

Ruin Probability in a Generalized Risk Process under Rates of Interest with Homogenous Markov Chain Claims and Homogenous Markov Chain Interests

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Abstract The aim of this paper is to give recursive and integral equations for ruin probabilities of generalized risk processes under assumption that both sequences of claims and rates of interest are homogenous Markov chains. Generalized Lundberg inequalities for ruin probabilities of these processes are derived by using recursive technique. Firstly, we give a recursive equations for finite – time probability and ultimate ruin probability. By using these equations, we can derive probability inequalities for finite – time probability and ultimate ruin probability. The above results give upper bounds for finite – time probability and ultimate ruin probability. A numerical example is given to illustrate results.

Keywords Integral equation, Recursive equation, Ruin probability, Homogeneous Markov chain

1. Introduction

For over a century, there has been a major interest in actuarial science. Since a large portion of the surplus of insurance business from investment income, actuaries have been studying ruin problems under risk models with rates of interest. For example, Teugels and Sundt[10],[11] studied the effects of constant rate on the ruin probability under the compound Poisson risk model. Yang[13] established both exponential and non – exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Cai[3],[4] investigated the ruin probabilities in two risk models, with independent premiums and claims and used a first – order autoregressive process to model the rates of in interest. Cai and Dickson[5] obtained Lundberg inequalities for ruin probabilities in two discrete- time risk process with a Markov chain interest model and independent premiums and claims.

In this paper, we study the models considered by Cai and Dickson[5] to the case homogenous Markov chain claims and homogenous Markov chain rates of interest and independent premiums. The main difference between the model in our paper and the one in Cai and Dickson[5] is that claims and rates of interest in our model are assumed to follow homogeneous Markov chains.

We let $X = \{X_n\}_{n \geq 0}$ be premiums, $Y = \{Y_n\}_{n \geq 0}$ be

claims, $I = \{I_n\}_{n \geq 0}$ be interests and they define on probability space (Ω, \mathcal{A}, P) . To establish probability inequalities for ruin probabilities of these models, we study two styles of premium collections. On the one hand of the premiums are collected at the beginning of each period then the surplus process $\{U_n^{(1)}\}_{n \geq 1}$ with initial surplus u can be written as

$$U_n^{(1)} = U_{n-1}^{(1)}(1 + I_n) + X_n - Y_n \quad (1.1)$$

which can be rearranged as

$$U_n^{(1)} = u \cdot \prod_{k=1}^n (1 + I_k) + \sum_{k=1}^n (X_k - Y_k) \prod_{p=k+1}^n (1 + I_p), \quad (1.2)$$

On the other hand, if the premiums are collected at the end of each period, then the surplus process $\{U_n^{(2)}\}_{n \geq 1}$ with initial surplus u can be written as

$$U_n^{(2)} = (U_{n-1}^{(2)} + X_n)(1 + I_n) - Y_n, \quad (1.3)$$

which is equivalent to

$$U_n^{(2)} = u \cdot \prod_{k=1}^n (1 + I_k) + \sum_{k=1}^n [X_k(1 + I_k) - Y_k] \prod_{p=k+1}^n (1 + I_p), \quad (1.4)$$

where throughout this paper, we denote $\prod_{t=a}^b x_t = 1$ and

$$\sum_{t=a}^b x_t = 0 \text{ if } a > b.$$

We assume that:

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Assumption 1.1. $U_o^{(1)} = U_o^{(2)} = u > 0$

Assumption 1.2. $X = \{X_n\}_{n \geq 0}$ is a sequence of independent and identically distributed non - negative continuous random variables with the same distributive function $F(x) = P(\omega \in \Omega : X_0(\omega) \leq x)$.

Assumption 1.3. $Y = \{Y_n\}_{n \geq 0}$ is a homogeneous Markov chain, Y_n take values in a finite set of non - negative numbers $E_Y = \{y_1, y_2, \dots, y_M\}$ with $Y_o = y_i$ and $p_{ij} = P\left[\omega \in \Omega : Y_{m+1}(\omega) = y_j \mid Y_m(\omega) = y_i\right], (m \in N); y_i, y_j \in E_Y$

where $0 \leq p_{ij} \leq 1, \sum_{j=1}^M p_{ij} = 1$.

Assumption 1.4. $I = \{I_n\}_{n \geq 0}$ is homogeneous Markov chain, I_n take values in a finite set of non - negative numbers $E_I = \{i_1, i_2, \dots, i_N\}$ with $I_o = i_r$ and $q_{rs} = P\left[\omega \in \Omega : I_{m+1}(\omega) = i_s \mid I_m(\omega) = i_r\right], (m \in N); i_r, i_s \in E_I$ where $0 \leq q_{rs} \leq 1, \sum_{s=1}^N q_{rs} = 1$.

Assumption 1.5. X, Y and I are assumed to be independent.

