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Saffman-Taylor Problem for a Non-Newtonian Fluid

Gelu Paşa

Simion Stoilow Institute of Mathematics of Romanian Academy, Calea Grivitei 21, Bucharest S1, Romania

Abstract We study the linear stability of a steady displacement of an Oldroyd-B fluid by air in a Hele- Shaw cell. We obtain the perturbations equations from the full basic flow equations and we perform a depth-average procedure (across the Hele-Shaw gap) in the dynamic boundary condition at the interface. The new element is an exact formula of the growth rate (in time) of perturbations, obtained in the range of small *Deborah* numbers which appear in the constitutive relations. If the *Deborah* numbers are equal, then our growth rate is quite similar to the Saffman-Taylor formula for a Newtonian liquid displaced by air. We prove the destabilizing effect of the elasticity properties of the Oldroyd-B fluid, in agreement with some previous numerical results is this field.

Keywords Hele-Shaw flow, Hydrodynamic instability, Oldroyd-B fluid, Small Deborah numbers, Dispersion relation

1. Introduction

Some fluids, as polymeric liquids, biological liquids, magma, are described by non-linear relations between the Cauchy stress tensor and strain-rate tensor and are called non-Newtonian fluids. The viscous and elastic properties of the non-Newtonian fluids are studied in a large number of papers - see [1-5].

The Oldroyd-B model was introduced by J. Oldroyd [6] in 1950. Such models of rate type, as Maxwell upper convected and Oldroyd-B, can be used to describe the polymeric flows, often related with the secondary oil recovery process in a porous medium, approximated by the Hele-Shaw model. In this field we refer to [7-9].

Numerical simulations of an Oldroyd-B flow in pipes are given in [10], [11]. Some results related with the stability of a thin film flow with visco-elasticity effect are given in [12].

In this paper we consider the steady displacement of an Oldroyd-B fluid by air in a horizontal Hele-Shaw cell and study the interface stability. The Hele-Shaw plates are parallel with the xOy plane; the gap between is denoted by b. The Oz axis is orthogonal on the plates. We use the constitutive equations (2). The perturbations equations (14), (15), (24), (25) are derived from the basic equations (8), (11), (12). A scaling procedure allows us to neglect the vertical component w of the perturbed velocity - see relations (18) and (19). We use the Fourier mode decomposition (22) with amplitude f (z) for the perturbed velocities (u, v). An approximate formula of the amplitude f is given in formula (46), by using the particular flow geometry (a very “thin” Hele-Shaw cell). This allows us to perform a depth-average procedure in the dynamic boundary conditions (33) - (34) (i.e. Laplace law) as a final step, and we obtain the dispersion relation (52), given by a ratio. This explicit formula for the growth rate (in time) of perturbations in terms of the problem data is the novelty of our paper.

The denominator of the ratio (52) contains a term depending on ($a_1 - a_2$), where a_1, a_2 are the relaxation and the retardation (time) constants appearing in the constitutive relations (2) of the Oldroyd-B fluid. When $a_1 = a_2$, our formula (52) is quite similar with the Saffman-Taylor formula [13] for a Newtonian liquid displaced by air (see the last part of section 5).

Our dispersion relation shows us that the displacement process of an Oldroyd-B fluid by air can be more unstable, compared with the displacement of a Newtonian liquid by air, so the elasticity properties have a destabilizing effect - see section 6, in agreement with some numerical previous results (see [7] below).

Wilson [7] considered a different scaling procedure and numerically solved a quite similar set of perturbation equations from which he obtained numerical values of the growth rate σ, for *Deborah* numbers near 1.

Mora and Manna [8], [9] studied the linear stability of the displacement of a Maxwell upper-convected fluid and generalized non-Newtonian fluids in a Hele-Shaw cell and obtained numerical values for the growth rate of perturbations, in the range of large *Deborah* numbers.

The growth rate (52) can be unbounded in the range of small *Deborah* numbers. The possible singularities may be related with the fractures observed in the flows of some complex fluids in Hele-Shaw cells - see [14-18].

