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# Optimal Policy on the Possible Rate of Returns of Contingent Claim by Fractal Dispersion on Hausdorff Measure to Market Signal

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**Abstract** We studied the rate of returns on investment as the net gain in wealth over the cumulative investment in continuous time. Dynamic asset allocations are continuously rebalanced so as to always keep a fixed constant proportion of wealth invested in various assets at each point in time play a fundamental role in the theory of optimal portfolio strategy. We proved that: (i) the limiting distribution of this measure of return is gamma distribution if the returns follow a geometric Brownian motion; (ii) if returns follow Weibull distribution, then it results to asymptotic power-law behavior of assets returns. For example, the mean return on investment is maximized by the same strategy that maximizes logarithm utility which is also known to minimize the exponential rate at which wealth grows and the return from this policy turns out to have stochastic dominance properties as well. We consider the logistic function in large market financial crashes corresponding to values of packing dimension of $R^n$ $or$ $\alpha_{max}$ by the fractal dispersion of Hausdorff measure prior to market signal with constraint of a zero heat capacity, the existence of a unique solution to the associated Hausdorff is established and optimal policy is characterized. Also advocated is a procedure for locating optimal market crises signal relative to the heat equation to give an early warning.

**Keywords** Optimal policy, Contingent claim, Power Law, Hausdorff measure, Market signal, Fractal dispersion

## 1. Introduction

Ethier et al, [2] studied the rate of return on investment in discrete time gambling method, where the total return on the individual gambler is assumed to follow a random walk. They also showed that the asymptotic distribution of the return, as the mean increment in the random walk goes to zero is a gamma distribution. Also Kelly (1956) in Thorp, [11] observed the relationship between the logarithm of wealth and expected asymptotic rate at which wealth compounds. The object is to let one know how one should invest in each equities of ones highly diversified stocks portfolio to maximize the capital growth. In finance, the rate of return (ROR) which is known as return on investment (ROI) is the ratio of money gained or lost whether realized or unrealized on an investment, relative to the amount of money invested. The amount of money gained or lost may be referred to as interest, profit/loss, gain/loss or net income/loss. There are several ways to determine ROI, but the most frequently used method is net gain divided by total assets. Constant proportions investment strategies also play a fundamental role in portfolio theory. Under these policies, an investor follows a dynamic trading strategy that continually rebalances the portfolio so as to always allocate fixed constant proportions of the investor's wealth across the investment opportunities. These strategies are widely used in practice and are also referred to as continuously rebalanced strategy [9]. Given the fundamental nature of policies in theoretical portfolio practice, it is of interest to know what the stochastic behavior of the rate of return on investment (RROI), defined as the net of gain over the cumulative investment [3]. Merton, [5] introduced the setting in the continuous time financial model as used in Black-Scholes, [1].

In this paper, we obtain some limit theorems for RROI which allows us to compare and derive some specific optimality properties for certain portfolio strategies. We also established and proved that the return on investment for such policies converges to a limiting stochastic distribution and the result provides a basis upon which to compare different strategies on Hausdorff measure prior to the heat equation to locate market crises and give early warning.

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## 2. The Rate of Return from Total Investment

The rate of return on investment is defined as the ratio of net gain in wealth over the cumulative investment. Our interest here is the rate of return from the total investment (RROI), which for the fixed policy will be denoted by the process $(pf(t): t \geq 0)$

Hence

$$pf(t) = \frac{X_t^f - X_0}{\int_s^t X_s^f ds}, \quad t \geq 0, \qquad (2.1)$$

which is equivalent to the least square estimator $\phi_t$ of $\phi$ studied by Hu, et al [4] on the continuous parametric estimation of observed fractional Ornstein-Uhlenbeck process $X = \{X_{t,} t \geq 0\}$ defined as

$$X_0 = 0 \text{ and } dX_t = \phi X_t dt + dB_t^H, t \geq 0. \qquad (2.2)$$

$B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion with Hurst parameter $H\epsilon(\frac{1}{2}, 1)$given as

$$\phi_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds} \quad t \geq 0, \qquad (2.3)$$

Where $pf(t)$ is a measure of the wealth it takes to finance a gain. If $pf(t)$ is large; it means that the investor is accumulating gains at a faster rate than if it is small. Note that if we divide the numerator and the denominator by t in equation (2.1) we also interpret $pf(t)$ as the average net gain over the average wealth level.

