluid in a thin-walled elastic tube with induced magnetic

field. This can be considered as a proposed therapy for treating localized diseases such as cancer.

The equations of motion of the tube are:

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{\rho_0 v}{\rho h R} \left[\frac{\partial w}{\partial y} + R \frac{\partial u}{\partial z} / y = 1 \right] + \frac{B}{\rho} \left[\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma}{R} \frac{\partial \xi}{\partial z} \right] \quad (1)$$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{P}{h \rho} - \frac{B}{\rho} \left[\frac{\sigma}{R} \frac{\partial \xi}{\partial z} + \frac{\xi}{R^2} \right] \quad (2)$$

with the boundary conditions for the motion of the fluid

$$u = \frac{\partial \xi}{\partial t} \quad \text{at } y = 1 \quad (3)$$

$$w = \frac{\partial \zeta}{\partial t} \quad \text{at } y = 1 \quad (4)$$

The equations of motion of the fluid are [12]

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} &= \frac{1}{\rho_0} \frac{\partial p}{\partial r} \\ &+ v \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial r^2} \right] - \sigma e B_0^2 u \end{aligned} \quad (5)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = \frac{1}{\rho_0} \frac{\partial p}{\partial z} + v \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right] \quad (6)$$

together with the continuity equation as

$$\frac{\partial u}{\partial t} + \frac{u}{r} + w \frac{\partial w}{\partial z} = 0 \quad (7)$$

Since the fluid is subjected to a pressure-gradient which is a periodic function of time, we can make the following assumptions as in [12]

$$P = P_1 \exp[in(t - z/c)] \quad (8)$$

$$u = u_1 \exp[in(t - z/c)] \quad (9)$$

$$w = w_1 \exp[in(t - z/c)] \quad (10)$$

with the Womersley Number $\alpha^2 = \frac{n^2 R}{v}$ and

$$Y = \frac{r}{R} \quad (11)$$

and B_0^2 is a constants.

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Neglecting the inertia terms in eqn (5) and (6) and substitute (8) to (10) into (5) and (6) respectively, we obtain

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{y} \frac{\partial u_1}{\partial y} + i^3 \alpha^2 u_1 - \frac{u_1}{y^2} = \frac{1}{R} \cdot \frac{R^2}{\rho} \frac{\partial \rho_1}{\partial y} - \sigma e B_0^2 u_1 \quad (12)$$

$$\frac{\partial^2 w_1}{\partial y^2} + \frac{1}{y} \frac{\partial w_1}{\partial y} + i^3 \alpha^2 w_1 = \frac{in}{c} \cdot \frac{R^2}{\mu} \rho_1 \quad (13)$$

where $\alpha^2 = \frac{n^2 R}{v}$

where the terms in $\frac{\partial^2 w}{\partial z^2}$, $\frac{\partial u}{\partial z^2}$ have been omitted, since $\frac{n^2 R}{C^2}$ is small.

Similarly, the equation of continuity (7) becomes

$$\frac{1}{y} \frac{d}{dy} (u_1 y) = \frac{inR}{c} w_1 \quad (14)$$

If now it is assumed that $P_1 = A_0(ky)$, where k is to be determined, the equation of motion can be integrated as in [13], to give

$$u_1 = \frac{c_{10}(i^{3/2} \gamma y)}{o(i^{3/2} \gamma)} + \frac{1}{R} \frac{R^2}{\mu} \frac{A_1 k_1(ky)}{(i^{3/2} \gamma - k^2)} \quad (15)$$

$$w_1 = \frac{c_{20}(i^{3/2} \gamma y)}{o(i^{3/2} \gamma)} + \frac{B_1 A_1}{(i^{3/2} \gamma - k^2)} o(ky) \quad (16)$$

where $\gamma^2 = i^3 \alpha^2 \beta^2$ and $C_2 = A_{10}(ky)$

C_1 and C_2 are arbitrary constants. If these values of u_1 and w_1 are inserted in the equation (7) it should reduce to an identity, i.e from (15) and (16), using (7), we get

$$\begin{aligned} & \frac{1}{y} \frac{d}{dy} \left[\frac{y c_{11}(i^{3/2} \gamma)}{o(i^{3/2} \gamma)} - \frac{y}{R} \cdot \frac{R^2}{\mu} \frac{M A_1 k_1(ky)}{(i^{3/2} \gamma - k^2)} \right] \\ &= \frac{inR}{c} \left[\frac{c_{20}(i^{3/2} \gamma y)}{o(i^{3/2} \gamma y)} - \frac{inR^2}{c \mu} \frac{A_{10}(ky)}{(i^{3/2} \gamma - k^2)} \right] \end{aligned}$$