We define the finite time and ultimate ruin probabilities of model (1.1) with assumption 1.1 to assumption 1.5, respectively, by

$$\psi_n^{(1)}(u, y_i, i_r) = P\left(\omega \in \Omega : \bigcup_{k=1}^n (U_k^{(1)}(\omega) < 0) \mid U_0^{(1)}(\omega) = u, Y_o(\omega) = y_i, I_o(\omega) = i_r\right) \quad (1.5)$$

$$\psi^{(1)}(u, y_i, i_r) = \lim_{n \rightarrow \infty} \psi_n^{(1)}(u, y_i, i_r) = P\left(\omega \in \Omega : \bigcup_{k=1}^{\infty} (U_k^{(1)}(\omega) < 0) \mid U_0^{(1)}(\omega) = u, Y_o(\omega) = y_i, I_o(\omega) = i_r\right) \quad (1.6)$$

Similarly, we define the finite time and ultimate ruin probabilities of model (1.3) with assumption 1.1 to assumption 1.5, respectively, by

$$\psi_n^{(2)}(u, y_i, i_r) = P\left(\omega \in \Omega : \bigcup_{k=1}^n (U_k^{(2)}(\omega) < 0) \mid U_0^{(2)}(\omega) = u, X_o(\omega) = y_i, I_o(\omega) = i_r\right) \quad (1.7)$$

$$\psi^{(2)}(u, y_i, i_r) = \lim_{n \rightarrow \infty} \psi_n^{(2)}(u, y_i, i_r) = P\left(\omega \in \Omega : \bigcup_{k=1}^{\infty} (U_k^{(2)}(\omega) < 0) \mid U_0^{(2)}(\omega) = u, Y_o(\omega) = y_i, I_o(\omega) = i_r\right) \quad (1.8)$$

In this paper, we derive probability inequalities for $\psi^{(1)}(u, y_i, i_r)$ and $\psi^{(2)}(u, y_i, i_r)$. The paper is organized as follows; in Section 2, we give recursive and integral equations for $\psi_n^{(1)}(u, y_i, i_r)$ and $\psi_n^{(2)}(u, y_i, i_r)$. In Section 3 we derive probability inequalities for $\psi^{(1)}(u, y_i, i_r)$ and $\psi^{(2)}(u, y_i, i_r)$ by an inductive approach. A numerical example is given to illustrate these results in Section 4. Finally, we conclude our paper in Section 5.

2. Integral Equation for Ruin Probabilities

We first give recursive equations for $\psi_n^{(1)}(u, y_i, i_r)$ and an integral equation for $\psi^{(1)}(u, y_i, i_r)$.

Theorem 2.1. If model (1.1) satisfies the assumptions 1.1 to 1.5 then for $n = 1, 2, \dots$

$$\psi_{n+1}^{(1)}(u, y_i, i_r) = \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \int_{y_j - u(1+i_s)}^{+\infty} \psi_n^{(1)}(u(1+i_s) + x - y_j, y_j, i_s) dF(x) + F[y_j - u(1+i_s)] \right\} \quad (2.1)$$

and

$$\psi^{(1)}(u, y_i, i_r) = \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \int_{y_j - u(1+i_s)}^{+\infty} \psi^{(1)}(u(1+i_s) + x - y_j, y_j, i_s) dF(x) + F[y_j - u(1+i_s)] \right\} \quad (2.2)$$

Proof.

Give $Y_1(\omega) = y_j \in E_Y, I_1(\omega) = i_s \in E_I (\omega \in \Omega)$.

Let

$$A = \left\{ \omega \in \Omega : U_o^{(1)}(\omega) = u, Y_o(\omega) = y_i, Y_1(\omega) = y_j, I_o(\omega) = i_r, I_1(\omega) = i_s \right\}$$

$$A_1 = \left\{ \omega \in \Omega : X_1(\omega) < y_j - u(1 + i_s) \right\},$$

$$A_2 = \left\{ \omega \in \Omega : X_1(\omega) \geq y_j - u(1 + i_s) \right\}.$$

From (1.1), we have $U_1^{(1)}(\omega) = u(1 + i_s) + X_1(\omega) - y_j$ and

$$P\left(\omega \in \Omega : U_1^{(1)}(\omega) < 0 \mid A_1 \cap A\right) = 1 \Rightarrow P\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \mid A_1 \cap A\right) = 1 \quad (2.3)$$

In addition,

$$P\left(\omega \in \Omega : U_1^{(1)}(\omega) < 0 \mid A_2 \cap A\right) = 0. \quad (2.4)$$

Let $\{\tilde{X}_n\}_{n \geq 0}, \{\tilde{Y}_n\}_{n \geq 0}, \{\tilde{I}_n\}_{n \geq 0}$ be independent copies of $\{X_n\}_{n \geq 0}, \{Y_n\}_{n \geq 0}, \{I_n\}_{n \geq 0}$ respectively with $\tilde{X}_o(\omega) = X_1(\omega), \tilde{Y}_o(\omega) = Y_1(\omega) = y_j, \tilde{I}_o(\omega) = I_1(\omega) = i_s$.

Thus, (2.4) and (1.2) imply that

$$\begin{aligned} P\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \mid A_2 \cap A\right) &= P\left(\omega \in \Omega : \bigcup_{k=2}^{n+1} (U_k^{(1)}(\omega) < 0) \mid A_2 \cap A\right) \\ &= P\left(\omega \in \Omega : \bigcup_{k=2}^{n+1} \left[u(1 + i_s) + X_1(\omega) - y_j \right] \prod_{m=2}^k (1 + I_m(\omega)) + \sum_{m=2}^k (X_m(\omega) - Y_m(\omega)) \prod_{p=m+1}^k (1 + I_p(\omega)) \right\} \\ &\quad \left(U_1^{(1)}(\omega) = u(1 + i_s) + X_1(\omega) - y_j, Y_1(\omega) = y_j, I_1(\omega) = i_s \right) \cap A_2 \\ &= P\left(\omega \in \Omega : \bigcup_{k=1}^n \left\{ \tilde{U}_o^{(1)}(\omega) \prod_{m=1}^k (1 + \tilde{I}_m(\omega)) + \sum_{m=1}^k (\tilde{X}_m(\omega) - \tilde{Y}_m(\omega)) \prod_{p=m+1}^k (1 + \tilde{I}_p(\omega)) < 0 \right\} \right) \\ &\quad \left(\tilde{U}_o^{(1)}(\omega) = u(1 + i_s) + X_1(\omega) - y_j, \tilde{Y}_o(\omega) = y_j, \tilde{I}_o(\omega) = i_s \right) \cap A_2 \end{aligned} \quad (2.5)$$

