

Analytical Solution of Unsteady Flow of a Viscoelastic Fluid due to an Oscillating Porous Wall

Abdulsalam Ya'u Gital^{1,*}, Muhammed Abdulhameed², Bala Ma'aji Abdulhamid¹,
Muhammad Sirajo Aliyu³, Aliyu Danladi Hina⁴

¹Department of Mathematical Sciences, Faculty of Science, Abubakar Tafawa University, Bauchi, Nigeria

²Department of Mathematics Faculty of Science, Universiti Tun Hussein Onn, Malaysia

³Department of Science, Collage of Science and Technology, Hassan Usman Katsina Polytechnic, Katsina, Nigeria

⁴Department of Basic Studies, The Federal Polytechnic, Bauchi, Nigeria

Abstract In this paper, a problem of unsteady flow of a viscoelastic fluid due to an oscillating porous wall is considered. Analytical solutions are obtained by using a modified version of the variable separation technique combined with similarity arguments. Results are validated by comparing the limiting cases of the present work with the results of related published articles and found in the excellent agreement. The variations of fluid behavior on various physical parameters are shown graphically and analyzed.

Keywords Viscoelastic fluid, Oscillating wall, Porous wall

1. Introduction

The theoretical investigation of the flow of second-grade fluids continues to receive special status in the literature due to growing importance of such fluid in modern technology and industries. The flows of second-grade fluid occur in some applications, with important examples of industrial processes that involve synthetic fibers, extrusion of molten plastic, flows of polymer solutions. Relevant contributions dealing with second grade fluid are found in Ali et al. (2012), Nazar et al. (2010), Yao and Liu (2010), Tan and Masuoka (2005) and Asghar et al. (2006).

A more general solution for the second grade fluid can be derived when fluid transpiration at the wall is considered. The governing equations must then be modified with the addition of new term representing the momentum introduced into the flow by the transpiration of fluid. The governing equation of motions of second grade fluid with transpiration is much complicated compare with second grade fluid without transpiration. This complexity of the governing equations for second grade with wall transpiration makes the task of acquiring the exact solution difficult to handle.

Most of the interest in the study of wall transpiration is due to its significant applications in boundary layer control, for instance manufacturing techniques, aeronautical systems, mechanical and chemical engineering processes. Recent published articles on this line, are related to Cruz and Lins

[6] which study the first and second problem of Stokes for unsteady flow of Newtonian fluid bounded by plate with wall transpiration, Fakhar *et al.* (2008) investigated the unsteady flow of an incompressible third grade fluid on a porous plate and also Hayat *et al.* (2010) study the Stokes' first problem of Sisko fluid over a wall with suction or blowing. Ahmed et. al. solved the problems of unsteady Magnetohydrodynamic Free convection flow of a second grade fluid in a porous medium with Ramped wall temperature using the Laplace transform method.

In this paper, the steady state solution of second grade fluid with wall transpiration is presented. To the best of the author's knowledge this is the first time that this type of steady state solution is presented. The solution shown here in the case of Newtonian fluid reduces to the solution of Cruz and Lins (2010).

The organisation of the paper is as follows. In section 2, the governing equations for the second grade fluid with transpiration are shown. In section 3, the solutions of the problem are developed. Section 4, Graphical result and discussion. Finally, conclusions are drawn.

2. Governing Equation

We consider a Cartesian coordinate system $r = (x, z)$ with x - axis is taken along the plate in the upward direction and z - axis normal to the plane of the plate in the fluid. An incompressible second grade fluids fills the space above an infinitely extended plate in the xz - plane. Initially, both the fluid and plate are at rest. Suddenly, the plate is jolted into motion in its own plane with a periodic

* Corresponding author:
abdulsalamgital@yahoo.com (Abdulsalam Ya'u Gital)
Published online at <http://journal.sapub.org/am>

Copyright © 2015 Scientific & Academic Publishing. All Rights Reserved

velocity $u_0 \exp(i\omega t)$ and transmits the motion into the fluid. Then, the fluid brought a motion through the action of viscous stress at the plate.