The paper is laid out as follows. The constitutive equations for the Oldroyd-B fluid are given in section 2. In section 3 we describe the basic flow and we obtain the equations of perturbations. In section 4 we describe the kinematic and dynamic boundary conditions on the interface air-liquid, that

* Corresponding author:
Gelu.Pasa@imar.ro (Gelu Paşa)
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means *the Laplace law*, used also in [7]. The dispersion relation (52) is given in section 5. We conclude in section 6, where some plots are given, proving the destabilizing effect of the elasticity properties of the Oldroyd-B fluid.

2. The Oldroyd-B Fluid

We consider the following definitions. The extra-stress tensor, the fluid viscosity, the relaxation and retardation (time) constants and the pressure are denoted by τ, η, a_1, a_2, p.

The 3 × 3 matrix containing the partial derivatives of the velocity components u, v, w is denoted by L and the strain-rate tensor is denoted by D:

$$D = (L + L^T),\ (L_{ij})T = L_{ji}.$$

The flow equations are:

$$-\nabla p + \nabla \cdot \tau = 0. \quad (1)$$

The partial x derivative of the components τ_{ij} are denoted by $\tau_{ij,x}$ etc.

We have the following constitutive relations between τ and D

$$\tau + a_1 \tau^{\nabla} = \eta[D + a_2 D^{\nabla}],\ a_1 > a_2 > 0, \quad (2)$$

where τ^{∇} and D^{∇} are the upper convected derivatives

$$\tau^{\nabla} = \tau_t + \mathrm{v}\nabla\tau - (L\tau + \tau L^{\mathrm{T}}),$$
$$D^{\nabla} = D_t + \mathrm{v}\nabla D - (LD + DL^{\mathrm{T}}),$$

where τ_t, D_t are partial time derivatives. As we consider a steady displacement, these time derivatives will be neglected.

We consider an incompressible fluid, then we have the *free divergence condition*:

$$u_x + v_y + w_z = 0. \quad (3)$$

The boundary conditions are:

i) the non-slip condition for the velocity components on the plates;
ii) the Laplace's law on the air-fluid inter- face - see relations (33) - (34) below.

3. Basic Steady Flow and Perturbations

We first consider the flow driven by the constant pressure gradients in the x, y directions and two velocities u, v depending only on z. The elements of this basic flow are denoted by the superscript ^{0}. The subscripts denote the partial derivatives with respect to x, y, z. The basic pressure and velocity are denoted by

$$\nabla p^0 = (p_x^0(x), p_y^0(y), 0),$$
$$\mathrm{v}^0 = (u^0(z), v^0(z),\ 0). \quad (4)$$

The steady state basic solution depends only on z and is verifying the following constitutive relations

$$\tau^0 - a_1(L^0\tau^0 + \tau^0 L^{0\mathrm{T}}) = \eta[D^0 - a_2(L^0 D^0 + D^0 L^{0T})], \quad (5)$$

where only two elements of L^0 are non-zero:

$$L_{13}^0 = u_z^0,\ L_{23}^0 = u_z^0 \quad (6)$$

$$D^0 = L^0 + L^{0T}. \quad (7)$$

Then the components τ_{ij}^0 of the basic extra-stress tensor are given in terms of the basic velocity components by the equations

$$(\tau^0) - a_1 E^0 = \eta D^0 - 2\eta a_2 F^0, \quad (8)$$

where

$$E^0 = L^0\tau^0 + \tau^0 L^{0T}$$
$$= \begin{pmatrix} 2u_z^0\tau_{31}^0 & (u_z^0\tau_{32}^0 + \tau_{13}^0 v_z^0) & u_z^0\tau_{33}^0 \\ (u_z^0\tau_{32}^0 + \tau_{13}^0 v_z^0) & 2u_z^0\tau_{32}^0 & u_z^0\tau_{33}^0 \\ u_z^0\tau_{33}^0 & u_z^0\tau_{33}^0 & 0 \end{pmatrix} \quad (9)$$

$$F^0 = L^0 D^0 + D^0 L^{0T}$$
$$\begin{pmatrix} (u_z^0)^2 & u_z^0\,\upsilon_z^0 & 0 \\ u_z^0\upsilon_z^0 & (\upsilon_z^0)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (10)$$

