

Surface Charge Density in Different Types of Lorentz Transformations

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Abstract Lorentz transformation is a group transformation that is used to transform the space and time coordinates of one inertial frame of reference into another one with travelling at a relative speed. Lorentz transformation is the basic tool for the study of Relativistic Mechanics. There are different types of Lorentz transformations such as Special, Most general, Mixed number, Geometric product and Quaternion Lorentz transformations. In this paper we have studied the surface charge density in different types of Lorentz Transformations.

Keywords Lorentz Transformation (LT), Most General Lorentz Transformation (MGLT), Mixed Number Lorentz Transformation (MNLTL), Geometric Product Lorentz Transformation (GPLT), Quaternion Lorentz Transformation (QLT), Surface Charge Density (SCD)

1. Introduction

In most treatments on special relativity [1], the line of motion is aligned with x-axis. In such a situation the (y-z) coordinates are invariant under the LT. However, it is of interest to study the case when the line of motion does not coincide with any of the coordinate axes. In this paper we have derived the formula for SCD in different types of LTs when the line of motion is not aligned with any of the coordinate axes.

1.1. Special Lorentz Transformation

Let us consider two inertial frames of references S and S', where the frame S is at rest and the frame S' moving along the X axis with the velocity \vec{v} with respect to the S frame. The space and time coordinates of S and S' are (x, y, z, t) and (x', y', z', t') respectively. The relation between then coordinates of S and S', which is called the special Lorentz transformation, can be written as [1]

$$x' = \gamma(x - vt), y' = y, z' = z \quad (1)$$

$$t' = \gamma(t - vx), \text{ where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } c = 1$$

And

$$x = \gamma(x' + vt'), y = y', z = z' \quad (2)$$

$$t = \gamma(t' + vx')$$

1.2. Most General Lorentz Transformation

When the velocity \vec{v} of S' with respect to S is not along the X axis, i.e. the velocity \vec{v} has three components v_x, v_y and v_z then the relation between the coordinates of S and S', which is called the MGLT, can be written as [1]

$$\vec{X}' = \vec{X} + \vec{v} \left[\frac{\vec{X} \cdot \vec{v}}{v^2} (\gamma - 1) - t\gamma \right] \quad (3)$$

$$t' = \gamma(t - \vec{X} \cdot \vec{v}) \text{ where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } c = 1$$

The inverse MGLT can be written as [1]

$$\vec{X} = \vec{X}' + \vec{v} \left[\frac{\vec{X}' \cdot \vec{v}}{v^2} (\gamma - 1) + t'\gamma \right] \quad (4)$$

$$t = \gamma(t' + \vec{X}' \cdot \vec{v})$$

1.3. Mixed Number Lorentz Transformation

MNLTL can be generated using Mixed number algebra. [1, 5-9].

Let us consider the velocity \vec{v} of S' with respect to S is not along the X axis, i.e. the velocity \vec{v} has three components v_x, v_y and v_z . Let in this case \vec{z} and \vec{z}' can be the space part in S and S' frame respectively. Then the MNLTL can be written as [1, 3, 5-9]

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Published online at <http://journal.sapub.org/am>

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$$\begin{aligned}\vec{z}' &= \gamma(\vec{z} - t\vec{v} - i\vec{z} \times \vec{v}) \\ t' &= \gamma(t - \vec{z} \cdot \vec{v})\end{aligned}\quad (5)$$

The inverse MNLT can be written as [1]

$$\begin{aligned}\vec{z} &= \gamma(\vec{z}' + t'\vec{v} + i\vec{z}' \times \vec{v}) \\ t &= \gamma(t' + \vec{z}' \cdot \vec{v})\end{aligned}\quad (6)$$

1.4. Geometric Product Lorentz Transformation

Bidyut Kumar Datta and his co-workers defined the geometric product of vectors as [2, 14, 15]

$$\vec{A}\vec{B} = \vec{A} \cdot \vec{B} + \vec{A} \wedge \vec{B}$$

Where \vec{A} and \vec{B} are two vectors. $\vec{A} \wedge \vec{B}$. i.e A wedge B, it has magnitude $AB\sin\theta$ and it is different from the usual cross product. It is not a scalar or vector. Geometric product of two perpendicular vectors is an area or a bivector oriented in the plane containing the vectors and Geometric product of two parallel vectors is simply the scalar product of the vectors [16]. We use the symbol \times instead of the symbol \wedge . Therefore the geometric product of vectors can be written as [2]

$$\vec{A}\vec{B} = \vec{A} \cdot \vec{B} + \vec{A} \times \vec{B} \quad (7)$$

We can also generate a type of most general Lorentz transformation using this geometric product.

In this case the velocity \vec{v} of S' with respect to S also has three components v_x , v_y and v_z as the MGLT. Let in this case \vec{Z} and \vec{Z}' be the space parts in the S and S' frames respectively. Then the GPLT can be written as [2]

$$\begin{aligned}\vec{z}' &= \gamma(\vec{z} - t\vec{v} - \vec{z} \times \vec{v}) \\ t' &= \gamma(t - \vec{z} \cdot \vec{v})\end{aligned}\quad (8)$$

The inverse GPLT can be written as [2]

$$\begin{aligned}\vec{z} &= \gamma(\vec{z}' + t'\vec{v} + \vec{z}' \times \vec{v}) \\ t &= \gamma(t' + \vec{z}' \cdot \vec{v})\end{aligned}\quad (9)$$

1.5. Quaternion Lorentz Transformation

The Quaternion can also be written as the sum of a scalar and a vector [2] i.e, $\vec{A} = a + \vec{A}$. The multiplication of any two quaternion's $\vec{A} = a + \vec{A}$ and $\vec{B} = b + \vec{B}$ is given by [2]

$$\begin{aligned}\vec{A}\vec{B} &= (a + \vec{A})(b + \vec{B}) \\ &= ab - \vec{A} \cdot \vec{B} + a\vec{B} + b\vec{A} + \vec{A} \times \vec{B}\end{aligned}\quad (10)$$

Taking $a = b = 0$, we get

$$\begin{aligned}\vec{A}\vec{B} &= (a + \vec{A})(b + \vec{B}) \\ &= -\vec{A} \cdot \vec{B} + \vec{A} \times \vec{B}\end{aligned}\quad (11)$$

This product is called quaternion product. We can also generate a type of most general Lorentz transformation using this quaternion product.