The stress in a second grade fluid can be modelled by Rivlin-Ericksen constitutive equation

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \quad (1)$$

where p is the pressure, I is the unit tensor, T is the Cauchy stress tensor, and μ , α_1 , α_2 are respectively, viscosity, elasticity and cross-viscosity. Rivlin-Ericksen tensors denoted by A_1 and A_2 respectively, the rate of strain and acceleration defined by

$$A_1 = (\text{grad } V) + (\text{grad } V)^T, \quad (2)$$

$$A_2 = \frac{dA_1}{dt} + (\text{grad } V)^T A_1 + A_1 (\text{grad } V) \quad (3)$$

where $\frac{d}{dt}$ is the material time derivative.

From the thermodynamic consideration they assumed

$$\mu \geq 0, \quad \alpha_1 > 0, \quad \alpha_1 + \alpha_2 = 0 \quad (4)$$

The basic governing equations are the conservation of mass and the conservation of linear momentum. These are

$$\text{div } V = 0 \quad (5)$$

$$\rho \frac{dV}{dt} = \text{div } T + \rho b \quad (6)$$

where ρ is the density and b is the body force.

We seek the velocity field of the form

$$u = u(y, t), \quad v = V_0, \quad (7)$$

where u and v are velocity components in the x - and z - coordinates direction, respectively. Also $V_0 < 0$ is the injection velocity and $V_0 > 0$ is the suction velocity.

Using Eq. (7), (5) is identically satisfied and Eq. (6) in the absence of body forces yields

$$\mu \frac{\partial^2 u}{\partial z^2} + \alpha_1 \frac{\partial^3 u}{\partial z^2 \partial t} - \alpha_1 V_0 \frac{\partial^3 u}{\partial z^3} + \rho V_0 \frac{\partial u}{\partial z} - \rho \frac{\partial u}{\partial t} = 0, \quad (8)$$

$$z > 0, \quad t > 0,$$

where μ is the kinematic viscosity of the fluid, α_1 is the second grade parameter, V_0 is the transpiration velocity and ρ is the density of the fluid.

Subject to the following initial and boundary conditions

$$u = u_0(t) = u_0 \exp(i\omega t), \quad \text{at } z = 0, \quad t > 0 \quad (9)$$

$$u = 0 \quad \text{at } z \rightarrow \infty, \quad t > 0 \quad (10)$$

$$u = 0 \quad \text{at } z = 0, \quad t = 0, \quad (11)$$

where ω is the oscillating frequency of the fluid.

We introduce the following set of non-dimensional variables

$$U = \frac{u}{u_0}, \quad \tau = \omega t, \quad y = z \left(\frac{\omega}{\nu} \right)^{\frac{1}{2}}, \quad 2\xi = \frac{V_0}{\sqrt{\omega \nu}}, \quad a = \frac{\alpha_1 \omega}{\mu} \quad (12)$$

and obtain the following non-dimensional problems

$$2a\xi \frac{\partial^3 U}{\partial y^3} - a \frac{\partial^3 U}{\partial y^2 \partial \tau} - \frac{\partial^2 U}{\partial y^2} - 2\xi \frac{\partial U}{\partial y} + \frac{\partial U}{\partial \tau} = 0, \quad (13)$$

$$y > 0, \quad \tau > 0$$

$$U = \exp(i\tau) \quad \text{at } y = 0, \quad \tau > 0 \quad (14)$$

$$U = 0 \quad y \rightarrow \infty, \quad \tau > 0 \quad (15)$$

$$U = 0 \quad \text{at } y = 0, \quad \tau > 0 \quad (16)$$

The governing Eq. (13) is the equation of motion of second grade fluids with transpiration. For $a = 0$, it reduces to Newtonian fluid with wall transpiration whose steady state solution was given by Cruz and Lins (2010).

3. Solution of the Problem

We seek a horizontal velocity as a linear combination function of the form

$$U = U(\phi), \quad \text{where } \phi = A\tau + By, \quad (17)$$

where A and B are complex constants.