An immediate consequence of the above relation is $\tau_{33}^0 = 0$. Other components of the basic extra-stress tensor are

$$\begin{aligned} &\tau_{13}^0 = \eta u_z^0, \quad \tau_{23}^0 = \eta v_z^0, \\ &\tau_{11}^0 = 2(a_1 - a_2)\eta(u_z^0)^2, \\ &\tau_{12}^0 = 2(a_1 - a_2)\eta u_z^0 v_z^0, \\ &\tau_{22}^0 = 2(a_1 - a_2)\eta(v_z^0)^2. \end{aligned} \quad (11)$$

As τ_{ij}^0 are depending only on z, we have

$$p_x^0 = \tau_{13,z}^0,\ p_y^0 = \tau_{23,z}^0,\ p_z^0 = 0. \quad (12)$$

From the equations (11)-(12) it follows

$$p_x^0 = \tau_{13,z}^0 = G = \eta u_{zz}^0, \quad (13)$$

where G is a negative constant.

The perturbations of the basic steady flow are denoted by τ, D, p, u, v, w, dropping the super index ^{0}. In the following we will refer only to the perturbed quantities, then no confusion is possible with the notations of the previous section 2.

The basic relations (11) are giving us

$$\begin{aligned} &\tau_{13} = \eta u_z,\ \tau_{23} = \eta v_z, \\ &\tau_{11} = 2\mu u_x + 4(a_1 - a_2)\eta u_z^0 u_z, \\ &\tau_{12} = \mu(u_y + v_x) + 2(a_1 - a_2)\eta u_z^0 v_z + 2(a_1 - a_2)\eta v_z^0 u_z, \\ &\tau_{22} = 2\mu v_y + 4(a_1 - a_2)\eta v_z^0 v_z. \end{aligned} \quad (14)$$

We can see that only the perturbations τ_{12}, τ_{22} are depending on v^0. If $w = 0$ (see the basic flow (16)) we get $\tau_{33} = 0$ - see the justification (21) below.

The perturbed flow equations are

$$\begin{aligned} p_x &= \tau_{11,x} + \tau_{12,y} + \tau_{13,z}, \\ p_y &= \tau_{21,x} + \tau_{22,y} + \tau_{23,z}, \\ p_z &= \tau_{31,x} + \tau_{32,y} + \tau_{33,z}. \end{aligned} \tag{15}$$

Consider now the flow driven by constant pressure gradient only in the x direction and velocity u depending only on z:

$$\begin{aligned} \nabla p^0 &= (p_x^0(x), 0, 0), \\ \mathrm{v}^0 &= (u^0(z),, 0). \end{aligned} \tag{16}$$

The first three perturbations (14) are not depending on v^0. Then we follow Mora and Manna [8], [9] and consider that formulas $(14)_1$ - $(14)_3$ still hold. Only $(14)_4$ and $(14)_5$ become

$$\begin{aligned} \tau_{12} &= \mu(u_y + v_x) + 2(a_1 - a_2)\eta u_z^0 v_z, \\ \tau_{22} &= 2\eta v_y. \end{aligned} \tag{17}$$

Let *l, b* be the width and gap of the Hele-shaw cell. We introduce the scaling

$$\begin{aligned} &x' = x / l, \quad y' = y / l, \quad z' = z / b, \\ &u_i' = u_i / U, \ \epsilon = b / l \in (10^{-3}, 10^{-2}) \end{aligned} \tag{18}$$

where the characteristic velocity U will be defined latter in the formula (30).