In this case the velocity \vec{v} of S' with respect to S also has three components v_x , v_y and v_z as the MGLT. Let in this case \vec{Z} and \vec{Z}' be the space parts in the S and S' frames respectively. Then the QLT can be written as [2]

$$\begin{aligned}\vec{z}' &= \gamma(\vec{z} - t\vec{v} - \vec{z} \times \vec{v}) \\ t' &= \gamma(t + \vec{z} \cdot \vec{v})\end{aligned}\quad (12)$$

The inverse QLT can be written as [2]

$$\begin{aligned}\vec{z} &= \gamma(\vec{z}' + t'\vec{v} + \vec{z}' \times \vec{v}) \\ t &= \gamma(t' - \vec{z}' \cdot \vec{v})\end{aligned}\quad (13)$$

2. Surface Charge Density

The charge density is the amount of electric charge (the number of electrons, ions) per unit volume of space in one, two or three dimensions. The surface charge density (σ) is the amount of electric charge per unit surface area (A).

$$\therefore \sigma = \frac{q}{A}$$

It is measured in coulombs per square meter (C/m^2) [1]. Since there are positive as well as negative charges, the charge density can take on negative values. Like any density it can depend on position. We know that the charge on the electron or proton is the minimum, called the elementary charge e ($= 1.6 \times 10^{-19}$ coul). The electric charge is discrete which may be determined by counting the number of elementary charged particles. As the total number of elementary charges cannot depend on the state of the motion of the observer, we may conclude that the electric charge is relativistically invariant. Based on this information we want to discuss the surface charge density (σ) in different types of Lorentz transformation.

3. Surface Charge Density in Different Types of Lorentz Transformation

3.1. Surface Charge Density in Special Lorentz Transformation

Let us consider two system S and S', S' is moving with uniform velocity \vec{v} relative to S along negative direction of X-axis as shown in Figure 1. Let there be a stationary sheet of uniform charge density $+\sigma$ coul/ m^2 at rest in system S having one edge parallel to X'-axis. Let the sheet be a square of side L_0 and placed parallel to X-Y plane. The observer in the system S' will observe that the sheet is moving with velocity v along (+) ve X-axis.

We have the transformation formula for the surface charge density in Special Lorentz transformation which is given [1]

as

$$\sigma' = \frac{\sigma}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\sigma \quad (14)$$

Where (σ') is the surface charge density of the moving frame.

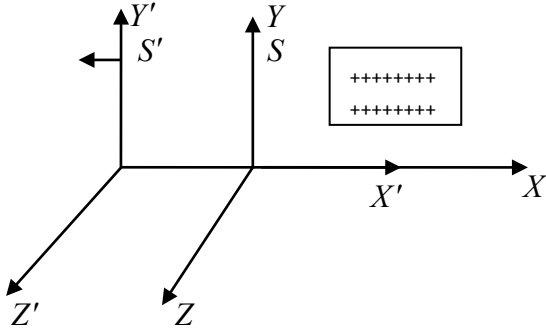


Figure 1. The system S' is moving along X-axis with velocity v relative to the system S

3.2. Surface Charge Density in Most general Lorentz Transformation

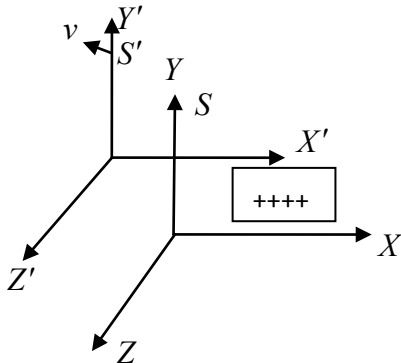


Figure 2. The system S' is moving in X-Y plane with velocity v relative to the system S

Let us consider two system S and S' where the frame S' is moving with velocity \vec{v} relative to the system S along X-Y plane as shown in Figure 2. Thus the velocity of \vec{v} has two components v_x and v_y . Let us consider a stationary sheet of uniform charge density $+\sigma$ coul/ m^2 at rest in system S having one edge parallel to X-axis and let the sheet be a square of side L_0 placed parallel to X-Y plane. The observer in the system S' will observe that the sheet is moving in opposite direction.

Therefore the surface charge density in MGLT is discussed in [1] and the transformation equation of surface charge density in MGLT is derived as

$$\sigma' = [1 + \cos^2 \theta (\gamma - 1) + \cos^2 \theta (\gamma - 1)^2] \sigma \quad (15)$$

Where σ' represent the surface charge density when the system S' is moving along X-Y plane.

3.3. Surface Charge Density in Mixed Number Lorentz Transformation

Again according to the systems described in 3.2 we have the transformation formula for the surface charge density in MNLT which is derived in [1] can be written as

$$\sigma' = \gamma^2 [1 - v^2 (1 - \cos^2 \theta)] \sigma \quad (16)$$

3.4. Surface Charge Density in Geometric product Lorentz Transformation

Let us consider two system S and S' where the frame S' is moving with velocity \vec{v} relative to the system S along any arbitrary direction as shown in the Figure 2. Thus the velocity of \vec{v} has two components v_x and v_y .