On the other hand, (1.5) implies

$$\psi_{n+1}^{(1)}(u, y_i, i_r) = P\left\{ \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \mid U_o^{(1)}(\omega) = u, Y_o(\omega) = y_i, I_o(\omega) = i_r \right\}$$

Thus, we have

$$\begin{aligned} \psi_{n+1}^{(1)}(u, y_i, i_r) &= \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} P\left\{ \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \mid A \right\} \\ &= \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ P\left\{ \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \mid A_1 \cap A \right\} \cdot P(A_1) \right. \\ &\quad \left. + P\left\{ \omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \mid A_2 \cap A \right\} \cdot P(A_2) \right\} \end{aligned} \quad (2.6)$$

From (2.3), we have

$$P\left\{\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \middle| A_1 \cap A\right\}.P(A_1) = \int_0^{y_j - u(1+i_s)} dF(x).$$

From (2.5), we have

$$P\left\{\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k^{(1)}(\omega) < 0) \middle| A_2 \cap A\right\}.P(A_2) = \int_{y_j - u(1+i_s)}^{+\infty} \psi_n^{(1)}(u(1+i_s) + x - y_j, y_j, i_s) dF(x)$$

Therefore, (2.6) is written as

$$\begin{aligned} \psi_{n+1}^{(1)}(u, y_i, i_r) &= \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \int_0^{y_j - u(1+i_s)} dF(x) + \int_{y_j - u(1+i_s)}^{+\infty} \psi_n^{(1)}(u(1+i_s) + x - y_j, y_j, i_s) dF(x) \right\} \\ &= \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \int_{y_j - u(1+i_s)}^{+\infty} \psi_n^{(1)}(u(1+i_s) + x - y_j, y_j, i_s) dF(x) + F\left[y_j - u(1+i_s)\right] \right\} \end{aligned} \quad (2.7)$$

Thus, the integral equation for $\psi^{(1)}(u, y_i, i_r)$ in Theorem 2.1 follows immediately from the dominated convergence theorem by letting $n \rightarrow \infty$ in (2.7).

This completes the proof

Similarly, the following recursive equation for $\psi_n^{(2)}(u, y_i, i_r)$ and integral equation for $\psi^{(2)}(u, y_i, i_r)$ are hold.

Theorem 2.2. If model (1.3) satisfies assumptions 1.1 to 1.5 then, for $n = 1, 2, \dots$

$$\psi_{n+1}^{(2)}(u, y_i, i_r) = \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \int_{\frac{y_j - u(1+i_s)}{1+i_s}}^{+\infty} \psi_n^{(2)}\left[(u+x)(1+i_s) - y_j, y_j, i_s\right] dF(x) + F\left(\frac{y_j - u(1+i_s)}{1+i_s}\right) \right\} \quad (2.8)$$

and

$$\psi^{(2)}(u, y_i, i_r) = \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \int_{\frac{y_j - u(1+i_s)}{1+i_s}}^{+\infty} \psi^{(2)}\left[(u+x)(1+i_s) - y_j, y_j, i_s\right] dF(x) + F\left(\frac{y_j - u(1+i_s)}{1+i_s}\right) \right\} \quad (2.9)$$

Next, we establish probability inequalities for ruin probabilities of model (1.1) and model (1.3).

3. Probability Inequalities for Ruin Probabilities

To establish probability inequalities for ruin probabilities of model (1.1), we first proof the following Lemma.

Lemma 3.1. Let model (1.1) satisfy assumptions 1.1 to 1.5 and $E(X_1^k) < +\infty (k = 1, 2)$.

If, any $y_i \in E_Y$,

$$E(Y_1 | \omega \in \Omega : Y_o(\omega) = y_i) < E(X_1) \quad \text{and}$$

$$P(\omega \in \Omega : (Y_1 - X_1)(\omega) > 0 | Y_o(\omega) = y_i) > 0 \quad (3.1)$$

then there exists a unique positive constant R_i satisfying:

$$E\left(e^{R_i(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right) = 1 \quad (3.2)$$

Proof.