We use the free-divergence condition, the above scalings *and get* $w = 0$. Indeed, we have

$$\frac{U}{l}\left\{u'_{x'} + v'_{y'}\right\} + \frac{U}{b}w'_{z'} = 0$$

and because $\epsilon = b/l \in (10^{-3}, 10^{-3})$ we can conclude

$$u'_{x'} + v'_{y'} = 0, \ w'_{z'} = 0 \tag{19}$$

therefore we obtain $w_z = 0$,

$$u_x + v_y = 0 \tag{20}$$

and from non-slip conditions on the Hele-Shaw plates we get $w = 0$.

As $w = 0$, we insert the perturbations in equations (8) and conclude that in this case

$$\tau_{33} = 0. \tag{21}$$

The following Fourier mode decomposition for velocities is considered

$$\begin{aligned} u &= f(z) exp(\alpha x + \sigma t) cos(ny), \\ v &= f(z) \ exp(\alpha x + \sigma t) sin(ny), \end{aligned} \tag{22}$$

then *the free divergence condition* (20) is giving

$$\alpha = -n. \tag{23}$$

The free-divergence condition is also giving

$$\begin{aligned} &2u_{xx} + (u_y + v_x)_y = u_{xx} + u_{yy} = 0, \\ &u_{zx} + v_{zy} = 0. \end{aligned}$$

Then from the formulas (14), (15), $(17)_1$, (22) it follows

$$p_x = \tau_{11,x} + \tau_{12,y} + \tau_{13,z} = 2\eta(a_1 - a_2)u_z^0 u_{zx} + \eta u_{zz}, \tag{24}$$

$$\begin{aligned} p_x - \tau_{11,x} &= \tau_{12,y} + \tau_{13,z} \\ &= \eta(u_y + v_x)_y + 2\eta(a_1 - a_2)u_z^0 u_{zy} + \tau_{13,z}. \end{aligned} \tag{25}$$

From (14), (15), (20), (21) it follows

$$p_z = 0. \tag{26}$$

4. The Laplace Law

We consider non-slip conditions for the velocities on the Hele-Shaw plates, then from the relations (13) we get

$$u^0 = \frac{G}{\eta}\left(z^2 - bz\right) / 2, \tag{27}$$

which are positive: $p_x^0 < 0$ and $z(z-b) < 0$ in the range $z \in (0, b)$.

We follow Wilson [7], then the pressure can depend on the time t:

$$p^0 = G\left(x - \langle u^0 \rangle t\right), \quad x > \ \langle u^0 \rangle t. \tag{28}$$

This is equivalent with formula (7) given in Wilson [7]. We average across the plates in the above relation and obtain

$$\langle u^0 \rangle = \frac{1}{b}\int_0^b u^0 dz = -\frac{Gb^2}{12\eta}; \tag{29}$$

this last relation is used to introduce the characteristic velocity U

$$U = \langle u^0 \rangle = -\frac{Gb^2}{12\eta}. \tag{30}$$

Even if our basic velocity is depending on z, we follow Wilson [7] and consider the kinetic and dynamic boundary condition on the steady air-liquid interface given by the straight line

$$x = \langle u^0 \rangle t.$$

The perturbed interface is

$$\psi = x - \langle u^0 \rangle t; \tag{31}$$

as the interface is material, we have

$$\psi_t = u = \textit{perturbation of } u^0. \tag{32}$$

From (28) we get the *Laplace law* on the air-liquid interface

$$(G\psi + p) - \tau_{11} = \gamma \cdot \left\{\psi_{yy} + \psi_{zz}\right\}, \tag{33}$$

where γ is the surface tension on the interface air-liquid and ψ is the perturbation of the straight initial interface (31). The relation (32) is giving

$$\psi = u / \sigma. \tag{34}$$

The total curvature of ψ in the plane xOy is approximated by $\{\psi_{yy} + \psi_{zz}\}$.