On substituting Eq. (17) into Eq. (13), one arrives at third order ordinary differential equation

$$aB^2(2\xi - A) \frac{d^3 U}{d\phi^3} - B^2 \frac{d^2 U}{d\phi^2} - (2\xi B - A) \frac{dU}{d\phi} = 0 \quad (18)$$

$$U = \exp(i\tau) \quad \text{at } y = 0, \quad \tau > 0 \quad (19)$$

$$U = 0 \quad y \rightarrow \infty, \quad \tau > 0 \quad (20)$$

$$U = 0 \quad \text{at } y = 0, \quad \tau > 0 \quad (21)$$

The general solution of Eq. (18) for $a > 0$ is given by

$$U(\phi) = C_1 e^{\lambda_1 \phi} + C_2 e^{\lambda_2 \phi} + C_3 \quad (22)$$

where C_1 , C_2 and C_3 are arbitrary constants and

$$\lambda_{1,2} = \frac{B \pm \sqrt{B^2 + 4a(2\xi B - A)^2}}{2aB(2\xi B - A)} \quad (23)$$

Apply condition (21) we get $C_1 = C_3 = 0$ and Eq. (22) becomes

$$U(\phi) = C_2 e^{\lambda_2(A\tau + By)}$$

$$= C_2 \exp \left[(\tau + ky) \frac{k - \sqrt{k^2 + 4a(2\xi k - 1)^2}}{2ak(2\xi k - 1)} \right] \quad (24)$$

where $k = \frac{B}{A}$

Next, apply condition (19) we get $C_2 = 1$ and k is a root of the following equation cubic equation

$$(2a\xi\omega^2)k^3 - \omega(a\omega - i)k^2 + 2\xi k - 1 = 0 \quad (25)$$

Hence, the require solution is

$$U = \exp[i\omega(\tau + ky)] \quad (26)$$

where k is a root of the Eq. (25).

Using Routh-Hurwitz criteria (1989), it shows that Eq. (25) has only one root whose imaginary part is positive.

Consider a case where $a = 0$ (for a Newtonian fluid), Eq. (25) reduces to

$$\omega i k^2 + 2\xi k - 1 = 0, \quad (27)$$

whose roots are

$$k_{1,2} = \frac{i}{\omega} \left(\xi \pm \sqrt{\xi^2 + i\omega} \right) \quad (28)$$

By Routh-Hurwitz criteria (1989), Eq. (28) has only one root with imaginary greater than zero. Then we have

$$U = \exp \left(i\tau - y \left(\xi - \sqrt{\xi^2 + i} \right) \right) \quad (29)$$

The above equation has the same form of that revisited by Cruz and Lins [2010, Eq. (18)].

4. Graphical Result

In this section, the variation of the steady state velocity profiles due to oscillatory nature of the wall boundary which generates harmonic motion is discussed over various immerging flow parameters. Eq. (26) are displayed graphically for the velocity field u versus the boundary layer coordinate y . Figs. 1-4 are plotted in which panel (a) is cosine oscillation and (b) sine oscillation.

Figs. 1 and 2 illustrate the influences of second grade parameter a on the velocity profiles when transpiration is considered. By comparing the velocity of second fluid with velocity correspond to Newtonian fluid ($a = 0$). As in Fig. 1 the case of injection for cosine oscillation, we found that the Newtonian fluid with injection flows faster than the second grade fluid with injection. For large values of time t the second grade fluid is faster near the plate and decreases as the distance from the wall increases. In this case the influence of the second grade parameter a on the fluid is significant only for large values of time. However, as in Fig. 2, the case of suction for sine oscillation the second grade fluid flow more rapidly than the Newtonian fluid.

Figs. 3 and 4 show the variation of velocity profiles for different values of the transpiration parameter for the cases of sine and cosine oscillation. As in Fig. 3 the case of injection for cosine oscillation, we found that the second grade fluid with injection ($\xi < 0$) flows faster near the plate than the second grade fluid without transpiration ($\xi = 0$). However, as in Fig. 4, the case of suction for cosine oscillation the second grade fluid with suction ($\xi > 0$) flow slowly when compare with second grade fluid without transpiration ($\xi = 0$).

