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Second Derivatives Free Fourth-Order Iterative Method Solving for Nonlinear Equation

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Abstract In this paper, we suggest and analyze a new two-step iterative method for solving nonlinear equations, which modified Noor method [6] without second derivatives for nonlinear equation. This Modified method has a fourth-order convergence and efficiency index of this method is $\sqrt[3]{4} \approx 1.5874$. Several examples are given to illustrate the efficiency and the performance of this new method. We also present a comparison among several recent fourth-order convergent methods free from second derivatives.

Keywords Predictor-Corrector method, Iterative method, Convergence criteria, Numerical examples

1. Introduction

In resent years, several iterative type methods have been developed for solving nonlinear equations $f(x_n) = 0$. These methods have been suggested and analyzed by using a variant of different techniques including Taylor series, decomposition and quadrature and homotopy methods, see [1-16] and the references therein for more information. Using the appropriate Taylor series expansion, we first derive an implicit type method for solving the nonlinear equations [9-16]. It is well known that to implement the implicit method, one has to use the prediction and correction type arguments. This facts motivated us to develop and suggest a two-step iterative methods. It is interesting to mention that by combining the predictor and corrector steps, we can obtain new method [6]. We would like to mention that we have to calculate the second derive of the functions, which may create some implementation problems. To overcome this drawback, we modify Noor [6] method by replacing the second derivatives of the function f with new technique in [3]. In this way, we have a modified iterative method for solving nonlinear equations free from the second derivatives. This new method is robush and easy to implement. We also prove that the new modified method is of fourth-order convergence. Several examples are given to illustrate the efficiency and advantage of this two-step method.

2. Iterative Methods

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We consider iterative methods to find a simple root of a nonlinear equation

$$\boldsymbol{f(x) = 0}. \tag{1}$$

where $f: D \subset R \to R$ for an open interval D is a scalar function.

We assume that α is simple root of the equation (1) and γ is an initial guess sufficiently close to α. Use the Taylor's series, we have

$$f(\gamma) + f'(\gamma)(x-\gamma) + f''(\gamma)\frac{(x-\gamma)^2}{2!} + \cdots = 0,, \tag{2}$$

where γ is the initial approximation for a zero.

From (2), we have

$$x = \gamma - \frac{f(\gamma)}{f'(\gamma)} . \tag{3}$$

This allows us to suggest the following iterative method for solving the nonlinear equation (1).

Algorithm 2.1. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Algorithm 2.1 is the well-known Newton method and it is one-step method; which has a quadratic convergence and an efficiency index of$\sqrt{2} \approx 1.414$.

We can rewrite (2), we have

$$x = \gamma - \frac{2f(\gamma)f'(\gamma)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)} \tag{4}$$

This formulation allows us to suggest the following an iterative method for solving the nonlinear equation (1).

Algorithm 2.2: For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme.

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}. \quad n = 0,1,2,\ldots$$

This is known as Halley's method, which has cubic convergence [2]. The efficiency of this method is $\sqrt[3]{3} \approx 1.4422$.

We can rewrite the equation (2) in a different way. This alternative equivalence formula has been used to a different class of iterative methods for solving nonlinear equation $f(x) = 0$.

$$x = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{(x-\gamma)^2 f''(\gamma)}{2f'(\gamma)}. \tag{5}$$

This fixed point formulation allows us to suggest the following method.

Algorithm 2.3: For a given x_0, compute approximates solution x_{n+1} by the iterative scheme.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(x_{n+1} - x_n)^2 f''(x_n)}{2f'(x_n)}.$$

which is an implicit method, since x_{n+1} occurs on both sides. In order to implement Algorithm 2.3, Noor [6, 7] has suggested a predictor-corrector method. We use Algorithm 2.2 as a predictor and then use Algorithm 2.3 as a corrector. The Algorithm of this method is as follows:

Algorithm 2.4: For a given x_0, compute the approximate solution x_{n+1} by iterative schemes.