The relations (31) - (34) *are used also in Wilson [7].*

5. The Dispersion Relation

We use the notation

$$I = \exp(\alpha x + \sigma t)\cos(ny). \tag{35}$$

As the derivative with respect to x is equivalent with multiplication with $(-n)$, the relations (22), $(14)_3$, (31) are giving

$$\begin{aligned} \psi_x &= -(n/\sigma) f(z) I, \\ \psi_{yy} &= -(n^2/\sigma) f(z) I, \\ \psi_{zz} &= (2/\sigma) I. \end{aligned} \tag{36}$$

Recall the partial derivative with respect to x is equivalent with multiplication with $(-n)$, then from the relation (25) it follows

$$p - \tau = \left(-\frac{1}{n}\right)\left\{-2n^2\eta f + 2n\eta(a_1 - a_2)\mathrm{u}_z^0 f_z + \eta f_{zz}\right\} I \tag{37}$$

The relations (33) and (37) are giving

$$\frac{Gnf}{\sigma} + 2n^2\eta f - 2n\eta(a_1 - a_2)u_z^0 f_z - 2\eta = \gamma\left[\frac{-n^3}{\sigma} f + 2\frac{n}{\sigma}\right]. \tag{38}$$

The last step is to average across the plates in the above relation. We need the expression of the amplitude function $f(z)$ *appearing in the Fourier modes decomposition (11). For this we use our particular flow geometry and obtain the approximate formula* (46) *below.*

We introduce the dimensionless quantities

$$p' = p\frac{1}{Gl}, \quad n' = nl, \tag{39}$$

$$Q_1 = a_1\frac{Gl}{\mu} < 0, \quad Q_2 = a_2\frac{Gl}{\mu} < 0. \tag{40}$$

Recall (2), $a_1 - a_2 > 0$, then

$$Q_1 - Q_2 < 0. \tag{41}$$

We have

$$u_z^0 = \frac{G(z - b/2)}{\mu} = \frac{Gb(z' - 1/2)}{\mu}$$

then the last two relations and (24) are giving

$$Glp'_{x'}\frac{1}{l} = \mu(Q_1 - Q_2)\frac{b}{l}(2z' - 1)Uu'_{x'z'}\frac{1}{bl} + \mu Uu'_{z'z'}\frac{1}{b^2}. \tag{42}$$

In the left part of the above relation we put $G = -12\mu U/b^2$ (see the relation (30)), we simplify with U, then it follows

$$-12p'_{x'} = (Q_1 - Q_2)(2z' - 1)u'_{x'z'}\epsilon^2 + u'_{z'z'}$$

and

$$-12p' = (Q_1 - Q_2)(2z' - 1)u'_{z'}\epsilon^2 - \frac{u'_{z'z'}}{n}. \tag{43}$$

Recall (26), then $p_z = 0, p'_{z'} = 0$ the above relation (43) is giving

$$(Q_1 - Q_2)\left((2z' - 1)u'_{z'}\right)_{z'}\epsilon^2 - \frac{1}{n'}u'_{z'z'z'} = 0 \tag{44}$$

where

$$u'(z') = \frac{1}{U} f(bz')\exp(-n'x' + \sigma't')\cos(n'y'),$$
$$z' \in (0,1), \quad t' = tl/U,$$

and σ' is given in the relation (48) below.

We can see that:

- if $Q_1 - Q_2 \approx O(\epsilon^0)$, then $f_{z'z'z'} = 0$ is verifying equation (44) with the precision order $O(\epsilon^2)$;
- if $Q_1 - Q_2 \approx O(\epsilon^{-1})$, , then $f_{z'z'z'} = 0$ is verifying equation (44) with the precision order $O(\epsilon)$.