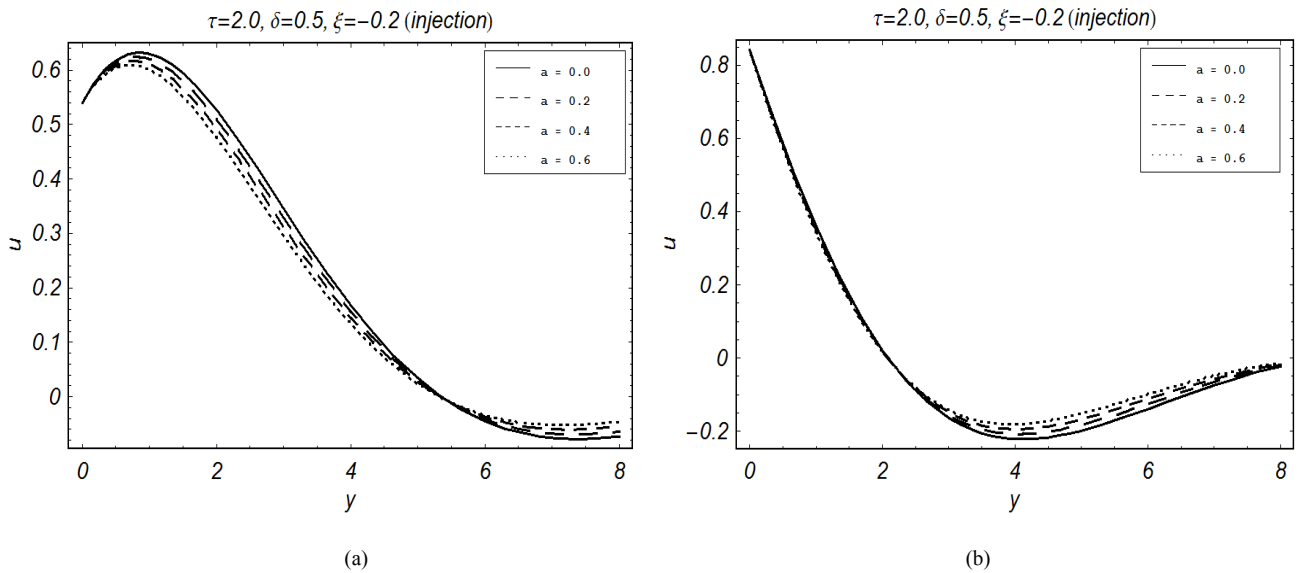


Figure 1. Velocity profiles for various values of second grade parameter with injection parameter