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \tag{6}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(y_n - x_n)^2 f''(x_n)}{2f'(x_n)}. \tag{7}$$

This is a two-step method. Algorithm 2.4 can be rewritten as:

Algorithm 2.5: For a given x_0, compute the approximate solution x_{n+1} by iterative scheme.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f^2(x_n)f'(x_n)f''(x_n)}{4f'^4(x_n) - 4f(x_n)f'^2(x_n)f''(x_n) + f^2(x_n)f''^2(x_n)}. \tag{8}$$

In order to implement this method, one has to find the second derivative of this function, which may create some problems. To overcome this drawback, one usually replaces the second order derivative by its finite difference. This technique enables us to suggest iterative method free from second derivatives. To be more precise, we the following finite difference technique:

$$f''(x_n) \cong \frac{f'(y_n) - f'(x_n)}{y_n - x_n}. \tag{9}$$

Combining (8) and (9), we suggest the following new iterative method for solving the nonlinear equation.

Algorithm 2.6: For a given x_0, compute the approximate solution x_{n+1} by iterative schemes.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{2f(x_n)[f'(y_n) - f'(x_n)]}{f'^2(y_n) + f'^2(x) + 2f'(x_n)f'(y_n)}. \tag{11}$$

Recently, another approach is used to remove the second derivative from Halley's method and some third-order iterative methods free from second derivative are obtained [3].

Here, we will apply the approach use in [3] to Algorithm 2.7 and obtain a new method free from second derivative. Analysis of convergence shows that this modified method is of fourth-order convergence. And efficiency index of this method is $\sqrt[4]{4} \approx 1.4142$.

Let $y_n = x_n - f(x_n)/f'(x_n)$, we consider Taylor expansion of $f(y_n)$ about x_n

$$f(y_n) \cong f(x_n) + f'(x_n)(y_n - x_n) + f''(x_n)\frac{(y_n - x_n)^2}{2!},$$

which implies

$$f(y_n) \cong \frac{f''(x_n)f(x_n)}{2f'^2(x_n)} \tag{12}$$

using (12) in (8), we obtain

$$x_{n+1} = x_n - \left[\frac{f^2(x_n) - f(x_n)f(y) + f^2(y)}{f^2(x_n) - 2f(x_n)f(y) + f^2(y)}\right]\frac{f(x_n)}{f'(x_n)}.$$

Algorithm of this method can be written as follows

Algorithm 2.7: For a given x_0, compute the approximate solution x_{n+1} by iterative schemes.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{13}$$

$$x_{n+1} = x_n - \left[\frac{f^2(x_n) - f(x_n)f(y) + f^2(y)}{f^2(x_n) - 2f(x_n)f(y) + f^2(y)}\right]\frac{f(x_n)}{f'(x_n)} \tag{14}$$

This method does not require the second derivative. Important characteristic of this method is that per iteration they require two evaluations of the function and one of its first derivatives. The efficiency of this method is $\sqrt[3]{4} \approx 1.5874$, which is better than that of the well-known other methods involving the second-order derivative of the function.

We recall these methods for the sake of completeness.

Kou et al. [4] have suggested a foruth-order methods. The Algorithm of these fourth-order methods can be written as follows:

Algorithm 2.8: (K1, [4]) For a given x_0, compute the approximate solution x_{n+1} by iterative scheme.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \left(1 + \frac{f(y_n)}{f(x_n)} + 2\left(\frac{f(y_n)}{f(x_n)}\right)^2 + \gamma\left(\frac{f(y_n)}{f(x_n)}\right)^3\right)\frac{f(x_n)}{f'(x_n)}. \qquad \gamma \in R.$$

Noor and V. Gupta [5] have suggested a iterative method of fourth-order. The Algorithm of this method can be written as follows:

Algorithm 2.9: (NR, [5]) For a given x_0, compute the approximate solution x_{n+1} by iterative scheme.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{1}{2}\left(\frac{f(y_n)}{f'(y_n)}\right)^2\left(\frac{f'(y_n)}{f(x_n)}\right)\left(\frac{f'(y_n) - f'(x_n)}{f'(y_n)}\right).$$

3. Convergence Criteria

Now we consider the convergence criteria of Algorithm 2.7.

Theorem 1: Let $\alpha \in D$ be a simple zero of sufficiently differentiable function $f : D \subset R \to R$ for an open interval D. If x_0 is sufficiently close to α, then Algorithm 2.7 has fourth-order convergences.