Let us consider a stationary sheet of uniform charge density $+\sigma$ coul/ m^2 at rest in system S having one edge parallel to X-axis and let the sheet be a square of side L_0 placed parallel to X-Y plane. The observer in the system S' will observe that the sheet is moving in opposite direction with velocity \vec{v} along any arbitrary direction as shown in Figure 2.

If L_0 is the length of the square charged sheet in S , then the length contraction in the moving frame for the GPLT can be written as [4]

$$\vec{L}_0 = \gamma (\vec{L} - \vec{L} \times \vec{V})$$

$$\therefore L_0^2 = \gamma (\vec{L} - \vec{L} \times \vec{v}) \cdot \gamma (\vec{L} - \vec{L} \times \vec{v}) + \gamma (\vec{L} - \vec{L} \times \vec{v}) \times \gamma (\vec{L} - \vec{L} \times \vec{v})$$

$$\text{or, } L_0^2 = \{\gamma (\vec{L} - \vec{L} \times \vec{v})\}^2$$

or,

$$L_0^2 = \gamma^2 [L^2 - \vec{L} (\vec{L} \times \vec{v}) - (\vec{L} \times \vec{v}) \cdot \vec{L} + (\vec{L} \times \vec{v}) (\vec{L} \times \vec{v})]$$

$$= \gamma^2 [L^2 + (L v \sin \theta)^2]$$

$$= \gamma^2 [L^2 + L^2 v^2 \sin^2 \theta]$$

$$= \gamma^2 [L^2 + L^2 v^2 (1 - \cos^2 \theta)]$$

$$= \gamma^2 L^2 [1 + v^2 (1 - \cos^2 \theta)]$$

$$\therefore L^2 = \frac{L_0^2}{\gamma^2 [1 + v^2 (1 - \cos^2 \theta)]}$$

Considering L has two components L_x and L_y .

$$L_x^2 = \frac{L_{0x}^2}{\gamma^2 [1 + v^2 (1 - \cos^2 \theta)]}$$

$$\text{and } L_y^2 = \frac{L_{0y}^2}{\gamma^2 [1 + v^2 (1 - \cos^2 \theta)]}$$

The total charge observed by an observer in S' is

$$\begin{aligned}
Q' = L_x L_y \sigma' &= \sqrt{\frac{L_{0x}^2}{\gamma^2 [1 + v^2 (1 - \cos^2 \theta)]}} \sqrt{\frac{L_{0y}^2}{\gamma^2 (1 + v^2 (1 - \cos^2 \theta))}} \sigma' \\
&= \frac{L_0^2}{\gamma^2 [1 + v^2 (1 - \cos^2 \theta)]} \sigma' \quad [\text{for square charge sheet}]
\end{aligned} \tag{17}$$

Where σ' is the surface charge density in system S'.
According to the principle of conservation of charge

$$\begin{aligned}
Q' &= Q_0 \\
\text{or, } \frac{L_0^2}{\gamma^2 [1 + v^2 (1 - \cos^2 \theta)]} \sigma' &= L_0^2 \sigma \\
\text{or, } \sigma' &= \gamma^2 [1 + v^2 (1 - \cos^2 \theta)] \sigma
\end{aligned} \tag{18}$$

This equation represents the transformation equation for surface charge density in GPLT.

3.5. Surface Charge Density in Quaternion Lorentz Transformation

Let us consider two system S and S' where the frame S' is moving with velocity \vec{v} relative to the system S along any arbitrary direction as shown in the Figure 2. Thus the velocity of \vec{v} has two components v_x and v_y .

Let us consider a stationary sheet of uniform charge density $+\sigma$ coul/m² at rest in system S having one edge parallel to X-axis and let the sheet be a square of side L_0 placed parallel to X-Y plane. The observer in the system S' will observe that the sheet is moving in opposite direction with velocity \vec{v} along any arbitrary direction as shown in Figure 2.

If L_0 is the length of the square charged sheet in S, then the length contraction in the moving frame for the QLT can be written as [4]

$$\begin{aligned}
\bar{L}_0 &= \gamma (\bar{L} - \bar{L} \times \vec{V}) \\
\text{or, } L_0^2 &= \gamma^2 [L^2 - (\bar{L} \times \vec{v})^2 - \bar{L} (\bar{L} \times \vec{v}) - (\bar{L} \times \vec{v}) \bar{L} + (\bar{L} \times \vec{v}) \times (\bar{L} \times \vec{v})] \\
&= \gamma^2 [L^2 - (Lv \sin \theta)^2] \\
&= \gamma^2 [L^2 - L^2 v^2 \sin^2 \theta] \\
&= \gamma^2 [L^2 - L^2 v^2 (1 - \cos^2 \theta)] \\
&= \gamma^2 L^2 [1 - v^2 (1 - \cos^2 \theta)] \\
\therefore L^2 &= \frac{L_0^2}{\gamma^2 [1 - v^2 (1 - \cos^2 \theta)]} \\
\text{or, } L^2 &= \frac{L_0^2}{\gamma^2 [1 - v^2 (1 - \cos^2 \theta)]} \\
L_x^2 + L_y^2 &= \frac{L_{0x}^2 + L_{0y}^2}{\gamma^2 [1 - v^2 (1 - \cos^2 \theta)]} \\
\therefore L_x^2 &= \frac{L_{0x}^2}{\gamma^2 [1 + v^2 (1 - \cos^2 \theta)]} \quad \text{and} \\
L_y^2 &= \frac{L_{0y}^2}{\gamma^2 [1 - v^2 (1 - \cos^2 \theta)]}
\end{aligned}$$

