

Oscillation Results for Third Order Nonlinear Neutral Delay Difference Equations

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Abstract In this paper, via comparison with first order oscillatory difference equations and by a Riccati transformation technique, we will study the oscillatory behavior of third order nonlinear neutral difference equations. We establish some new sufficient conditions which insure that every solution of this equation either oscillates or converges to zero. Our results improve and extend some known results in the literature. Examples are given to illustrate the importance of the results.

Keywords Oscillatory Solutions, Third Order, Nonlinear, Neutral, Difference Equation

1. Intrduction

By comparison with some first difference equations whose oscillatory characters are known and by means of a Riccati transformation technique, we obtain several new sufficient conditions for the oscillation of all solutions of the nonlinear neutral difference equation of the form

$$\Delta \left(a(n) \left(\Delta \left(b(n) \left(\Delta \left(x(n) + p(n)x(\tau(n)) \right) \right)^{\alpha_1} \right) \right)^{\alpha_2} \right) + q(n)f \left(x(g(n)) \right) = 0, \quad n \geq n_0. \quad (1.1)$$

Where $n_0 \in N$ is a fixed integer, Δ denotes the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$ and $\Delta^i x(n) = \Delta(\Delta^{i-1} x(n))$. Throughout this paper, we will assume the following hypotheses:

- (A₁) $a(n), b(n) > 0$ and $q(n) \geq 0$ for $n \in N(n_0)$.
- (A₂) α_1, α_2 are quotient of odd positive integers.
- (A₃) $\{p(n)\}_{n=n_0}^\infty$ is positive, $0 \leq p(n) \leq p < 1$.
- (A₄) $g: N(n_0) \rightarrow N$ satisfies $g(n) \leq n, \Delta g(n) \geq 0, \tau(n) \leq n$ and $\lim_{n \rightarrow \infty} \tau(n) = \lim_{n \rightarrow \infty} g(n) = \infty$.
- (A₅) $f \in C(R, R)$ such that $xf(x) > 0, f'(x) > 0$ for all $x \neq 0$ and $-f(-xy) \geq f(xy) \geq f(x)f(y)$ for $xy > 0$.

In addition, we will make use of the following conditions:

- (S₁) $f(u)/u^\alpha \geq K > 0, K$ is a real constant, $u > 0$ and $\alpha := \alpha_1 \alpha_2$.
- (S₂) $f(u_n) - f(v_n) = B(u_n, v_n)(u_n + p_n u_{n-\tau}) - (v_n + p_n v_{n-\tau})$ for $u_n, v_n \neq 0, n > \tau > 0$,

B is a nonnegative real valued function,

$f^{\frac{1}{\alpha}-1}(u)B(u_n, v_n) \geq \mu > 0$ and μ is a constant.

We set $z(n) := x(n) + p(n)x(\tau(n))$. By a solution of equation (1.1) we mean a nontrivial sequence $\{x(n)\}$ defined on $N(n_0)$, which satisfies equation (1.1) for all $n \geq n_0$. A solution $\{x(n)\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. Equation(1.1) is called oscillatory if all its solutions are oscillatory. In recent years, there has been an increasing interest in the study of the problem of determining the oscillation and non-oscillation of solutions of difference equations of the form (1.1) and its special cases. In[1], Graef et al. proved several theorems provided sufficient conditions for oscillation of all solutions of the third order difference equation of the form

$$\Delta \left(a(n) \Delta \left(b(n) \Delta x(n) \right) \right) + q(n)f \left(x(g(n+1)) \right) = 0, \quad (1.2)$$

depend on condition

$$\sum_{n=n_0}^{\infty} a^{-1}(n) = \sum_{n=n_0}^{\infty} b^{-1}(n) = \infty. \quad (1.3)$$

In[2], via comparison with first order oscillatory difference equations, Agarwal et al. proved several theorems provided sufficient conditions for oscillation of all solutions of the third order difference equation of the form

$$\Delta \left(\frac{1}{a_2(n)} \left(\Delta \left(\frac{1}{a_1(n)} \right) (\Delta x(n))^{\alpha_1} \right)^{\alpha_2} \right) + q(n)f \left(x(g(n)) \right) = 0, \quad (1.4)$$

depend on condition

$$\sum_{n=n_0}^{\infty} \left(a_i(n) \right)^{\frac{1}{\alpha_i}} = \infty, i = 1, 2. \quad (1.5)$$

In[3], by a Riccati transformation technique, Schmeidel studied the oscillatory and asymptotic behavior of solutions of the third order difference equation

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$\Delta(a(n)\Delta(b(n)(\Delta y(n)))) + p(n)f(y(n+l)) = 0$, (1.6)
using the condition

$$\sum_{n=1}^{\infty} a^{-1}(n) = \infty, \sum_{n=1}^{\infty} b^{-1}(n) = \infty. \quad (1.7)$$