Define

$$f_i(t) = E\left\{e^{t(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right\} - 1; t \in (0, +\infty)$$

We have

$$f_i(t) = E\left\{e^{tY_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right\} \cdot E\left(e^{-tX_1}\right) - 1 = g_i(t) \cdot h(t) - 1$$

From Y_1 is discrete random variables and it takes values in $E_Y = \{y_1, y_2, \dots, y_M\}$ then

$$g_i(t) = E\left\{e^{tY_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right\} = \sum_{j=1}^M p_{ij} e^{ty_j} \text{ has } n\text{-th derivative function on } (0, +\infty) \text{ (any } n \in N^* = N \setminus \{0\} \text{)}.$$

In addition, $h(t) = E\left(e^{-tX_1}\right) = \int_0^{+\infty} e^{-tx} f(x) dx$ with $f(x) = F'(x)$ satisfying :

$$h(t) = \int_0^{+\infty} e^{-tx} f(x) dx \leq \int_0^{+\infty} f(x) dx = 1 \text{ and } \int_0^{+\infty} x^k e^{-tx} f(x) dx \leq \int_0^{+\infty} x^k f(x) dx = E(X_1^k) < +\infty (k = 1, 2).$$

This implies that $h(t)$ has n -th derivative function on $(0, +\infty)$ with $n = 1, 2$. Thus, $f_i(t)$ has n -th derivative function on $(0, +\infty)$ with $n = 1, 2$ and

$$\begin{aligned} f_i'(t) &= E\left\{(Y_1 - X_1)e^{t(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right\} \\ f_i''(t) &= E\left\{(Y_1 - X_1)^2 e^{t(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right\} \geq 0. \end{aligned}$$

This implies that

$$f_i(t) \text{ is a convex function with } f_i(0) = 0 \quad (3.3)$$

and

$$f_i'(0) = E\left\{(Y_1 - X_1) \middle| \omega \in \Omega : Y_o(\omega) = y_i\right\} = E(Y_1 | \omega \in \Omega : Y_o(\omega) = y_i) - E(X_1) < 0 \quad (3.4)$$

By $P(\omega \in \Omega : (Y_1 - X_1)(\omega) > 0 | Y_o(\omega) = y_i) > 0$, we can find some constant $\delta > 0$ such that

$$P(\omega \in \Omega : (Y_1 - X_1)(\omega) > \delta > 0 | Y_o(\omega) = y_i) > 0$$

Then, we can get that

$$\begin{aligned} f_i(t) &= E\left\{e^{t(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right\} - 1 \\ &\geq E\left\{e^{t(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right\} \cdot 1_{\{\omega \in \Omega : (Y_1 - X_1)(\omega) > \delta | Y_o(\omega) = y_i\}} - 1 \\ &\geq e^{t\delta} \cdot P\left\{\omega \in \Omega : (Y_1 - X_1)(\omega) > \delta \middle| Y_o(\omega) = y_i\right\} - 1. \end{aligned}$$

Imply

$$\lim_{t \rightarrow +\infty} f_i(t) = +\infty. \quad (3.5)$$

From (3.3), (3.4) and (3.5) there exists a unique positive constant R_i satisfying (3.2).

This completes the proof.

$$\text{Let: } R_o = \min \left\{ R_i > 0 : E\left(e^{R_i(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right) = 1 (y_i \in E_Y) \right\}$$

Using Lemma 3.1 and Theorem 2.1, we obtain a probability inequality for $\psi^{(1)}(u, y_i, i_r)$ by an inductive approach.

Theorem 3.1. If model (1.1) satisfies assumptions 1.1 to 1.5, $E(X_1^k) < +\infty (k = 1, 2)$ and (3.1) then

for any $u > 0, y_i \in E_Y$ and $i_r \in E_I$

$$\psi^{(1)}(u, y_i, i_r) \leq \beta_1 \cdot E \left[e^{-R_o u(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right], \quad (3.6)$$

where

$$\beta_1^{-1} = \inf_{z>0} \frac{\int_0^z e^{R_o x} dF(x)}{F(z)}, \beta_1 \leq 1.$$

Proof.

Firstly, we have

$$\beta_1^{-1} = \inf_{z>0} \frac{\int_0^z e^{R_o(z-x)} dF(x)}{F(z)} \geq \inf_{z>0} \frac{\int_0^z dF(x)}{F(z)} = 1 \Leftrightarrow \frac{1}{\beta_1} \geq 1 \Leftrightarrow \beta_1 \leq 1.$$

For any $z > 0$, we have

$$F(z) = \left[\frac{\int_0^z e^{R_o x} dF(x)}{F(z)} \right]^{-1} \cdot e^{R_o z} \cdot \int_0^z e^{-R_o x} dF(x) \leq \beta_1 \cdot e^{R_o z} \cdot \int_0^z e^{-R_o x} dF(x) \quad (3.7)$$

$$\leq \beta_1 \cdot e^{R_o z} \cdot \int_0^{+\infty} e^{-R_o x} dF(x) = \beta_1 \cdot e^{R_o z} \cdot E \left[e^{-R_o X_1} \right]. \quad (3.8)$$

Then, for any $u > 0$, $y_i \in E_Y$ and $i_r \in E_I$, we can write

$$\psi_1^{(1)}(u, y_i, i_r) = P(\omega \in \Omega : U_1^{(1)}(\omega) > 0 \mid U_o^{(1)}(\omega) = u, Y_o(\omega) = y_i, I_o(\omega) = i_r) = \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} F[y_j - u(1+i_s)] \quad (3.9)$$

Thus, combining (3.8) and (3.9), we have

$$\begin{aligned} \psi_1^{(1)}(u, y_i, i_r) &= \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} F(y_j - u(1+i_s)) \leq \beta_1 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} e^{R_o[y_j - u(1+i_s)]} E \left[e^{-R_o X_1} \right] \\ &= \beta_1 E \left[e^{R_o(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-R_o u(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right] \\ &= \beta_1 E \left[e^{-R_o u(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right]. \end{aligned} \quad (3.10)$$

Applying an inductive hypothesis, we assume for any $u > 0$, $y_i \in E_Y$ and $i_r \in E_I$,

$$\psi_n^{(1)}(u, y_i, i_r) \leq \beta_1 E \left[e^{-R_o u(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right]. \quad (3.11)$$

Then (3.10) implies that (3.11) holds with $n = 1$.