The total charge observed by an observer in S' is

$$\begin{aligned}
 Q' &= L_x L_y \sigma' = \sqrt{\frac{L_{0x}^2}{\gamma^2[1-v^2(1-\cos^2\theta)]}} \sqrt{\frac{L_{0y}^2}{\gamma^2(1-v^2(1-\cos^2\theta))}} \sigma' \\
 &= \frac{L_0^2}{\gamma^2[1-v^2(1-\cos^2\theta)]} \sigma' \quad [\text{for square charge sheet}]
 \end{aligned} \quad (19)$$

Where σ' is the surface charge density in system S' .
According to the principle of conservation of charge

$$\begin{aligned}
 Q' &= Q_0 \\
 \text{or, } \frac{L_0^2}{\gamma^2[1-v^2(1-\cos^2\theta)]} \sigma' &= L_0^2 \sigma \\
 \text{or, } \sigma' &= \gamma^2[1-v^2(1-\cos^2\theta)] \sigma
 \end{aligned} \quad (20)$$

This equation represents the transformation equation for surface charge density in QLT.

4. Comparative Study

4.1. Table 1

Table 1. Surface charge density in different types of Lorentz transformations (LT)

Name of LT	Surface charge density(SCD)
S LT	$\sigma' = \gamma \sigma$
MG LT	$\sigma' = [1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2] \sigma$
Q LT	$\sigma' = \gamma^2[1 - v^2(1-\cos^2\theta)] \sigma$
MNLT	$\sigma' = \gamma^2[1 - v^2(1-\cos^2\theta)] \sigma$
GP LT	$\sigma' = \gamma^2[1 + v^2(1-\cos^2\theta)] \sigma$

4.2. Graphical Representation of SCD in Different Types of LTs

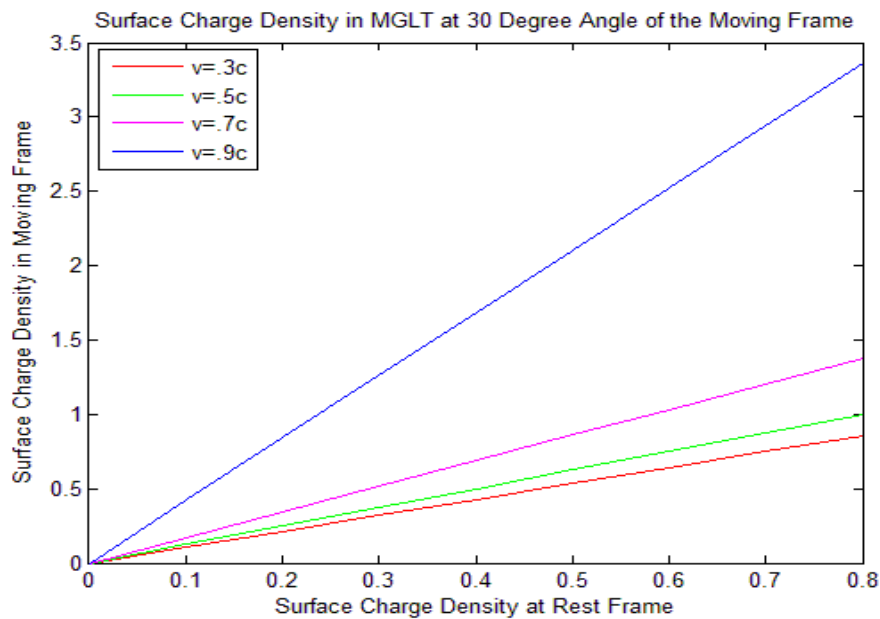


Figure 3. Surface Charge Density in MGLT at 30 Degree Angle of the moving Frame

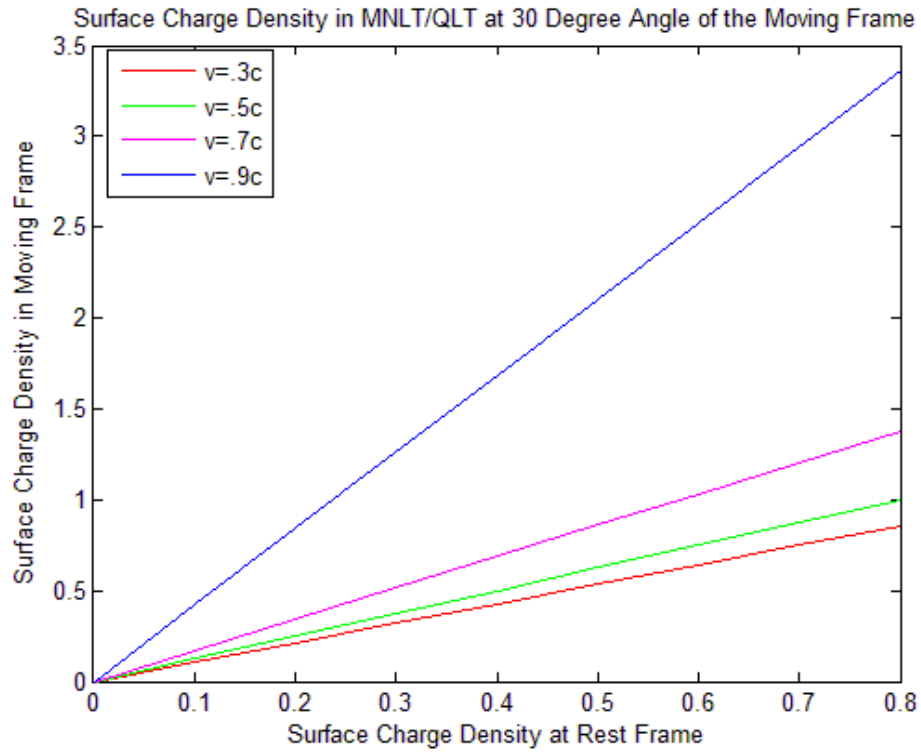


Figure 4. Surface Charge Density in MNLT/QLT at 30 Degree Angle of the Moving Frame