**Theorem 2.1**

If the returns of contingent claim $pf(t) = \frac{X_t^f - X_0}{\int_s^t X_s^f ds} f_t$, $t \geq 0$ follow geometric Brownian motion $X_t^f = X_0 exp\left[\left(r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2\right)t + f\sigma W_t\right]$, then the resulting distribution is Gamma distribution, that is, $pf(t) \xrightarrow{d} pt \sim gamma\left(\frac{2(r+f\tilde{\mu})}{\sigma^2 f^2} - 1, \frac{2}{\sigma^2 f^2}\right)$ $as\ t \to \infty$.

**Proof**

Recall that $X_t^f = X_0 exp\left[(r + f\tilde{\mu})dt + f\sigma X_t^f dW_t\right]$ then

$$\int_0^t X_t^f ds = X_0 \int_0^t exp\left[(r + f\tilde{\mu})dst + f\sigma X_s^f dW_s\right] ds \qquad (2.4)$$

From geometric Brownian motion,

$$X_t^f = X_0 exp\left[\left(r + f\tilde{\mu} - {}^1\!/_2 f^2\sigma^2\right)t + f\sigma W_t\right] \qquad (2.5)$$

and

$$\int_0^t X_t^f ds = X_0 \int_0^t exp\left[\left(r + f\tilde{\mu} - {}^1\!/_2 f^2\sigma^2\right)s + f\sigma W_s\right] ds \qquad (2.6)$$

Substitute (2.5) and (2.6) in (2.1) to get

$$pf(t) = \frac{X_0 exp\left[\left(r+f\tilde{\mu}-{}^1\!/_2 f^2\sigma^2\right)t+f\sigma W_t\right]-X_0}{X_0 \int_0^t exp\left[\left(r+f\tilde{\mu}-{}^1\!/_2 f^2\sigma^2\right)s+f\sigma W_s\right]ds},$$

which implies

$$pf(t) = \frac{X_0\{exp\left[\left(r+f\tilde{\mu}-{}^1\!/_2 f^2\sigma^2\right)t+f\sigma W_t\right]-1\}}{X_0 \int_0^t exp\left[\left(r+f\tilde{\mu}-{}^1\!/_2 f^2\sigma^2\right)s+f\sigma W_s\right]ds},$$

and simplifies to

$$pf(t) = \frac{exp\left[\left(r+f\tilde{\mu}-{}^1\!/_2 f^2\sigma^2\right)t+f\sigma W_t\right]-1}{\int_0^t exp\left[\left(r+f\tilde{\mu}-{}^1\!/_2 f^2\sigma^2\right)s+f\sigma W_s\right]ds} \qquad (2.7)$$

If $f = 0$ we have that from equation (2.7) it shows that for that policy under which the total wealth is always invested in the risk-free asset is

$$p_0(t) = \frac{e^n - 1}{\int_0^t e^{rs} ds}.$$

But $\int_0^t e^{rs} ds = {}^1\!/_r e^{rs}|_0^t$, so that

$$p_0(t) = {}^1\!/_r (e^{rs} - 1)$$
$$= \frac{e^n - 1}{{}^1\!/_r (e^{rt} - 1)}.$$

If $n = rt$ then $p_0(t) = r$ is the risk free interest rate as expected but if $f \neq 0$, the rate of return on investment (RROI) process $pf(t) = t > 0$ is complicated and does not yield to a simple direct analysis.

NOTE: a random variable X$\sim$ gamma $(\alpha, \beta)$ means that X is a random variable with density function

$$\varphi(x) = \frac{e^{-\beta x} x^{\alpha-1} \beta^\alpha}{\gamma(\alpha)},$$

with $E(x) = {}^\alpha\!/_\beta$ and the $Var(x) = {}^\alpha\!/_{\beta^2}$.