For $y_j \in E_Y, i_s \in E_I$, $u(1+i_s) + x - y_j > 0$ and $I_1(\omega) \geq 0 (\forall \omega \in \Omega)$, we have

$$\begin{aligned} &\psi_n^{(1)}(u(1+i_s) + x - y_j, y_j, i_s) \\ &\leq \beta_1^* E \left[e^{-R_o^*[u(1+i_s) + x - y_j](1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_s \right] \leq \beta_1^* e^{-R_o^*[u(1+i_s) + x - y_j]} \end{aligned}$$

where

$$\beta_1^{*-1} = \inf_{z>0} \frac{e^{R_o^* z} \int_0^z e^{-R_o^* x} dF(x)}{F(z)}, E\left(e^{R_o^*(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_j\right) = 1$$

and $R_o^* \geq R_o > 0$.

For any $z > 0$:

$$\frac{e^{R_o z} \int_0^z e^{-R_o x} dF(x)}{F(z)} = \frac{\int_0^z e^{R_o(z-x)} dF(x)}{F(z)} \leq \frac{\int_0^z e^{R_o^*(z-x)} dF(x)}{F(z)} = \frac{e^{R_o^* z} \int_0^z e^{-R_o^* x} dF(x)}{F(z)}$$

then

$$\beta_1^{-1} = \inf_{z>0} \frac{e^{R_o z} \int_0^z e^{-R_o x} dF(x)}{F(z)} \leq \beta_1^{*-1} = \inf_{z>0} \frac{e^{R_o^* z} \int_0^z e^{-R_o^* x} dF(x)}{F(z)}$$

$$\Leftrightarrow \frac{1}{\beta_1} \leq \frac{1}{\beta_1^*} \Leftrightarrow \beta_1^* \leq \beta_1.$$

That $R_o^*[u(1+i_s) + x - y_j] \geq R_o[u(1+i_s) + x - y_j] > 0$ then

$$\psi_n^{(1)}(u(1+i_s) + x - y_j, y_j, i_s) \leq \beta_1 e^{-R_o[u(1+i_s) + x - y_j]}. \quad (3.12)$$

Therefore, by Lemma 3.1, (2.1), (3.7) and (3.12), we get

$$\begin{aligned} \psi_{n+1}^{(1)}(u, y_i, i_r) &= \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ F(y_j - u(1+i_s)) + \int_{y_j - u(1+i_s)}^{+\infty} \psi_n^{(1)}(u(1+i_s) + x - y_j, y_j, i_s) dF(x) \right\} \\ &\leq \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \beta_1 e^{R_o[y_j - u(1+i_s)]} \int_0^{y_j - u(1+i_s)} e^{-R_o x} dF(x) + \beta_1 \int_{y_j - u(1+i_s)}^{+\infty} e^{-R_o[u(1+i_s) + x - y_j]} dF(x) \right\} \\ &= \beta_1 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} e^{R_o[y_j - u(1+i_s)]} \int_0^{+\infty} e^{-R_o x} dF(x) \\ &= \beta_1 E\left[e^{R_o(Y_1 - X_1)} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right] \cdot E\left[e^{-R_o u(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r\right] \\ &= \beta_1 E\left[e^{-R_o u(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r\right] \end{aligned}$$

Thus

$$\psi_{n+1}^{(1)}(u, y_i, i_r) \leq \beta_1 E\left[e^{-R_o u(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r\right]$$

Consequently, for any $n = 1, 2, \dots$ (3.11) holds. Therefore, (3.6) follows by letting $n \rightarrow \infty$ in (3.11).

This completes the proof

Remark 3.1. Let $A(u, y_i, i_r) = \beta_1 \cdot E\left[e^{-R_o u(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r\right]$. From $I_1(\omega) \geq 0 (\forall \omega \in \Omega)$ and $\beta_1 \leq 1$,

we have

$$A(u, y_i, i_r) \leq \beta_1 \cdot E\left[e^{-R_o u} \middle| \omega \in \Omega : I_o(\omega) = i_r\right] = \beta_1 e^{-R_o u} \leq e^{-R_o u}$$

Therefore, upper bound for ruin probability in (3.6) is better than $e^{-R_o u}$.

Similar to Lemma 3.1, we have Lemma 3.2.

Lemma 3.2. Assume that model (1.3) satisfies assumptions 1.1 to 1.5 and $E(X_1^k) < +\infty (k=1,2)$. If any $y_i \in E_Y$ and $i_r \in E_I$,

$$E\left[(Y_1 - X_1(1 + I_1)) \middle| \omega \in \Omega : Y_o(\omega) = y_i, I_o(\omega) = i_r\right] < 0$$

and

$$P(Y_1 - X_1(1 + I_1) > 0 \middle| \omega \in \Omega : Y_o(\omega) = y_i, I_o(\omega) = i_r) > 0, \quad (3.13)$$

then there exists a unique positive constant $R_{ir} > 0$ satisfying:

$$E\left(e^{R_{ir}[Y_1 - X_1(1 + I_1)]} \middle| \omega \in \Omega : Y_o(\omega) = y_i, I_o(\omega) = i_r\right) = 1$$

Let

$$\bar{R}_o = \min\left\{R_{ir} > 0 : E\left(e^{R_{ir}[Y_1 - X_1(1 + I_1)]} \middle| \omega \in \Omega : Y_o(\omega) = y_i, I_o(\omega) = i_r\right) = 1 (y_i \in E_Y, i_r \in E_I)\right\}$$

Next, we use Lemma 3.2 and Theorem 2.2 to give a probability inequality for $\psi^{(2)}(u, y_i, i_r)$ by an inductive approach.