Proof.

If α is the root and e_n be the error at nth iteration, than $e_n = x_n - \alpha$, using Taylor's expansion, we have

$$f(x_n) = f'(\alpha)e_n + \frac{1}{2!}f''(\alpha)e_n^{\ 2} + \frac{1}{3!}f'''(\alpha)e_n^{\ 3} + \frac{1}{4!}f^{(iv)}(\alpha)e_n^{\ 4} + \frac{1}{5!}f^{(v)}(x_n)e_n^{\ 5} + \frac{1}{6!}f^{(vi)}(\alpha)e_n^{\ 6} + O(e_n^{\ 7}), \tag{15}$$

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^{\ 2} + c_3e_n^{\ 3} + c_4e_n^{\ 4} + c_5e_n^{\ 5} + O(e_n^{\ 6})], \tag{16}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^{\ 2} + 4c_4e_n^{\ 3} + 5c_5e_n^{\ 4} + 6c_6e_n^{\ 5} + O(e_n^{\ 6})], \tag{17}$$

where

$$c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}, \qquad k = 2,3,..., \quad \text{let} \quad e_n = x_n - \alpha.$$

From (16) and (17), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^{\ 2} - (2c_3 - 2c_2^2)e_n^{\ 3} - (3c_4 - 7c_2c_3 + 4c_2^3)e_n^{\ 4} + (-6c_3^{\ 2} + 20c_3c_2^2 - 10c_2c_4 + 4c_5 - 8c_2^4)e_n^{\ 5} + O(e_n^{\ 6}). \tag{18}$$

Using (18) in (13), we obtain

$$y_n = \alpha + c_2e_n^{\ 2} + (2c_3 - 2c_2^2)e_n^{\ 3} + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^{\ 4} + (6c_3^{\ 2} - 20c_3c_2^2 + 10c_2c_4 - 4c_5 + 8c_2^4)e_n^{\ 5} + O(e_n^{\ 6}). \tag{19}$$

From (19), we have

$$f(y_n) = f'(\alpha)[c_2e_n^{\ 2} - (-2c_3 + 2c_2^2)e_n^{\ 3} - (7c_3c_2 - 5c_2^3 - 3c_4)e_n^{\ 4} + O(e_n^{\ 5})], \tag{20}$$

from (16) and (20), we obtain

$$f^2(x_n) = f'^2(\alpha)[e_n{}^2 + 2c_2e_n{}^3 + (2c_3 + c_2^2)e_n{}^4 + (2c_4 + 2c_3c_2)e_n{}^5 + O(e_n{}^6)], \tag{21}$$

$$f^2(y_n) = f'^2(\alpha)[c_2^2e_n{}^4 - 4(c_2^3 - c_2c_3)e_n{}^5 + O(e_n{}^6)]. \tag{22}$$

Adding (21) and (22), we obtain

$$f^2(x_n) + f^2(y_n) = f'^2(\alpha)[e_n{}^2 + 2c_2e_n{}^3 + (2c_2^2 + 2c_3)e_n{}^4 + (6c_2c_3 - 4c_2^3 + 2c_4)e_n{}^5 + O(e_n{}^6), \tag{23}$$

from (16) and (20), we have

$$f(x_n)f(y_n) = f'^2(\alpha)[c_2e_n^3 + (2c_3 - c_2^2)e_n{}^4 + (-5c_2c_3 + 3c_2^3 + 3c_4)e_n^5 + O(e_n^6)]. \tag{24}$$

Subtracting (23) and (24), we obtain

$$f^2(x_n) - f(x_n)f(y_n) + f^2(y_n) = f'^2(\alpha)[e_n{}^2 + c_2e_n^3 + 3c_2^2e_n^4 + (11c_2c_3 - 7c_2^3 - c_4)e_n^5 + O(e_n^6)]. \tag{25}$$

$$f^2(x_n) - 2f(x_n)f(y_n) + f^2(y_n) = f'^2(\alpha)[e_n{}^2 + (4c_2^2 - 2c_3)e_n^4 + (16c_2c_3 - 10c_2^3 - 4c_4)e_n^5 + O(e_n^6)] \tag{26}$$