$$\text{i.e } \frac{1}{y} \frac{d}{dy} (u_1 y) = \gamma i^{3/2} \frac{c_{10}(i^{3/2} \gamma y)}{o(i^{3/2} \gamma)} - \frac{R k^2}{\mu} \frac{M A_{10}(ky)}{(i^{3/2} \gamma - k^2)} \quad (17)$$

$$\text{and } \frac{inR}{c} w_1 = \frac{inR}{c} \frac{c_{20}(i^{3/2} \gamma y)}{o(i^{3/2} \gamma)} - \frac{i^2 n^2 R^3}{c^2 \mu} \frac{A_{10}(ky)}{(i^{3/2} \gamma - k^2)} \quad (18)$$

For the equation of continuity (14) to be satisfied, (17) and (18) must be identical, i.e

$$\begin{aligned} & \gamma i^{3/2} \frac{c_{10}(i^{3/2} \gamma y)}{o(i^{3/2} \gamma)} - \frac{R k^2}{\mu} \frac{A_{10}(ky)}{(i^{3/2} \gamma - k^2)} \\ &= \frac{inR}{c} \frac{c_{20}(i^{3/2} \gamma y)}{o(i^{3/2} \gamma)} - \frac{i^2 n^2 R^3}{c^2 \mu} \frac{A_{10}(ky)}{(i^{3/2} \gamma - k^2)} \end{aligned} \quad (19)$$

Using (19), equating coefficients of $o(i^{3/2} \gamma y)$ and $A(ky)$, we obtain

$$\frac{\gamma i^{3/2} c_1}{o(i^{3/2} \gamma)} = \frac{inR c_2}{c o(i^{3/2} \gamma)} \quad (20)$$

and

$$\frac{-R k^2 A_1}{\mu(i^{3/2} \gamma^2 - k^2)} = -\frac{i^2 n^2 R^3 A_1}{c^2 \mu(i^{3/2} \gamma^2 - k^2)} \quad (21)$$

From (20) we have

$$\frac{c_1}{c_2} = \frac{inR}{c \gamma i^{3/2}} \quad (22)$$

$$\text{From (21)} \quad k^2 = \frac{i^2 n^2 R^3 A}{R A_1 c^2} = \frac{i^2 n^2 R^2 A_1}{A_1 c^2}$$

$$\frac{i^2 n^2 R^2}{c^2} \Rightarrow k = \frac{inR}{c} \quad (23)$$

From [1] we use the approximations,

$$o(ky) = I_0\left(\frac{nR}{c}\right), \quad i_1(ky) = i I_1\left(\frac{nR}{c}\right), \quad i^3 \gamma^2 - k^2 = i^3 \gamma^2,$$

$$I_1\left(\frac{nRy}{c}\right) = \frac{nRy}{2c}, \quad \text{inserting these approximations into}$$

(15) and (16) we get

$$w_1 = \frac{c_{20}(i^{3/2} \gamma y)}{(i^{3/2} \gamma)} + A_1 \quad (24)$$

$$u_1 = \frac{inR}{2c} \left[\frac{c_{121}(\gamma i^{3/2} \gamma y)}{\gamma i^{3/2} o(i^{3/2} \gamma)} + \frac{y A_1}{\rho_0 c} \right] \quad (25)$$

At the inner surface of the tube, i.e at $y = 1$,

$$w_1 = c_2 + \frac{A_1}{\rho_0 c} \quad (26)$$

$$u_1 = \frac{inR c_1}{2c} F_{10}(\gamma) + \frac{1}{2} \frac{inR}{c} \frac{A_1}{\rho_0 c} \quad (27)$$

$$\text{where} \quad F_{10}(\gamma) = \frac{2_1(\gamma i^{3/2} \gamma)}{\gamma i^{3/2} o(i^{3/2} \gamma)} \quad (28)$$

The value of $\frac{\partial w_1}{\partial y}$ at $y = 1$, is also needed to be substituted in the equation of motion of the tube.