And for any fixed proportion that satisfies $\pi_t^b(\delta) = \delta f^*\ for$ all $t \geq 0\ and\ all\ \delta > 0$, the (RROI) process $pf(t) = t > 0$ converges $(as\ t \to \infty)$ in distribution to random variable which has a gamma distribution, where $f^*$ is the constant vector given by

$$f^* = \Sigma(\mu - r)^{-1} (\mu - r) \equiv \Sigma\,\tilde{\mu}^{-1}.$$

If $\pi_t$ is a constant vector for all $t \geq 0$ such a policy is called constant proportion policy which is the optimal investment policy for any interesting objective function. A constant vector is also optimal policy for other objective criteria, such as minimizing the expected time t to reach a given level of wealth as well.

Specifically, as $t \to \infty$ we have

$$pf(t) \xrightarrow{d} pt \sim gamma\left(\frac{2(r+f\tilde{\mu})}{\sigma^2 f^2} - 1, \frac{2}{\sigma^2 f^2}\right), \qquad (2.8)$$

where $\xrightarrow{d}$ denotes convergence in distribution.

Therefore to get the expectation of $E(pf)$ we have

$$E(pf) = \frac{\frac{2(r + f\tilde{\mu})}{\sigma^2 f^2} - 1}{,\frac{2}{\sigma^2 f^2}}$$
$$= r + f\tilde{\mu} - \frac{\sigma^2 f^2}{2}. \qquad (2.9)$$

The expectation in (2.9) should not be confused with the ratio of the expected gain to the expected total investment which for any $t > 0$, is equal to

$$\frac{E(total\ gain)}{E(total\ investment)} = \frac{E(X_t^f - X_0)}{E(\int_0^t X_t^f ds)}$$
$$= \frac{X_o e - X_0}{E(\int_0^t X_t^f ds)} = (r + f\tilde{\mu})t$$
$$= \frac{X_o e - X_0}{E(\int_0^t X_t^f ds)} = (r + f\tilde{\mu})t$$
$$= \frac{X_o(e-1)}{X_0 \frac{1}{r+f\tilde{\mu}}(e^{(r+f\tilde{\mu})t} - 1)}$$
$$= r + f \qquad (2.10)$$

**Theorem 2.2**

If the returns $pf(t)$ as defined in (2.1) follow Weibull randomvariates $F(x)$ , then the resulting distribution $pt = \frac{\alpha}{\gamma[pf(t)]^{\gamma-1}}$ follows asymptotic power-law.

**Proof**

Let $pf(t) = \frac{X_t^f - X_0}{\int_s^t X_s^f ds}$, $t \geq 0$ be distributed according to the following probability density function

$$F(pf(t)) = \begin{cases} \frac{\beta}{\alpha}(pf(t)))^{\beta-1} e^{-(|pf(t)|)^{\beta}}, & pf(t)) \geq 0 \\ o & otherwise \end{cases}, \qquad (2.11)$$

where $\alpha, \beta > 0$ are the mean and the shape parameters of the Weibull distribution (2.11).

If $pf(t))$ has the Weibull density function, then $Z = \left(\frac{X_t^f - X_0}{\int_s^t X_s^f ds}\right)^{\gamma}$, has the exponential density function with $\alpha = 1$ as (using the formula in [7]);

$$f_Z(z) = \begin{cases} exp\left\{-\left(\frac{X_t^f - X_0}{\int_0^t X_s^f ds}\right)^{\gamma}\right\}, & pf(t) \geq 0 \\ 0 & , pf(t) < 0 \end{cases}. \qquad (2.12)$$

Thus the optimal investment strategy is (see [8]);

$$H(f_Z(z)) = \int_0^{\infty} f_Z(z) d\, f_Z(z). \qquad (2.13)$$

That is

$$H(f_Z(z)) = \int_0^{\infty} exp\left\{-\left(\frac{X_t^f - X_0}{\int_0^t X_s^f ds}\right)^{\gamma}\right\} dX_t^f. \qquad (2.14)$$