Theorem 3.2. If model (1.3) satisfies assumptions 1.1 to 1.5, $E(X_1^k) < +\infty (k=1,2)$ and (3.13) then, for any $y_i \in E_Y$ and $i_r \in E_I$

$$\psi^{(2)}(u, y_i, i_r) \leq \beta_2 E\left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i\right] E\left[e^{-\bar{R}_o(u + X_1)(1 + I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r\right] \quad (3.14)$$

where

$$\beta_2^{-1} = \inf_{z > 0} \frac{e^{\bar{R}_o z} \int_0^z e^{-\bar{R}_o x} dF(x)}{F(z)}, \beta_2 \leq 1.$$

Proof.

Similarly with Theorem 3.1, we have $\beta_2 \leq 1$ and any $z > 0$

$$F(z) \leq \beta_1 \cdot e^{\bar{R}_o z} \cdot \int_0^z e^{-\bar{R}_o x} dF(x) \quad (3.15)$$

$$\leq \beta_1 \cdot e^{\bar{R}_o z} E\left[e^{-\bar{R}_o X_1}\right]. \quad (3.16)$$

Then, for any $u > 0$, $y_i \in E_Y$ and $i_r \in E_I$

$$\begin{aligned} \psi_1^{(2)}(u, y_i, i_r) &= \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} F\left(\frac{y_j - u(1 + i_s)}{1 + i_s}\right) \\ &\leq \beta_2 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \int_0^{\frac{y_j - u(1 + i_s)}{1 + i_s}} e^{\bar{R}_o \left[\frac{y_j - u(1 + i_s)}{1 + i_s} - x\right]} dF(x) \end{aligned}$$

$$\begin{aligned}
&= \beta_2 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \int_0^{\frac{y_j - u(1+i_s)}{1+i_s}} e^{\bar{R}_o \left[\frac{y_j - u(1+i_s) - x(1+i_s)}{1+i_s} \right]} dF(x) \\
&\leq \beta_2 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \int_0^{\frac{y_j - u(1+i_s)}{1+i_s}} e^{\bar{R}_o [y_j - (u+x)(1+i_s)]} dF(x) \\
&\leq \beta_2 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \int_0^{+\infty} e^{\bar{R}_o [y_j - (u+x)(1+i_s)]} dF(x) \\
&= \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-\bar{R}_o (u+X_1)(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right]
\end{aligned}$$

Hence

$$\psi_1^{(2)}(u, y_i, i_r) \leq \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-\bar{R}_o (u+X_1)(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right] \quad (3.17)$$

Under an inductive hypothesis, we assume that

$$\psi_n^{(2)}(u, y_i, i_r) \leq \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-\bar{R}_o (u+X_1)(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right]. \quad (3.18)$$

Then, (3.17) implies that (3.18) holds with $n = 1$.

For $y_j \in E_Y, i_s \in E_I, x > \frac{y_j - u(1+i_s)}{1+i_s}$ and $I_1(\omega) \geq 0, (\omega \in \Omega)$, we have

$$\begin{aligned}
&\psi_n^{(2)}((u+x)(1+i_s) - y_j, y_j, i_s) \\
&\leq \beta_2^* E \left[e^{\bar{R}_o^* Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_j \right] \cdot E \left[e^{-\bar{R}_o^* [(u+x)(1+i_s) - y_j + X_1](1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_s \right] \\
&= \beta_2^* E \left[e^{\bar{R}_o^* Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_j \right] \cdot E \left[e^{-\bar{R}_o^* [(u+x)(1+i_s) - y_j](1+I_1) - \bar{R}_o^* X_1(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_s \right] \\
&\leq \beta_2^* E \left[e^{\bar{R}_o^* Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_j \right] \cdot E \left[e^{-\bar{R}_o^* X_1(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_s \right] \cdot e^{-\bar{R}_o^* [(u+x)(1+i_s) - y_j]} \\
&= \beta_2^* \cdot e^{-\bar{R}_o^* [(u+x)(1+i_s) - y_j]} (y_j \in E_Y, i_s \in E_I, (u+x)(1+i_s) - y_j > 0),
\end{aligned}$$

where

$$\beta_2^{*-1} = \inf_{z>0} \frac{e^{\bar{R}_o^* z} \int_0^z e^{-\bar{R}_o^* x} dF(x)}{F(z)}, E \left(e^{\bar{R}_o^* (Y_1 - X_1(1+I_1))} \middle| \omega \in \Omega : Y_o(\omega) = y_j, I_o(\omega) = i_s \right) = 1$$

and $\bar{R}_o^* \geq \bar{R}_o > 0$.