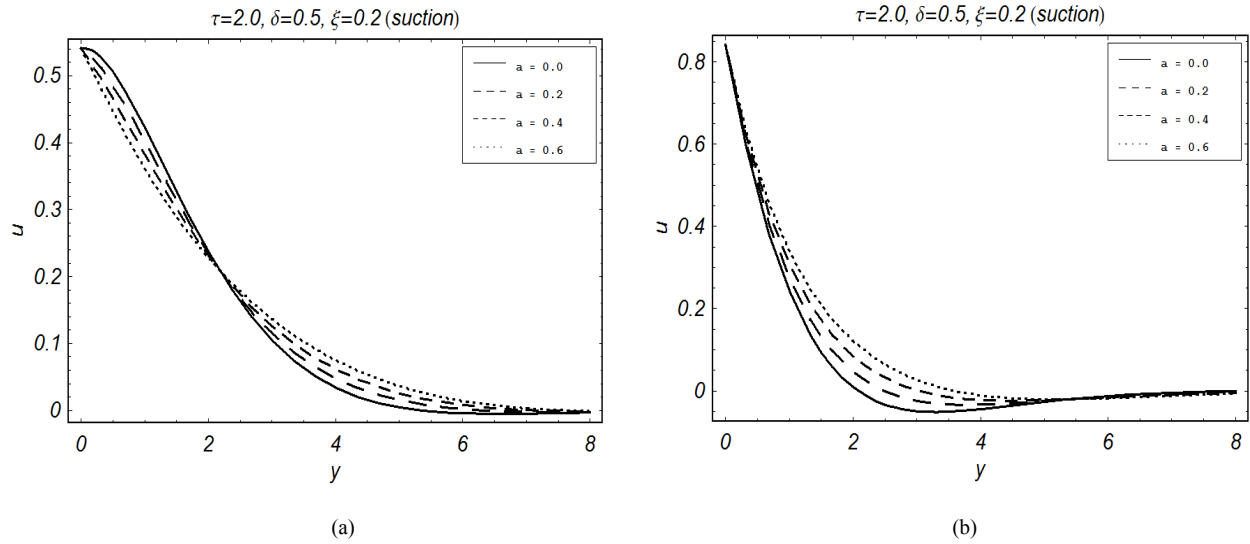


Figure 2. Velocity profiles for various values of second grade parameter with suction parameter