But $dX_t^f = \varphi(f) dpf(t)$ (where $\varphi(f)$ is as in (2.4) or (2.6)), so that

$$H(f_Z(z)) = \int_0^{\infty} exp\{-(pf(t))^{\gamma}\} dpf(t).$$

Hence the optimal strategy is the asymptotic power-law;

$$H = \frac{\varphi(f)}{\gamma}(pf(t))^{1-\gamma}, \qquad (2.15)$$

where$\gamma$ is the fractal exponent given by (see [6]) $\gamma = \frac{a q_n^2}{2}$, with $q_n > 0$ a Bessel function given as $\frac{J_n}{2-2(x)}$ and $a$ the singularity strength $\left(0 \leq a \leq \frac{4}{q_n^2}\right)$.

## 3. Optimal Growth Policy and Stochastic Dominance

Here we can see that the quantity (2.8) is maximized by the strategy that invests as much as possible in the risky asset. The mean of (2.7), is maximized at a finite value

$$f^* = \frac{\tilde{\mu}}{\sigma^2}, \qquad (3.1)$$

which is the same policy that is optimal for maximizing logarithmic utility of wealth at a fixed terminal time and hence for maximizing exponential rate of growth. Notice that the mean of the limiting distribution of (RROI) process is maximized at the value $f^* = \frac{\tilde{\mu}}{\sigma^2}$ with resulting mean $E(p^*) = r + \gamma$, for this strategy the RROI, $p^*(t)$, satisfies

$$p^*(t) \xrightarrow{d} p^* \sim gamma\left(\frac{r+\gamma}{r}, \frac{1}{r}\right). \qquad (3.2)$$

In fact, the distribution characterization of the limiting RROI allows for some-what stronger statement, in terms of stochastic orderings. Suppose that for two random variables, X,Y we say that $X \leq icxY\ if$

$$E(X - x)^* \leq E(Y - x)^*, \forall x.$$

This is equivalent to say that $E(g(x)) \leq E(g(y))$ for all increasing convex function $g$, and is referred to as the increasing convex ordering. We also say that (provided the expectations are finite) this is equivalent to saying that $E(h(x)) \leq E(h(Y))$ for all increasing concave function $h$, and is hence referred to as the increasing concave ordering.

We let$p^*$denote the RRIO obtained from using the policy $f^*$ defined in (2.9) and let $pf$ be the RROI. From any other constant proportion strategy $f = cf^*$, where c is an arbitrary constant, the following hold

$$pf \leq icxp^*\ for\ C \leq 1 \qquad (a)$$
$$pf \leq icvp^*\ for\ C \geq 1 \qquad (b)$$

Equation (a) is in effect for investor with greater relative risk aversion while equation (b) is in effect for an investor with less relative risk aversion. Then for a proportional strategy $f = Cf^*, with\ c \neq 0$
where $f^*$ is the optimal policy of (2.7), the relationship $pf \leq_{st} p^*$ holds if and only if C satisfies $1 - \sqrt{r + \gamma/_r} < C < -{}^r/_{r + 2\gamma}$.

Showing that the only type of constant proportion policy (other than $f^*$) for which RROI is stochastically dominated by the optimal growth policy is one that is shorting the stock to the degree required by $(f)$. To establish equation (2.1) in section 1 above, we have that the process $pf(t), t \geq 0$ does not admit a simple direct analysis, there is a related markov process amenable to analysis which holds the key for the limiting behavior of $pf(t)$, specifically the $Rf(t), t \geq 0$defined by

$$R_f(t) = \frac{X_t^f}{X_0+\int_0^t X_s^f ds}. \quad (3.3)$$

If t = 0 we have $Rf(0) = 1$, also we will first show that the limiting behavior of $pf(t)$ is equivalent to the limiting behavior of $Rf(t)$. But our interest here is to show the limiting behavior of the RROI process $pf(t)$. We show that the diffusion process $Rf(t)$ whose limiting behavior can be analyzed.