For any $z > 0$:
$$\frac{e^{\bar{R}_o z} \int_0^z e^{-\bar{R}_o x} dF(x)}{F(z)} = \frac{\int_0^z e^{\bar{R}_o(z-x)} dF(x)}{F(z)} \leq \frac{\int_0^z e^{\bar{R}_o^*(z-x)} dF(x)}{F(z)} = \frac{e^{\bar{R}_o^* z} \int_0^z e^{-\bar{R}_o^* x} dF(x)}{F(z)}$$

then

$$\beta_2^{-1} = \inf_{z>0} \frac{e^{\bar{R}_o z} \int_0^z e^{-\bar{R}_o x} dF(x)}{F(z)} \leq \beta_2^{*-1} = \inf_{z>0} \frac{e^{\bar{R}_o^* z} \int_0^z e^{-\bar{R}_o^* x} dF(x)}{F(z)}$$

$$\Leftrightarrow \frac{1}{\beta_2} \leq \frac{1}{\beta_2^*} \Leftrightarrow \beta_2^* \leq \beta_2$$

We get $\bar{R}_o^*[(u+x)(1+i_s) - y_j] \geq \bar{R}_o[(u+x)(1+i_s) - y_j] > 0$ then

$$\psi_n^{(2)}((u+x)(1+i_s) - y_j, y_j, i_s) \leq \beta_2 \cdot e^{-\bar{R}_o[(u+x)(1+i_s) - y_j]} \quad (3.19)$$

Therefore, by Lemma 3.2, (2.8), (3.15) and (3.19), we get

$$\begin{aligned} \psi_{n+1}^{(2)}(u, y_i, i_r) &= \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ F\left(\frac{y_j - u(1+i_s)}{1+i_s}\right) + \int_{\frac{y_j - u(1+i_s)}{1+i_s}}^{+\infty} \psi_n^{(2)}((u+x)(1+i_s) - y_j, y_j, i_s) dF(x) \right\} \\ &\leq \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \beta_2 \int_0^{\frac{y_j - u(1+i_s)}{1+i_s}} e^{\bar{R}_o \left[\frac{y_j - u(1+i_s)}{1+i_s} - x \right]} dF(x) + \beta_2 \int_{\frac{y_j - u(1+i_s)}{1+i_s}}^{+\infty} e^{-\bar{R}_o[(u+x)(1+i_s) - y_j]} dF(x) \right\} \\ &= \beta_2 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \int_0^{\frac{y_j - u(1+i_s)}{1+i_s}} e^{\bar{R}_o \frac{y_j - (u+x)(1+i_s)}{1+i_s}} dF(x) + \int_{\frac{y_j - u(1+i_s)}{1+i_s}}^{+\infty} e^{-\bar{R}_o[(u+x)(1+i_s) - y_j]} dF(x) \right\} \\ &\leq \beta_2 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \left\{ \int_0^{\frac{y_j - u(1+i_s)}{1+i_s}} e^{-\bar{R}_o[(u+x)(1+i_s) - y_j]} dF(x) + \int_{\frac{y_j - u(1+i_s)}{1+i_s}}^{+\infty} e^{-\bar{R}_o[(u+x)(1+i_s) - y_j]} dF(x) \right\} \\ &= \beta_2 \sum_{j=1}^M \sum_{s=1}^N p_{ij} q_{rs} \int_0^{+\infty} e^{\bar{R}_o[y_j - (u+x)(1+i_s)]} dF(x) \\ &= \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-\bar{R}_o(u+X_1)(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right]. \end{aligned}$$

Thus

$$\psi_{n+1}^{(2)}(u, y_i, i_r) \leq \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-\bar{R}_o(u+X_1)(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right]$$

Consequently, for any $n=1, 2, \dots$ (3.18) holds. Therefore, (3.14) follows by letting $n \rightarrow \infty$ in (3.18).

Remark 3.2.

Let

$$B(u, y_i, i_r) = \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-\bar{R}_o(u+X_1)(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right]$$

From $I_1(\omega) \geq 0, X_1(\omega) \geq 0, (\forall \omega \in \Omega)$ and $\beta_2 \leq 1$, we have

$$\begin{aligned} B(u, y_i, i_r) &= \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-\bar{R}_o u(1+I_1) - \bar{R}_o X_1(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right] \\ &\leq \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] \cdot E \left[e^{-\bar{R}_o u - \bar{R}_o X_1(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right] \\ &= \beta_2 E \left[e^{\bar{R}_o Y_1} \middle| \omega \in \Omega : Y_o(\omega) = y_i \right] E \left[e^{-\bar{R}_o X_1(1+I_1)} \middle| \omega \in \Omega : I_o(\omega) = i_r \right] \cdot e^{-\bar{R}_o u} \\ &= \beta_2 e^{-\bar{R}_o u} \leq e^{-\bar{R}_o u} \end{aligned}$$

Hence, upper bound for ruin probability in (3.14) is better than $e^{-\bar{R}_o u}$.

4. A Numerical Illustration

In this section we give a numerical example to illustrate the bounds of $\psi^{(1)}(u, y_i, i_r)$ derived in Section 3.

Let $X = \{X_n\}_{n \geq 0}$ be a sequence of independent and identically distributed non-negative continuous random variables with the same distributive function $F(x) = 1 - e^{-0.25x} (x \geq 0)$.