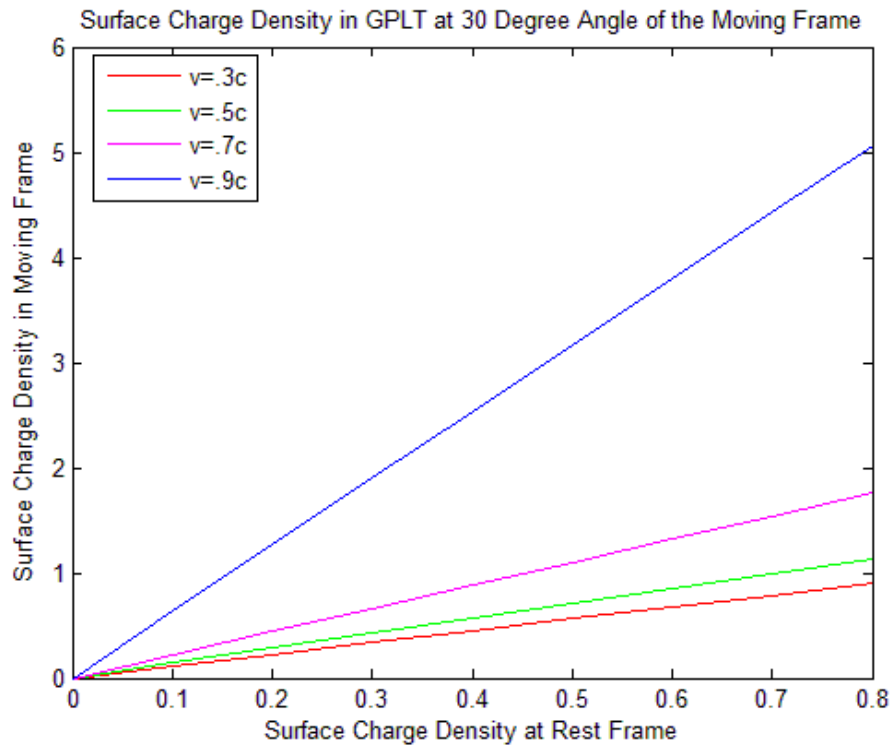


Figure 5. Surface Charge Density in GPLT at 30 Degree Angle of the Moving Frame

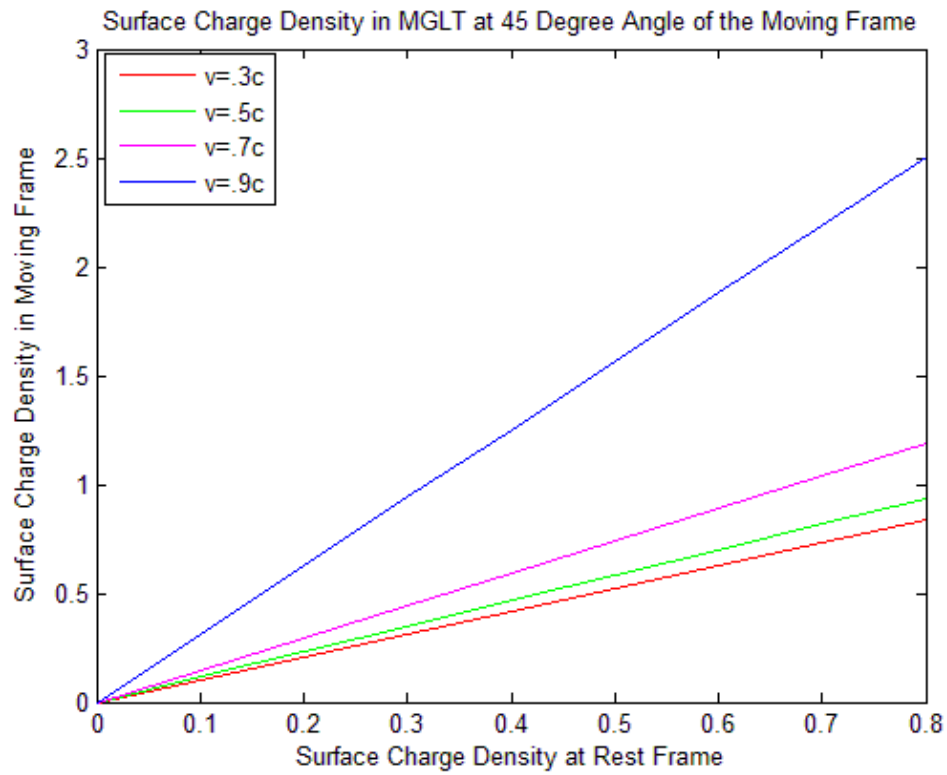


Figure 6. Surface Charge Density in MGLT at 45 Degree Angle of the Moving Frame

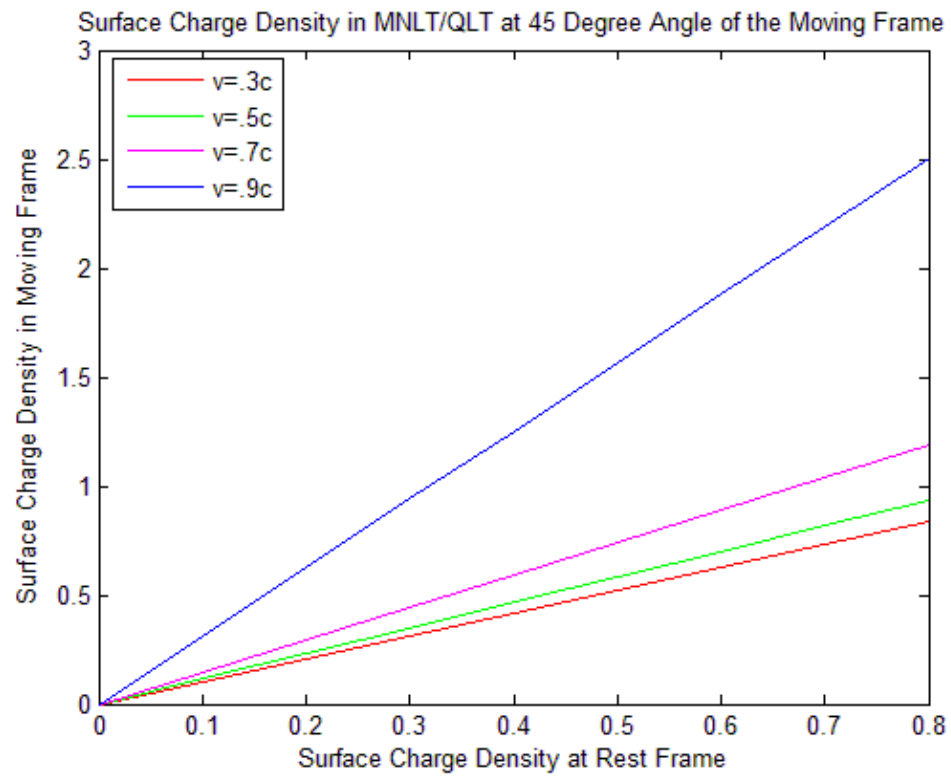