Suppose that for random variable $Rf$, we have $Rf(t) \xrightarrow{d} Rf \text{ as } t \to \infty$. Then for any $f \text{ as } t \to \infty$ we have $pf(t) \xrightarrow{d} Rf$

$$pf(t) = Rf(t)\frac{(X_t^f - X_0)}{X_t^f}\left[\frac{X_0 + \int_0^t X_t^f ds}{\int_0^t X_s^f ds}\right]$$

$$= Rf(t)(1-e^{-B_t})\left[1+\frac{1}{\int_0^t e^{B_s} ds}\right], \quad (3.4)$$

where $B_s, s \geq 0$ is the linear Brownian motion defined by

$$B_s = \left(r + f\tilde{\mu} - {}^1\!/_2 f^2\sigma^2\right) + f\sigma W_s,\ B_0 = 0.$$

Note that $B_s$ hasa positive drifts which implies that $\lim_{t\to\infty} e^{-B_t} = \lim_{t\to\infty}\frac{1}{e^{B_t}} = 0\ a.s$ as well as

$$\lim_{t\to\infty}\left(\int_0^1 e^{B_s} ds\right)^{-1} = \lim_{t\to\infty}\frac{1}{\int_0^1 e^{B_s} ds} = 0 \quad a.s.$$

Equation (2.1) will be completely establish if we can prove that $Rf(t) \xrightarrow{d} Rf$, for some random variable $Rf$ with $Rf = pf$.

**Lemma 3.1:** For some fixed proportion investment policy $f$, the process $Rf(t)$ follows the stochastic differential equation

$$dRf(t) = [(r + f\tilde{\mu})Rf(t) - R^2 f(t)]dt + f\sigma Rf(t)dW_t \quad (3.5)$$

$Rf(t)$ is a temporary homogeneous diffusion process with drift function

$$b(x) = (r + f\tilde{\mu})x - x^2$$

and diffusion function

$$V^2(x) = f^2\sigma^2 x^2.$$

**Proof**

Let $A_t = \int_0^t X_s^t ds$ be the cumulative wealth investment process, also let

$$Rf(t) = X_t^f (X_0 + A_t).$$

Since $A_t$ is a process of bounded variation, its Ito's rule show that $dA_t = X_t^f dt$ so applying Ito's rule to $Rf(t)$ gives

$$d\frac{X_t^f}{X_0 + A_t} = \frac{1}{X_0 + A_t} dX_t^f - \frac{X_t^f}{(X_0 + A_t)^2} dA_t$$

which upon substitution gives

$$d\frac{X_t^f}{X_0+A_t} = \frac{X_t^f}{X_0+A_t}(r + f\tilde{\mu})dt + \frac{X_t^f}{X_0+A_t} f\sigma dW_t - \left(\frac{X_t^f}{X_0+A_t}\right)^2 dt. \quad (3.6)$$

The result is equivalent to (3.5). Here, the policy is maximized by a strategy that invests as much as possible in the risky asset and the total return from this policy turns out to have stochastic dominance property as well. This continuous stochastic process can be used for modeling random behavior that evolves over time like fluctuation in an asset price.

# 4. Application on Logistic Financial Fractal Dispersion Function of the Hausdorff Priorto Crash Market Signal

One of the several distinct techniques of investigating the size of subsets of zero market in $R^n$ is the notion of packing dimension due to Taylor and Triort, [10].

Packing dimension is defined via the packing measure as a class φ of monotone functions h $h: (0, \delta) \to (0,1)$ which is non decreasing, right continuous satisfying h(0+) = 0 and for which there is a constant.

$$C > 0 \text{ such that } h(2s) = Ch(s) \text{ for } 0 < s < \frac{\delta}{2} \quad (4.1)$$