Let $Y = \{Y_n\}_{n \geq 0}$ be a homogeneous Markov chain such that for any n , Y_n take values in $E_Y = \{1, 3\}$ with Y_1 having a distribution:

Y_1	1	3
P	0,4	0,6

and matrix $P = [p_{ij}]_{2 \times 2}$ is given by

$$P = \begin{bmatrix} 0,3 & 0,7 \\ 0,2 & 0,8 \end{bmatrix}$$

Let $I = \{I_k\}_{k \geq 0}$ be a homogeneous Markov chain such that for any n , I_n take value in $E_I = \{0, 1; 0, 15\}$ with I_1 having a distribution:

I_1	0,1	0,15
P	0,35	0,65

and matrix $Q = [q_{rs}]_{2 \times 2}$ is given by

$$Q = \begin{bmatrix} 0,25 & 0,75 \\ 0,6 & 0,4 \end{bmatrix}$$

Then, we have

$$E(Y_1 | Y_o = 1) = 1 \cdot 0,3 + 3 \cdot 0,7 = 2,4$$

$$E(Y_1 | Y_o = 3) = 1 \cdot 0,2 + 3 \cdot 0,8 = 2,6; E(X_1) = \frac{1}{0,25} = 4$$

Therefore

$$E(Y_1 | Y_o = y_i) < E(X_1), y_i \in E_Y \quad (4.1)$$

In the other hand,

$$P(Y_1 - X_1 > 0 | Y_o = 1) > 0, \quad P(Y_1 - X_1 > 0 | Y_o = 3) > 0 \quad (4.2)$$

and

$$E(X_1^k) < +\infty (k = 1, 2) \quad (4.3)$$

Combining (4.1), (4.2) and (4.3) imply that Lemma 3.1 holds.

Next, we solve equation (3.2).

Firstly, we have

$$E[e^{R_i(Y_1 - X_1)} | Y_o = y_i] = E[e^{R_i Y_1} | Y_o = y_i] E[e^{-R_i X_1}] (i = 1, 2).$$

where

$$E[e^{-R_i X_1}] = 0,25 \int_0^{+\infty} e^{-(R_i + 0,25)x} dx = \frac{0,25}{R_i + 0,25} (i = 1, 2)$$

and

$$E[e^{R_1 Y_1} | Y_o = 1] = e^{R_1} . P[Y_1 = 1 | Y_o = 1] + e^{3R_1} . P[Y_1 = 3 | Y_o = 1] = 0,3e^{R_1} + 0,7e^{3R_1}$$

$$E[e^{R_2 Y_1} | Y_o = 3] = e^{R_2} . P[Y_1 = 1 | Y_o = 3] + e^{3R_2} . P[Y_1 = 3 | Y_o = 3] = 0,2e^{R_2} + 0,8e^{3R_2}$$

Respective equation (3.2) for R_1, R_2 , by

$$0,3e^{R_1} + 0,7e^{3R_1} = 4R_1 + 1 \quad (4.4)$$

$$0,2e^{R_2} + 0,8e^{3R_2} = 4R_2 + 1 \quad (4.5)$$

Using Maple, we find respective root of (3.2) for R_1, R_2 , by

$$R_1 \approx 0,33878; R_2 \approx 0,28124$$

Hence, $R_o = \min\{R_1, R_2\} = 0,28124$.

We can apply the result of Theorem 3.1 for $\psi^{(1)}(u, y_i, i_r)$

$$\psi^{(1)}(u, y_i, i_r) \leq E[e^{-R_o u(1+I_1)} | I_o = i_r] = g(u, i_r) (i_r \in E_I) \quad (4.6)$$

where

$$\begin{aligned} g(u; 0, 1) &= E[e^{-R_o u(1+I_1)} | I_o = 0, 1] \\ &= e^{-1,1R_o u} . P[I_1 = 0, 1 | I_o = 0, 1] + e^{-1,15R_o u} . P[I_1 = 0, 15 | I_o = 0, 1] \\ &= 0,25e^{-1,1R_o u} + 0,75e^{-1,15R_o u} \\ g(u; 0, 15) &= E[e^{-R_o u(1+I_1)} | I_o = 0, 15] \\ &= e^{-1,1R_o u} . P[I_1 = 0, 1 | I_o = 0, 15] + e^{-1,15R_o u} . P[I_1 = 0, 15 | I_o = 0, 15] \\ &= 0,6e^{-1,1R_o u} + 0,4e^{-1,15R_o u} \end{aligned}$$

Table 1 shows values upper bounds $g(u, i_r)$ of $\psi^{(1)}(u, x_i, i_r)$ for a range of value of u

Table 1. Upper bounds $g(u, i_r)$ of $\psi^{(1)}(u, y_i, i_r)$

u	$g(u, 0, 1)$	$g(u, 0, 15)$
1	0.726228	0.729814
2	0.527426	0.532654
3	0.38306	0.388775
4	0.27822	0.283774
5	0.202082	0.207141
6	0.146785	0.15121
7	0.106624	0.110387
8	0.077454	0.080588
9	0.056266	0.058836
10	0.040876	0.042958
15	0.008276	0.008919
20	0.001677	0.001854

5. Conclusions

Our main results in this paper are Theorem 2.1 and Theorem 2.2 giving recursive equations for $\psi_n^{(1)}(u, y_i, i_r)$ and $\psi_n^{(2)}(u, y_i, i_r)$ and integral equations for $\psi^{(1)}(u, y_i, i_r)$ and $\psi^{(2)}(u, y_i, i_r)$; Theorem 3.1 and Theorem 3.2 giving probability inequalities for $\psi^{(1)}(u, y_i, i_r)$ and $\psi^{(2)}(u, y_i, i_r)$ by an inductive approach. In addition, a numerical example is given illustrating Theorem 3.1.

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