We consider the following condition:

$$Q_1 - Q_2 \approx O(\epsilon^{-1}); \tag{45}$$

the non-slip conditions on the Hele-Shaw plates are giving the $O(\epsilon)$ *approximate solution*

$$f(z) = z(z - b). \tag{46}$$

We perform the average across the plates, we use (27), (30) and obtain

$$\langle f(z)\rangle = (1/b)\int z(z - b)dz = -b^2/6,$$

$$\langle u_z^0 f_z\rangle = \frac{2G}{\eta}\left\langle(z - b/2)^2\right\rangle = \frac{Gb^2}{6\eta} = -2U.$$

From (38) and the last two above relations it follows the dispersion relation

$$\sigma = \frac{Un - (\gamma/\eta)(b^2/12)n^3 - (\gamma/\eta)n}{1 - 2(a_1 - a_2)Un + b^2n^2/6}. \tag{47}$$

We introduce the following *dimensionless quantities*, denoted with the superscript ':

$$\sigma' = \sigma\frac{l}{U}; \quad \gamma' = \gamma\frac{1}{\eta U}; \quad n' = nl; \tag{48}$$

$$D_1 = a_1\frac{U}{l} > 0, \quad D_2 = a_2\frac{U}{l} > 0. \tag{49}$$

The dimensionless quantities D, D_1 are the *Deborah* or

Weissenberg numbers associated to our problem.

We compute now the order of the term $(a_1 - a_2)Un$ in the denominator of the ratio (47). From (30) we have

$$\frac{U}{l} = -\frac{\epsilon^2}{12}\frac{Gl}{\eta} > 0 \tag{50}$$

therefore (see also (40))

$$(a_1 - a_2)Un = (D_1 - D_2)n',$$

$$(a_1 - a_2)Un = -(Q_1 - Q_2)\frac{\epsilon^2}{12}n'$$

and the condition (45) is giving

$$(D_1 - D_2)n' \approx O(\epsilon)n'. \tag{51}$$

The last above relation and (47)-(48)-(49) are giving the dimensionless growth constant

$$\sigma' = \frac{n' - \gamma' n' - \gamma'\left(\epsilon^2/12\right)n'^3}{1 - 2(D_1 - D_2)n' + \epsilon^2 n'^2/6} \tag{52}$$

If $a_1 = a_2$ (that means $D_1 = D_2$) we obtain an expression quite similar with the Saffman-Taylor formula for air displacing a Newtonian fluid with viscosity η, but we have three new terms as follows:

- the new term $\gamma' n'$ in the numerator, due to the meniscus curvature across the plates;
- the new term $2(D - D_1)n'$ in the denominator, due to the non-Newtonian effect;
- the new term $\epsilon^2 n'^2/6$ in the denominator, due to the fact that Fourier mode decomposition (22) does not allow us to neglect the derivatives with respect to x.

6. Conclusions

We obtained an exact dispersion formula for the linear stability of an Oldroyd-B fluid displaced by air in a Hele-Shaw cell. The perturbations of the basic flow are verifying the very simple equations (14), (15), (17), (24), (25), (26).

When $a_1 = a_2$, the obtained dispersion relation (52) is quite similar with the well-known Saffman-Taylor formula for a Newtonian fluid displaced by air in a Hele-Shaw cell:

$$\sigma'_{ST} = n' - \gamma'(\epsilon^2/12)n'^3. \tag{53}$$

In the Figures 1 - 4 we compare our dispersion formula (52) with the Saffman-Taylor formula (53) (the curve appearing unchanged in all figures) in the case $\epsilon = 0.06$, $\gamma' = 0.1$. Then our formula (52) becomes

$$\sigma' = \frac{(1-0.1)x - (0.0003)(0.1)x^3}{1 - Mx + 0.0006 \cdot x^2}. \tag{54}$$

where x, M stands for n' and $2(D_1 - D_2)$.

In Figure 2 we used the following values of order $O(\epsilon)$ (in agreement with the estimate (51)):

$$M = 0.01, 0.02, 0.03, 0.04, 0.041, 0.042.$$

We can see the non-Newtonian destabilizing effects.