Dividing (25) and (26), we obtain

$$\frac{f^2(x_n) - f(x_n)f(y_n) + f^2(y_n)}{f^2(x_n) - 2f(x_n)f(y_n) + f^2(y_n)} = [1 + c_2e_n + (2c_3 - c_2^2)e_n{}^2$$

$$+(-3c_2c_3 - c_2^3 - 5c_4)e_n^3 + (22c_2^4 + 4c_3^2 - 3c_2^2c_3 + 4c_2c_4)e_n^4 + O(e_n^5)]. \tag{27}$$

Using (18) and (27), we have

$$\left(\frac{f^2(x_n) - f(x_n)f(y_n) + f^2(y_n)}{f^2(x_n) - 2f(x_n)f(y_n) + f^2(y_n)}\right)\frac{f(x_n)}{f'(x_n)} = [e_n + (-8c_4 + 3c_2c_3 - 3c_2^3)e_n{}^4 + O(e_n{}^5)]. \tag{28}$$

Put (28) in (14), we obtain

$$e_{n+1} = (8c_4 - 3c_2c_3 + 3c_2^3)e_n^4 + O(e_n^5),$$

which shows that Algorithm 2.7 has fourth-order convergence.

4. Numerical Examples

We present some examples to illustrate the efficiency of the new developed iterative methods. We compare the Newton method (NM), Noor method (NR, [5]), method of Kou (K1, [4]), method of Algorithm 2.6 (N1, [6]), and new method Algorithm 2.7. We used $\in = 10^{-15}$. The following stopping criteria are used for computer programs:

(i) $|x_{n+1} - x_n| < \in$.

(ii) $|f(x_{n+1})| < \in$.

The examples are the same as in Chun [1].

$f_1(x) = \sin^2 x - x^2 + 1,$ $f_2(x) = x^2 - e^x - 3x + 2,$

$f_3(x) = \cos x - x,$ $f_4(x) = (x-1)^3 - 1,$

$f_5(x) = x^3 - 10,$ $f_6(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$

$f_7(x) = e^{x^2+7x-30} - 1,$

Table 1. Numerical comparison

	IT	x_n	Efficiency Index
f_1, x_0 = 1.3			
NM	4	1.404491648215341226035086817 8	1.4142
NR	3	1.4044916482153412260350868178	1.4142
K1	3	1.4044916482153412260350868178	1.5874
Alg. 2.7	3	1.4044916482153412260350868178	1.5874
f_2, x_0 =2			
NM	5	0.2575302854398607604553673049 4	1.4142
NR	3	0.25753028543986076045536730494	1.4142
K1	3	0.25753028543986076045536730494	1.5874
Alg. 2.7	3	0.25753028543986076045536730494	1.5874
f_3, x_0 =1.7			
NM	4	0.73908513321516064165537208767	1.4142
NR	3	0.73908513321516064165537208767	1.4142
K1	3	0.73908513321516064165537208767	1.5874
Alg. 2.7	3	0.73908513321516064165537208767	1.5874
f_4, x_0 = 2.5			
NM	6	2	1.4142
NR	3	2	1.4141
K1	3	2	1.5874
Alg. 2.7	3	2	1.5874
f_5, x_0 = 2			
NM	4	2.1544346900318837217592935665	1.4142
NR	2	2.1544346900318837217592935665	1.4142
K1	3	2.1544346900318837217592935665	1.5874
Alg. 2.7	2	2.1544346900318837217592935665	1.5874
f_6, x_0 =−2			
NM	8	-1.2076478271309189270094167584	1.4142
NR	5	-1.2076478271309189270094167584	1.4142
K1	5	-1.2076478271309189270094167584	1.5874
Alg. 2.7	4	-1.2076478271309189270094167584	1.5874
f_7, x_0 = 3.2			
NM	8	3	1.4142
NR	5	3	1.4142
K1	4	3	1.5874
Alg. 2.7	4	3	1.5874

5. Conclusions

We have shown that it is possible to obtain modifications of Noor [6] method free from second derivative by using a new approach to remove the second derivative from Noor method. We prove that the order of convergence of this method is four. Analysis of efficiency shows that this method is preferable to the well-known other methods involving the second order derivative of the function. Numerical results show that these methods have definite practical utility.

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