Figure 7. Surface Charge Density in MNLT/QLT at 45 Degree Angle of the Moving Frame

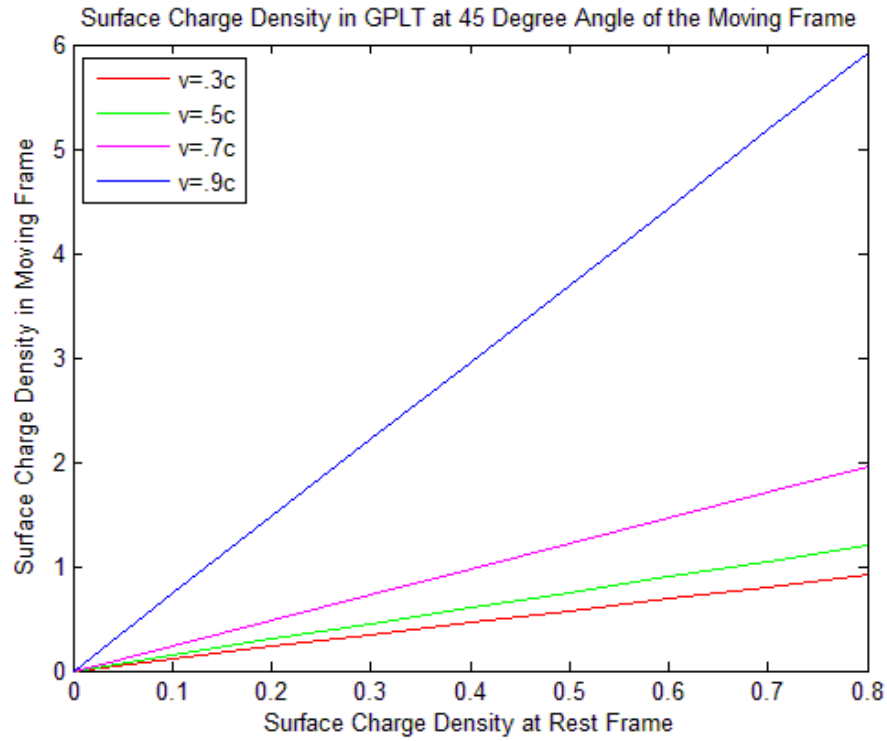


Figure 8. Surface Charge Density in GPLT at 30 Degree Angle of the Moving Frame

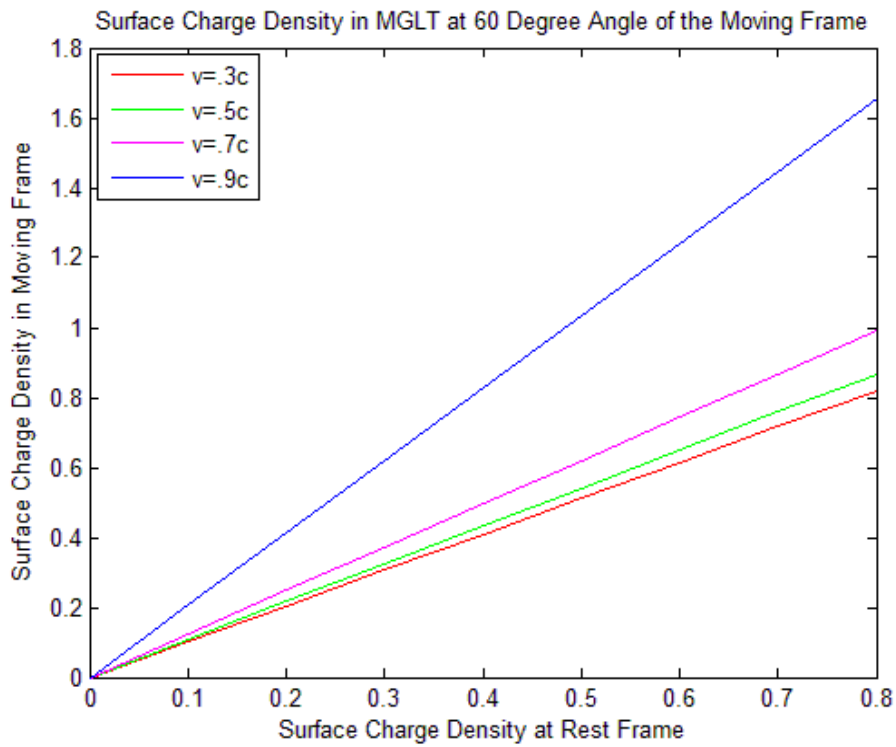


Figure 9. Surface Charge Density in MGLT/QLT at 30 Degree Angle of the Moving Frame