In Figures 3, 4 we used the values

$$M = 0.045, 0.048.$$

For $M = 0.05$ we get a blow-up of the growth rate (52), that means we have two real roots of the denominator in the ratio (54), because

$$\Delta = (0.05)^2 - 4(0.0006) = 0.0001 > 0.$$

These roots are giving a blow-up of the growth rate, which can be related with the fractures observed in the flows of some complex fluids in Hele-Shaw cells [14, 15, 17, 18], as we mentioned in Introduction. This phenomenon exceeds the linear stability frame and should be better analyzed by using a nonlinear theory.

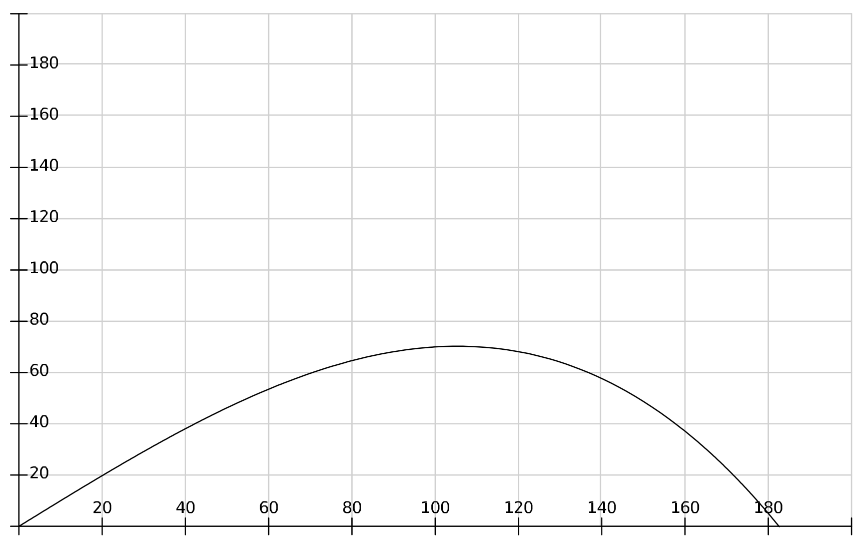

Figure 1. Saffman-Taylor formula (53): $\gamma' = 0.1$, $\epsilon = 0.06$

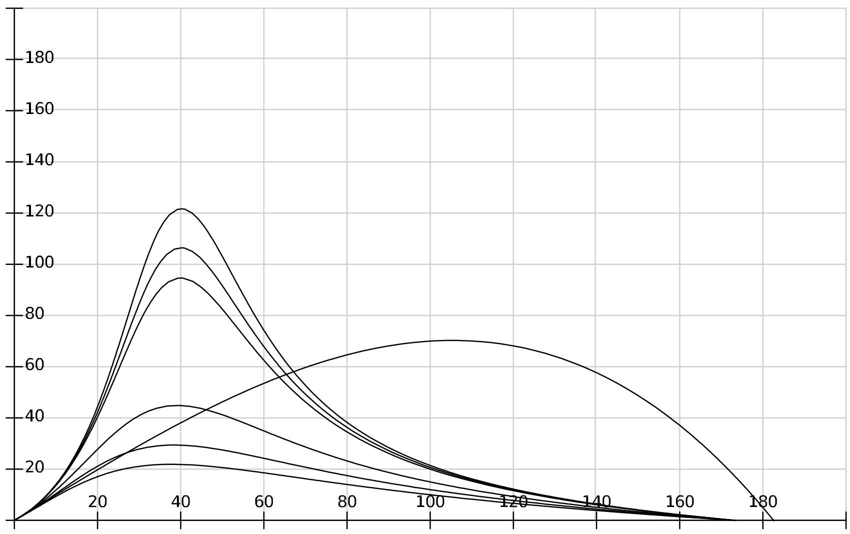

Figure 2. Eq. (54): $\gamma' = 0.1$, $\epsilon = 0.06$, $M = 0.01, 0.02, 0.03, 0.04, 0.041, 0.042$