We obtain the packing dimension by two definitions. First, we defined a pre-measure

$$h - p(E) = Lim\,sup \textstyle\sum_{i=1}^{\infty} h(2r): B_r\,(X) \quad (4.2)$$

disjoint $X \subseteq E.r.$

Where $B_r(X)$ denotes the open ball centered on X radius r Eq(4.2) is not an outer measure because it is not count ably Sub-additive. However it leads to an outer measure by defining

$$h - p(E) = \lim_{s\to n} Inf \textstyle\sum_{i=1}^{\infty} h - p(E): E \subset \bigcup_{i=1}^{\infty} E_1\, B_r\,(X) \quad (4.3)$$

which can be thought of as a generalization of Hausdorff measure using maximal packing of E by balls, so that if h(s) $= S^n$ then h –p( ) on $R^n$ is n-dimensional Hausdorff measure. Thus to measure the Borel subset of E $\subset R^n$ we need

$$\lim_{s\to n}\frac{S^n}{h(s)} = 0 \Rightarrow pf(t) = \frac{X_t^f - X_0}{\int_s^t X_s^f ds} f_r\ t \geq 0 \text{ from (2.1)} \quad (4.4)$$

So that if h(s) = $S^n$ then $\alpha > 0$, there is a unique value of $\alpha$ for which the packing measure h- P(E) drops from infinity to zero.

This in turn means that E is less occupied than if it were $\alpha - \varepsilon$ dimensional. For $\varepsilon > 0$. We define the next Dim E as

$$DimE = inf\{\alpha > 0: S^n - P(E) = o\} = \inf\frac{\mu(B_t(x))}{h(2r)} = V \quad (4.5)$$

$$DimE = sup\{\alpha > 0: S^n - P(E) = \infty\} = \sup\frac{\mu(B_t(x))}{h(2r)} = hs \quad (4.6)$$

which denotes the packing dimension E.

The underlying assets is paying nontrivial continuous dividends with an annualized yield $D \geq 0$. A holder of the underlying asset receives a dividends yield $DSdt$over any time interval with length$dt$. Paying dividends leads to the asset price decrease

$$dS = (\mu - D)Sdt + \hat{\sigma}SdW. \quad (4.7)$$

Comparing equations (3.5) and (4.7) implies that$Rf(t) = S, \mu = \tilde{\mu}f, D = r - Rf(t), \sigma f = \hat{\sigma}$. After an elapse of time $\Delta t$ the value of the portfolio will change by the rate $h(\Delta S + D\Delta t) - \Delta V$ in view of the dividend received on h units held. Using Ito's lemma for $dV$ we conclude with the equation

$$\frac{\partial V}{\partial t}+(r-D)S\frac{\partial V}{\partial S}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2-rV=0. \quad (4.8)$$

Using a new function $V(S,\tau)=Ke^{-(\alpha x+\beta t)}u(x,\tau)$, (where $\alpha,\beta\in R$are some constants) and the transformation $x=\ln {S}/{K}$ , $\tau=T-t$ we have; $\frac{\partial u}{\partial x}=S\frac{\partial V}{\partial S}$, $\frac{\partial^2 u}{\partial x^2}=S^2\frac{\partial^2 V}{\partial S^2}+S\frac{\partial V}{\partial S}=S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial u}{\partial x}$, which is substituted in (4.8) to get ;

$$\frac{\partial u}{\partial \tau}+A\frac{\partial u}{\partial x}-\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2+Bu=0,$$
$$u(x,0)=0,\ Ee^{\alpha x}\max(e^x-1,0), \quad (4.9)$$

where $A=\alpha\sigma^2+\frac{\sigma^2}{2}-r+D$ and $B=(1+\alpha)(r-D)-\beta-\frac{\alpha^2\sigma^2+\alpha\sigma^2}{2}$. By setting $\alpha=\frac{r-D}{\sigma^2}-\frac{1}{2}$, $\beta=\frac{r+D}{2}+\frac{\sigma^2}{8}+\frac{(r-D)^2}{2\sigma^2}$ we $A=B=0$, so that (4.9) reduces to finding the solution of the Cauchy problem

$$\frac{\partial u}{\partial \tau}-\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2=0\,,u(x,0)=u_0(x),$$
$$-\infty<x<\infty,\tau\in[0,T]. \quad (4.10)$$