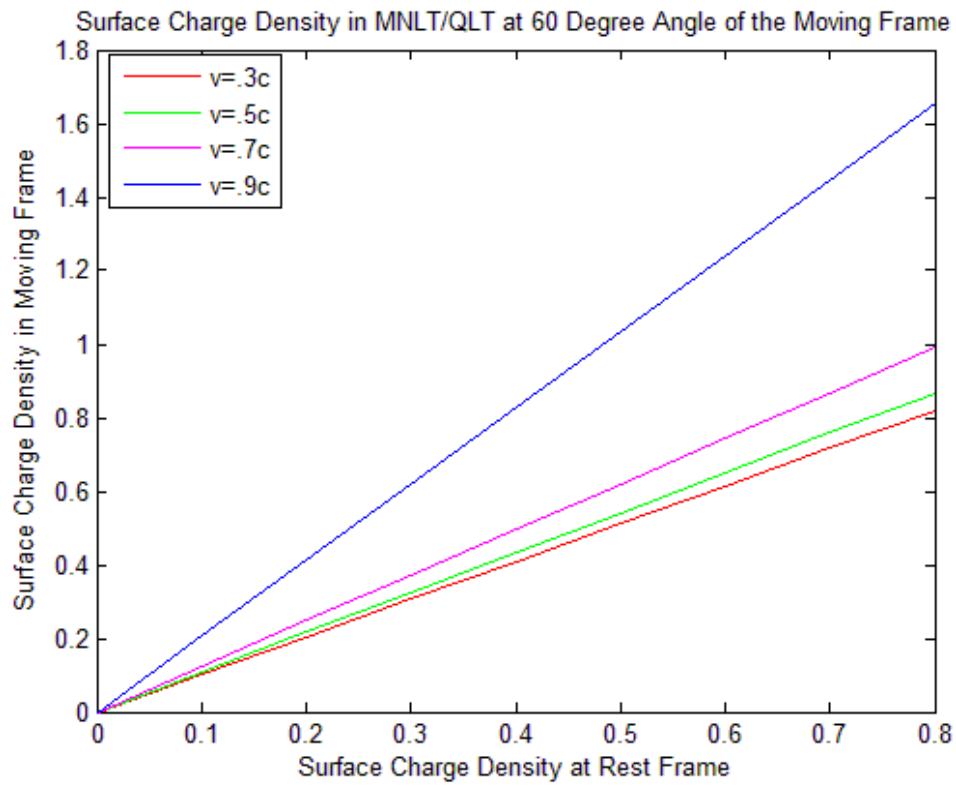


Figure 10. Surface Charge Density in MNLT/QLT at 30 Degree Angle of the Moving Frame