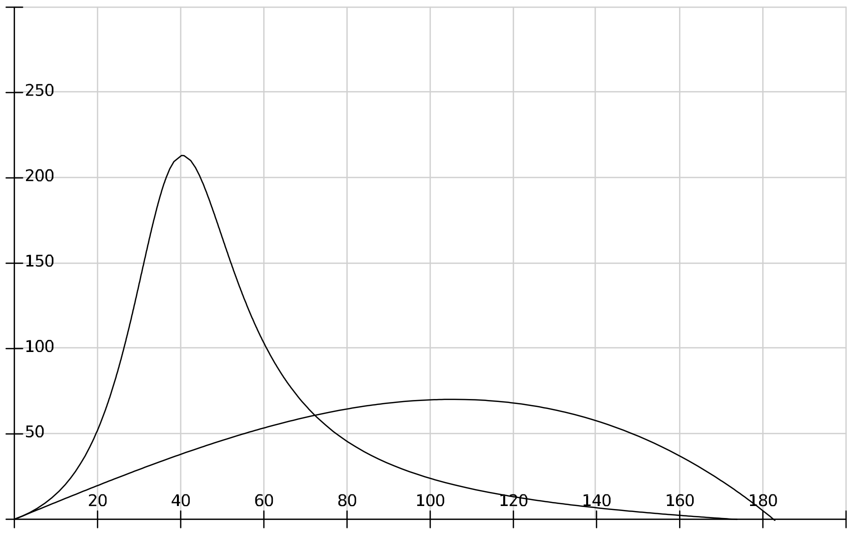

Figure 3. Formula (54): $\gamma' = 0.1$, $\epsilon = 0.06$, $M = 0.045$

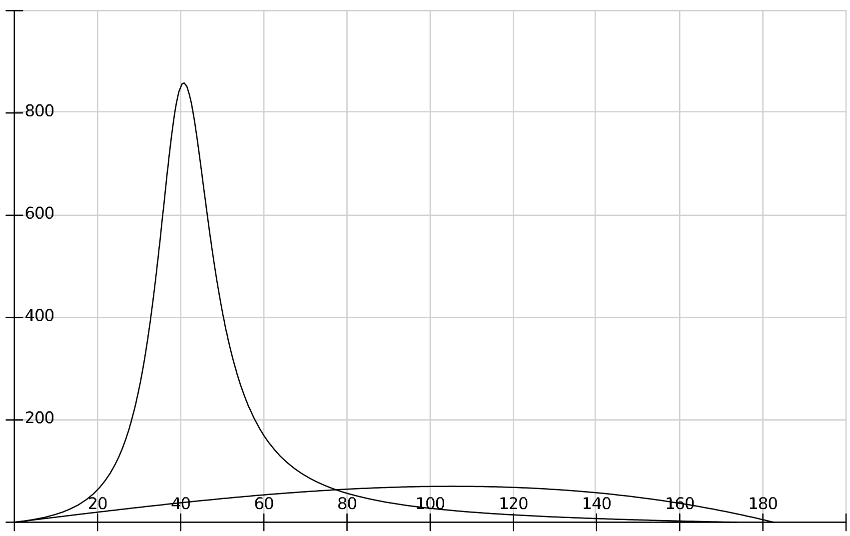

Figure 4. Formula (54): $\gamma' = 0.1$, $\epsilon = 0.06$, $M = 0.048$

We used twice the particular flow geometry due to the Hele-Shaw approximation:

- first, we obtained $w = 0$ - see the dimensionless quantities (18) and the relation (19), where we neglected the terms of order $O(\epsilon)$.
- second, we considered the hypothesis (45) and neglected the terms of order $O(\epsilon)$ in the relation (44).

Then from this point of view, our dispersion formula (52) can be considered as an $O(\epsilon)$ approximation of the growth rate.

We emphasize that in the formula (52) we not neglected the terms of order $O(\epsilon)$ and $O(\epsilon^2)$ which are multiplied with the dimensionless wave number n', n'^2 or n'^3. We can see that, *only for very small values* n' *of the wave numbers*, our formula (52) is quite close to the Saffman-Taylor formula (53).

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