Consider a market comprising of $h$ unit of wealth in long position (expected) and $u$ unit of the wealth in short position (actual), at time $\tau$ the market value is assumed to be $h-u$ after an elapse the value changes by the amount and the Hausdorff measure in this case is to determine which subsets of $R^{n+1}(R^n$ is the n-dimensional Euclidean space) are of zero heat capacity (that is where there is no market signal and hence market crash) with respect to the heat equation

$$\sum_j^n\frac{\partial^2 u}{\partial x_j^2}=\frac{\partial u}{\partial \tau}. \quad (4.11)$$

A solution $u(x,\tau)$ to the Cauchy problem (4.10) is given by the Green's formula

$$u(x,\tau)=\frac{1}{\sqrt{2\sigma^2\pi\tau}}\int_{-\infty}^{\infty}e^{-\frac{(x-S)^2}{2\sigma^2}}u(S,0)dS. \quad (4.12)$$

Equation (4.12) gives us the idea that the transition density of the Brownian motion in $R^n$ is just the heat kernel (for $\sigma^2=2$);

$$u(x,\tau)=\begin{cases}(4\pi\tau)^{-n/2}exp\left(\frac{\|x-s_0\|^2}{4\tau}\right), & \tau>0\\ 0, & \tau\le 0\end{cases},$$

which satisfies the heat equation (4.11). The measure zero is equivalent to the heat capacity zero on the hyper plane. It therefore follows that the solution of (4.8) is given as;

$$V(S,T-t)$$
$$=Ee^{-(\alpha\ln {S}/{K}+\beta t)}\left(4\pi(T-t)\right)^{-n/2}exp\left(\frac{\left\|ln\, {S}/{K}-s_0\right\|^2}{4(T-t)}\right),$$
$$T-t>0. \quad (4.13)$$

The market crash (bubble) source gives rise to a $V{:}\,R^{n+1}\to R$which assigns a value to a portfolio $V(S,\tau)$ to each point of $(x,\tau)\in R^{n+1}$ through a generating kernel $u(x,\tau)$ so that on a positive market strategy $\pi$ on $R^{n+1}$ is the portfolio growth rate if it is finite on a dense subset of $R^{n+1}$.

## 5. Conclusions

Equations (2.8) and (2.15) reveal that: (i) if assets returns follow a geometric Brownian motion, then the limiting distribution is gamma distribution, conversely (ii) if returns follow Weibull distribution, then it results to asymptotic power-law behavior. Furthermore the optimal strategy (2.15) depends on $\gamma$ which in turn depends on $a$ (the singularity strength). $\gamma\to 0$ as $a\to 0$ and $H$(the optimal claims) increases without bound. As $a\to\frac{4}{q_n^2}$, $\gamma\to 2$ and $H$ decrease constantly with $t$. Given (3.1), we have $\mu=\left(\frac{\tilde{\mu}}{\sigma}\right)^2$ (the square of the correlation coefficient) and $\hat{\sigma}=\frac{\tilde{\mu}}{\sigma}$.

The policy is maximized by a strategy that invest as much as possible in the risky asset and the total return from this policy turns out to have stochastic dominance property as well. This continuous stochastic process can be used for modeling random behavior that evolves over time like fluctuation in an asset price and With Hausdorff measure and the heat equation via packing dimension $h(s)-V$ and $\frac{h(s)}{s^n}$ there is no market signal as it tends to zero hence the market is likely to crash at that point indicating shortfall on the wealth investment. The exact packing dimension is determined for subset of $R^n.n\ge 3$ for which $u(X,t)$ is unbounded for $(x,t)\subset R^{n+1}$. Where $R^n$ is n-dimensional Euclidean space. (size of the market)

And for analysis of subset of B of $R^n$ of Zero Hausdorff measure, it is appropriate to assume that

$$\frac{h(s)}{S^\alpha}\to\infty\ as\ S\to 0$$

So that if $h(S)=S^\alpha\ \alpha>0$ but $h-P(E)$ turns out to be either zero or infinity.

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