Using eqn (16)

$$\frac{\partial w_1}{\partial y} = -\frac{c_1}{2} i^3 \gamma^2 F_{10}(\gamma) + \frac{1}{2} \frac{n^2 R^2 A_1}{c^2 \rho_0 c} \quad \text{at } y = 1 \quad (29)$$

If it is now assumed that

$$\xi = D_1 \exp \left[in \left(t - \frac{z}{c} \right) \right] \quad (30)$$

$$\zeta = E_1 \exp \left[in \left(t - \frac{z}{c} \right) \right] \quad (31)$$

where D_1 and E_1 are arbitrary constants, the boundary condition for u_1 and w_1 becomes

$$\frac{\partial \xi}{\partial t} = D_1 in \exp \left[in \left(t - \frac{z}{c} \right) \right] = u \text{ at } y = 1 \quad (32)$$

$$\frac{\partial \zeta}{\partial t} = E_1 \exp \left[in \left(t - \frac{z}{c} \right) \right] = w \text{ at } y = 1 \quad (33)$$

From eqn (32) and (33)

$$D_1 in = u_1 = \frac{1}{2} \frac{inR}{c} F_{10}(\gamma) c_1 + \frac{A_1}{\rho_0 c} \quad (34)$$

and similarly,

$$in E_1 = c_2 + \frac{A_1}{\rho_0 c} \quad (35)$$

and the equation of motion of the tube can be written as

$$-n^2 D_1 = \frac{A_1}{h\rho} - \frac{B}{\rho} \left[\frac{\sigma}{R} \left(\frac{-inE_1}{c} \right) + \frac{D_1}{R^2} \right] \quad \text{and} \quad (36)$$

$$-n^2 E_1 = \frac{\rho_0 v}{\rho h R} \left[-\frac{c_2}{2} i^3 \gamma^2 F_{10}(\gamma) + \frac{1}{2} \frac{A_1}{\rho_0 c} \frac{n^2 R^2}{c^2} \right] + \frac{B}{\rho} \left[\frac{-n^2 E_1}{c^2} + \frac{\sigma}{R} \left(\frac{-in}{c} \right) D_1 \right] \quad (37)$$

Eqns (34), (35), (36) and (37) are four homogeneous equations in the arbitrary constants A_1 , c_2 , D_1 and E_1 .

Eliminating them will give a “frequency equation” which will determine the wave-velocity, c , in terms of the elastic properties of the tube and the non-dimensional parameter γ . From [12], the result of the elimination will give

$$\begin{vmatrix} \frac{1}{\rho_0 c} & 1 & 0 & -in \\ \frac{inR}{2c^2 \rho_0} & \frac{F_{10}(\gamma)inR}{2c} & -in & 0 \\ \frac{1}{h\rho} & 0 & n^2 - \frac{B}{\rho R^2} & \frac{in\sigma B}{cR\rho} \\ \frac{-\rho_0 v}{2\rho} \frac{1}{hR\rho_0 c} \cdot \frac{n^2 R^2}{c^2} & \frac{-i/2\rho_0 v \gamma^2 F_{10}(\gamma)}{\rho h R} & \frac{-in\sigma B}{cR\rho} & n^2 \left(1 - \frac{B}{\rho c^2} \right) \end{vmatrix} = 0 \quad (38)$$

Operating on the columns and row of equation (38) several times we obtain

$$\begin{vmatrix} 1 - F_{10}(\gamma) & 2 & 1 \\ 1 & x & 1 + \sigma x \\ \frac{1}{2} F_{10}(\gamma) R & -\sigma x & (k - x) \end{vmatrix} = 0 \quad (39)$$

where $k = \frac{h\rho}{R\rho_0}$, $x = \frac{kB}{\rho_0 c^2}$

(39) becomes,

$$(1 - \sigma^2)(1 - F_{10}(\gamma))x^2 - x \left[2 + k(1 - F_{10}(\gamma)) + F_{10}(\gamma) \left(\frac{1}{2} - 2\sigma \right) \right] + F_{10}(\gamma) + 2k = 0 \quad (40)$$

Hence the roots of (40) are given by [12]

$$(1 - \sigma^2)x = P \pm [P^2 - (1 - \sigma^2)H] \quad (41)$$

where $P = \frac{1 + \frac{1}{4} - \sigma}{1 - F_{10}(\gamma)} + \left(\frac{k}{2} + \sigma - \frac{1}{4} \right)$ (42)

$$H = \frac{1 + 2k}{1 - F_{10}(\gamma)} - 1 \quad (43)$$

Now the details of the motion of the fluid at a particular value of Z , i.e over a short length of artery, will be studied. If the origin of Z is taken at the midpoint of this short length, the longitudinal velocity will be [1]

$$w = \left[\frac{A_1}{\rho_0 c} + c_3 \frac{o(\gamma i^{3/2} \gamma)}{o(i^{3/2} \gamma)} \right] e^{int} \quad (44)$$

where the value of c_3 is to be determined from the boundary conditions. From (34), (35) and (36), D_1 and E_1 can be eliminated. If in the resulting equation, \hat{f} is written for the ratio of c_3 to $\frac{A_1}{\rho_0 c}$, the value of \hat{f} is given by [1]

$$\hat{f} = \frac{2}{x(F_{10}(\gamma) - 2\sigma)} - \frac{1 - 2\sigma}{F_{10}(\gamma) - 2\sigma} \quad (45)$$

where x is the root of eqn (40), since $c_3 = \frac{A_n}{\rho_0 c}$ it follows that

$$w = \frac{A}{\rho_0 c} \left[1 + \hat{f} \frac{o(\gamma i^{3/2} \gamma)}{o(i^{3/2} \gamma)} \right] e^{int} \quad (46)$$

The first point to be noted is that for a given pressure-function, pressure gradient is inversely proportional

to the wave-velocity, so that the velocity of the fluid and there the amount of flow, will be proportionately greater, in that which is more elastic.