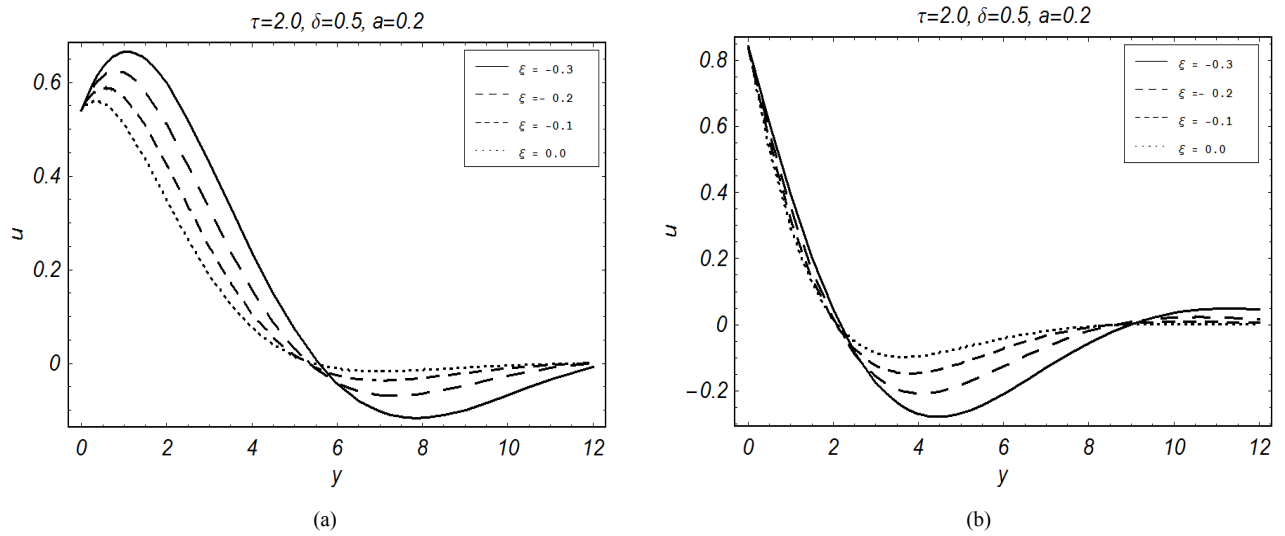


Figure 3. Velocity profile for different values of injection parameter

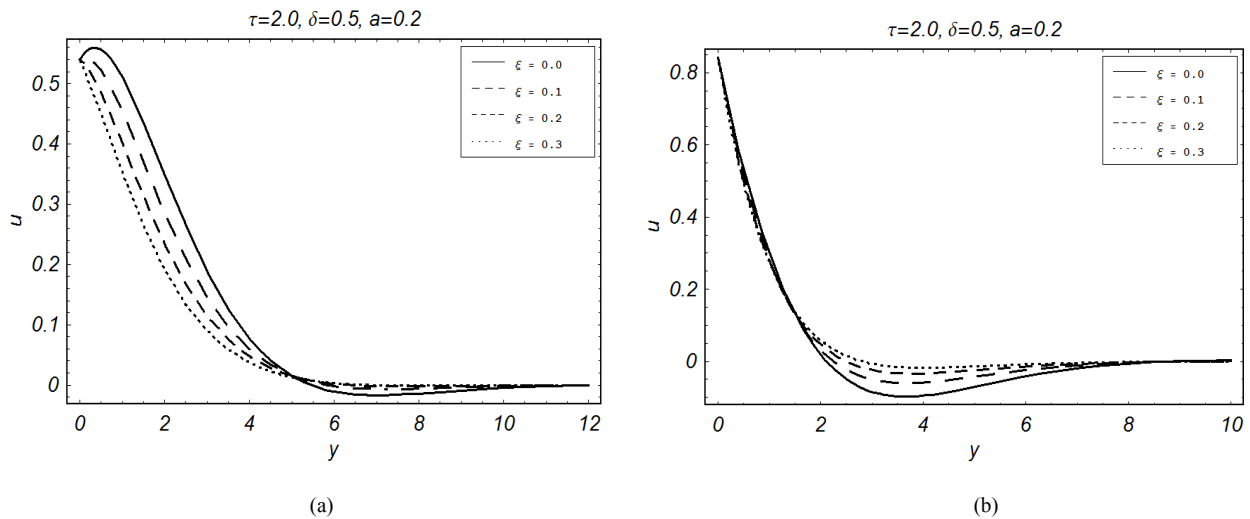


Figure 4. Velocity profile for different values of suction parameter

5. Conclusions

In this work, modified version of the variable separation technique was used to derived the steady state solution for unsteady second grade fluid with suction and injection. The present results are validated by comparing the limiting case of the present work with corresponding existing solution of Newtonian fluid with transpiration.

The analysis of the result has shown the influence of second grade fluids on cosine and sine oscillation. It found that the second grade parameter exert a great influence on the general flow pattern and the injection and suction velocity shows opposite effect on the velocity field.

REFERENCES

- [1] Ali, F., Norzieba, M., Shaidan, S., Khan, I., and Hayat, T. (2012). New Exact Solutions of Stokes Second for an MHD Second Fluid in a Porous Space. *Int. J. of Non-Linear Mech.* 47, 521-525.
- [2] Nazar, M., Fetecau, C., Vieru, D., Fetecau, C. (2010). New exact solutions corresponding to the second problem of Stokes for second grade fluids. *Non-Linear Anal. Real World Appl.* 11,584-591.
- [3] Yao, Y. and Liu, Y. (2010). Some unsteady flows of a second grade fluid over a plane wall, *Nonlinear Analysis: Real World Appl.*, 11(2010)4442-4450.
- [4] W. Tan and T. Masuoka, Stokes' first problem for a second grade fluid in a porous half-space with heated boundary, *Int. J. Non-Linear Mech.*, 40(2005)515-522.
- [5] Asghar, S. Nadeem, S. Harif, K., and Hayat, T. (2006) Analytical solution of Stokes second problem for second grade fluid, *Mathematical Problems in Engineering*., doi:10.1155/MPE/2006/72468, (2006) 1-8.
- [6] D. O. d. A. Cruz and E. F. Lins, The Unsteady Flow Generated by an Oscillating wall with Transpiration, *Int. J. Non-Linear Mech.* 45(2010)453-457.
- [7] K. Fakhar, Xu Zhenli and Yi Cheng, Exact solutions of a third grade fluid on a porous plate, *Appl. Math. Comput.* , 202(2008)376-382.
- [8] T. Hayat, R. J. Moitsheki, S. Abelman, Stokes' first problem for Sisko fluid over a porous wall, *Appl. Math. Comput.*, 217(2010) 622-628.
- [9] E. J. Barbeau, *Polynomials*, Springer-Verlag New York Inc. 1989.
- [10] S. Ahmad, D. Vieru, I. Khan, & S. Shafie, Unsteady magnetohydrodynamic free convection flow of a second grade fluid in a porous medium with ramped wall temperature. *PloS one*, (2014) 9(5).