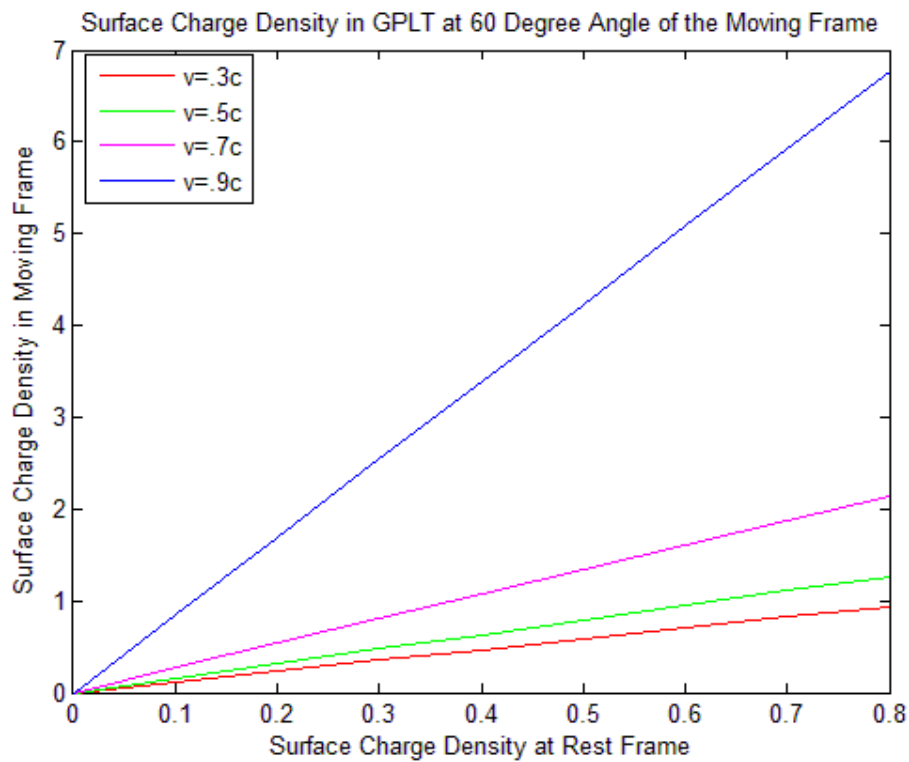


Figure 11. Surface Charge Density in GPLT at 60 Degree Angle of the Moving Frame

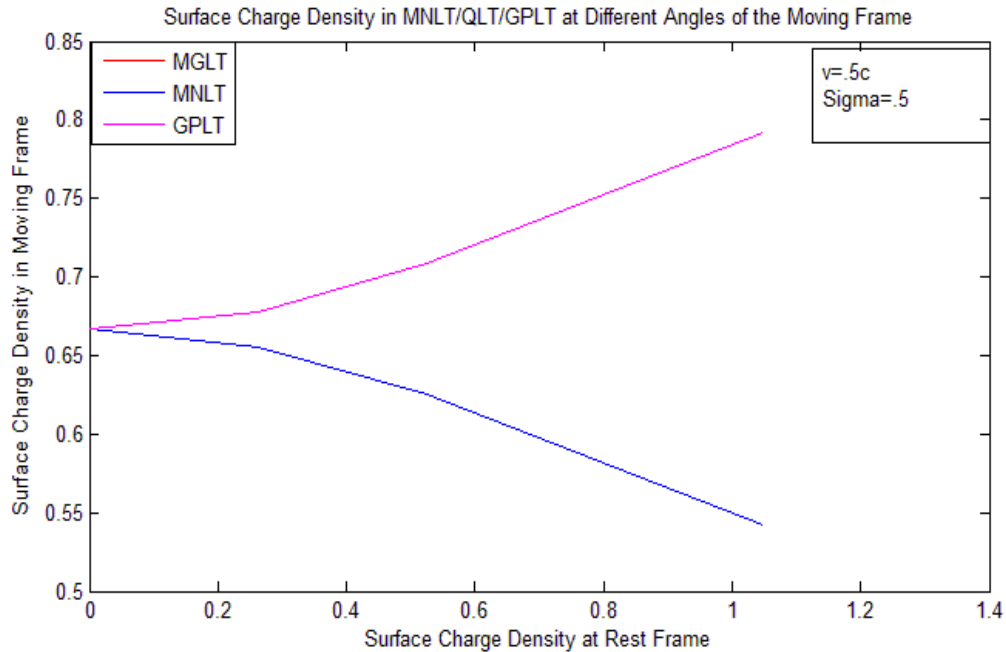


Figure 12. Surface Charge Density in MGLT/MNLT/GPLT at Different Angles of the Moving Frame

Discussion: For the less velocity of the moving frame there exist slightly changes of the surface charge density. But when the velocity of the moving frame is very close to the speed of light the change of surface charge density is highly remarkable.

Discussions: We see from the Figure 12 surface charge density is increasing in GPLT at fixed velocity and fixed surface charge density at rest frame when angle is increasing. But in MNLT it is decreasing. The surface charge density in MGLT of the moving frame is same as in MNLT. So it is coincide with MNLT and not visible in the graph.

5. Conclusions

The transformation formulae for surface charge density in SLT, MGLT, QLT, MNLT and GPLT are illustrated in Table 1. The numerical values of surface charge densities of the moving system in terms of a system at rest for different types of LTs have calculated and it has observed graphically that for same angle (θ) between system S and S', the values of surface charge density of moving system, (σ') increases with increasing velocity \vec{v} of the moving system and for the same velocity of the moving system (σ') decreases with increasing angle (θ) but in GPLT for the same velocity of the moving system (σ') increases with increasing angle (θ).

REFERENCES

- [1] S.B Rafiq and M.S Alam , Transformation of Surface Charge Density in Mixed Number Lorentz Transformation, Sri Lankan Journal of Physics, Vol.13(1)(2012) 17-25.
- [2] Atikur Rahman Baizid & Md. Shah Alam, Reciprocal Property of Different Types of Lorentz Transformation, International Journal of Reciprocal Symmetry and Theoretical Physics, volume1, No 1(2014).
- [3] M.S. Alam, S. Bauk and M. Ahmed, Agreements and disagreements between Lorentz transformation and von Peschke's comment, Physics Essays 24, (2011)1.
- [4] M.S. Alam, K. Begum, Different types of Lorentz transformation, Jahangirnagar Physics Studies. 15, (2009).
- [5] M. S. Alam, Mixed Product of vectors, Journal of Theoretics. 3, (2001) 4.
- [6] M.S. Alam, M.H. Ahsan and M. Ahmad, Mathematical Tools of Mixed-Number algebra J. Natn. Sci. Foundation, Sri Lanka. 33, (2005)1.
- [7] M.S. Alam, M.H. Ahsan, Mixed-Number Lorentz Transformation, Physics Essays, 16, (2003) 4.
- [8] M.S. Alam, Consequences of Mixed Number Lorentz Transformation, Proc. Pakistan Acad. Sci. 41, (2004) 2.
- [9] M. S. Alam, M.D. Chowdhury, Relativistic aberration of Mixed number Lorentz transformation, J. Natn. Sci. Foundation Sri Lanka. 34, (2005) 3.
- [10] M.S. Alam, Comparative Study of Mixed Product and Quaternion Product, Indian Journal of Physics A 77, (2003).
- [11] M.S. Alam, Comparative Study of Quaternions and Mixed Number, Journal of Theoretics. 3, (2001) 6.
- [12] Md. Shah Alam and Sabar Bauk, Comparative Study of Geometric product and Mixed product. Proceedings of the 6th IMT-GT conference on Mathematics, Statistics and its Applications (ICMSA2010) Universiti Tunku Abdul Rahman, Kuala Lumpur, Malaysia.
- [13] M. R. Spiegel, Theory and Problems of Vector Analysis and An Introduction to Tensor Analysis, Schaum's outline series,

McGraw-Hill book company.

- [14] Datta, B.K, De Sabbata V. and Ronchetti L. 1998. Quantization of gravity in real space time, II Nuovo Cimento, 113 B.
- [15] Datta, B.K., Datt a R. 1998. Einstein field equations in spinor formalism, Foundations of Physics letters, 11,1.