The constant $\frac{A}{\rho_0 c}$ may be written as [1]

$$\frac{-in}{\rho_0 c} \left(\frac{-1}{in} \right) A \quad (47)$$

and since $\alpha^2 = \frac{nR^2}{v}$, this may be written as

$$\frac{A}{\rho_0 c} = \frac{A' R^2}{i^3 \alpha^2 \mu} \quad (48)$$

where A' is the quantity that will be given by Fourier Analysis of the observed pressure-gradient. The average velocity across the tube is

$$\bar{w} = \frac{A}{\rho_0 c} [1 + \hat{f} F_{10}(\gamma)] e^{int} \quad (49)$$

This can be put in the form

$$\bar{w} = \frac{A' R^2}{\mu} \frac{M_{10}''}{\alpha^2} \sin(nt - \varphi \varepsilon_{10}) \quad (50)$$

For a pressure-gradient $M \cos(nt - \varphi)$ and the values of M_{10}'' and ε_{10} compared with those of M_{10}' and ε_{10}' for the right tube. That is [1]

$$\begin{aligned} \bar{w} &= \frac{A}{\rho_0 c} [1 + \hat{f} F_{10}(\gamma)] e^{int} \\ &= \frac{A}{\rho_0 c} M e^{i(nt - \varphi)}, \text{ now express in terms of phase.} \\ &= \frac{A}{\rho_0 c} M [\cos(nt - \varphi) + i \sin(nt - \varphi)] \end{aligned} \quad (51)$$

\therefore Expressing \bar{w} in real form, means

$$\bar{w} = \frac{A}{\rho_0 c} M \cos(nt - \varphi) \quad (52)$$

where

$$M = 1 + \hbar F_{10}(\gamma) \quad (53)$$

for $\sigma = \frac{1}{2}$ and $\sigma = 0$

$\hbar F_{10}(\gamma)$ are; when

$$\sigma = \frac{1}{2}, \quad \hbar = \frac{2}{x(F_{10}(\gamma)-1)} - \frac{(1-1)}{(F_{10}(\gamma)-1)}$$

$$\therefore \quad \hbar = \frac{2}{x(F_{10}(\gamma)-1)} \quad (54)$$

$$1 + \hbar F_{10}(\gamma) = \frac{1-2F_{10}}{x(1-F_{10}(\gamma))} \quad (55)$$

when $\sigma = 0, \quad \hbar = \frac{\left[1-\frac{2}{x}\right]}{F_{10}(\gamma)} \quad (56)$

$$1 + \hbar F_{10} = \frac{2}{x} \quad (57)$$

Using these formulae and the corresponding one for $\sigma = \frac{1}{4}$, the values of M''_{10} and ε''_{10} have been calculated by Womersley [1] for $\sigma = \frac{1}{2}, \frac{1}{4}, 0$ for the first four harmonics at the femoral artery. The formula for the rate of flow might at first sight be written as [1]

$$Q = \frac{A}{\rho_0 c} \pi R^2 [1 + \hbar F_{10}(\gamma)] e^{int} \quad (58)$$

But this is only an approximation, since at any time the value of the radices is not R, but $(R+\xi)$, and varies with the time. Therefore we obtained the approximation as

$$Q = \frac{A}{\rho_0 c} \pi \left(1 + \frac{2\xi}{R}\right)^2 R^2 [1 + \hbar F_{10}(\gamma)] e^{int},$$

which agrees with [1] (59)

2. Conclusions

We have shown that the flow rate, Q , contains $F_{10}(\gamma)$, showing that $\sigma e B_0^2$ the applied magnetic field affects the flow rate. Also the longitudinal velocity, contains a function of γ , thereby pointing out that over a short length of the artery, the velocity is affected by the applied magnetic field. Magnetic Field Therapy (MFT) is one of the world's oldest forms of healing. The first documented references to MFT in Medicine were made over 6,000 years ago (Google). For the study of Magnetic Field in the treatment of Cancer and hypertension, see [16] and [17] respectively. We therefore consider our model as a proposed therapy for Cancer and Hypertension treatment.

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