In[4], via comparison with first order oscillatory difference equations, Grace et al. discussed the oscillatory behavior of the solutions of the difference equation of the form

$$\Delta(a(n)(\Delta^2 x(n))^\alpha) + q(n)f(x(g(n))) = 0, \quad (1.8)$$

under the condition

$$\sum_{n=1}^{\infty} b^{-\frac{1}{\alpha}}(n) < \infty. \quad (1.9)$$

In[5], by a Riccati transformation technique, Selvaraj et al. established some sufficient conditions for oscillation of all solutions of the third order non-linear difference equation of the form

$$\Delta\left(\frac{1}{a(n)}\Delta^2 y(n)\right) + p(n)f(y(\sigma(n))) = 0, \quad (1.10)$$

using the condition

$$\sum_{n=n_0}^{\infty} a(n) = \infty. \quad (1.11)$$

In[6], Saker et al. studied the oscillatory behavior of solutions of the equation

$$\Delta(c(n)\Delta(d(n)(\Delta x(n)))^\alpha) + q(n)f(x(g(n))) = 0, \quad (1.12)$$

using the condition (1.7) and

$$\sum_{n=n_0}^{\infty} c^{-\alpha}(n) < \infty, \sum_{n=n_0}^{\infty} d^{-1}(n) < \infty. \quad (1.13)$$

In[7], Thandapani et al. considered the third order difference equation of the form

$$\Delta(a(n)(\Delta^2(x(n) + p(n)x(n-\delta)))^\alpha) + q(n)x^\alpha(n-\tau) = 0, \quad (1.14)$$

under the condition

$$\sum_{n=n_0}^{\infty} a^{-\frac{1}{\alpha}}(n) = \infty. \quad (1.15)$$

In[8], using condition (1.11) Selvaraj et al. considered nonlinear third-order difference equations of the form

$$\Delta^2\left(\frac{1}{a(n)}\Delta x(n)\right) + q(n)f(x(\sigma(n))) = 0, \quad (1.16)$$

and they study the oscillatory behavior of solutions of equation (1.16). In[9], Saker investigated the third-order difference equation (1.12) using the condition (1.13) and the author obtained some Hille and Nehari type criteria for the oscillation of equation (1.12). For further results concerning

the oscillatory and asymptotic behavior of third order difference equation we refer to the books[10, 11, 12] and the references cited therein. Our results improve and extend some known results in the literature. The paper is organized as follows. In Section 2, we will state and prove the main oscillation theorems, in Section 3, and Section 4, we will give some remarks and provide some examples to illustrate the main results.

2. Main Results

In this section, we establish some new oscillation criteria for the equation (1.1) under the following conditions

$$\sum_{n=n_0}^{\infty} a^{-\frac{1}{\alpha_2}}(n) = \infty, \sum_{n=n_0}^{\infty} b^{-\frac{1}{\alpha_1}}(n) = \infty. \quad (2.1)$$

$$\sum_{n=n_0}^{\infty} a^{-\frac{1}{\alpha_2}}(n) < \infty, \sum_{n=n_0}^{\infty} b^{-\frac{1}{\alpha_1}}(n) = \infty. \quad (2.2)$$

$$\sum_{n=n_0}^{\infty} a^{-\frac{1}{\alpha_2}}(n) < \infty, \sum_{n=n_0}^{\infty} b^{-\frac{1}{\alpha_1}}(n) < \infty. \quad (2.3)$$

In the following results, we shall use the following notations:

$$\delta(n, n_2) := \sum_{s=n_2}^{n-1} a^{-\frac{1}{\alpha_2}}(s), \varphi(n) := \frac{\rho(n)\delta^{\alpha_2}(g(n), n_2)}{\rho^2(n+1)b^{\alpha_2}(g(n))},$$

$$\theta(n) := \frac{\mu\rho(n)\delta^{\frac{1}{\alpha_1}}(g(n), n_2)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)b^{\frac{1}{\alpha_1}}(g(n))},$$

$$\vartheta(m, n) := \left(\frac{\Delta\rho(n)}{\rho(n+1)} - \frac{h(m, n)}{\sqrt{H(m, n)}}\right),$$

$$\Psi(n) := K(1 - p(g(n)))^\alpha q(n) \left(\sum_{s=n_2}^{g(n)-1} \frac{1}{(b(s))^{\frac{1}{\alpha_1}}}\right)^\alpha.$$

We assume that there exists a double sequence $\{H(m, n) | m \geq n \geq 0\}$ and $h(m, n)$ such that

(i) $H(m, m) = 0$ for $m \geq 0$,

(ii) $H(m, n) > 0$ for $m > n > 0$,

(iii) $\Delta_2 H(m, n) = H(m, n+1) - H(m, n) \leq 0$ for $m > n \geq 0$,

(iv) $h(m, n) = -\frac{\Delta_2 H(m, n)}{\sqrt{H(m, n)}}.$

We begin with some useful lemmas, which will be used in obtaining our main results. The proof of the following Lemmas are similar to that of (Lemma 2.1, 2.2 and 2.3 respectively in [13]) and hence the details are omitted.

Lemma 2.1. Let $\{x(n)\}$ be an eventually positive solution of the equation (1.1) and suppose that $\{z(n)\}$

satisfies $\Delta z(n) > 0$, $\Delta(b(n)(\Delta z(n))^{\alpha_1}) > 0$,

$$\Delta\left(a(n)\left(\Delta(b(n)(\Delta z(n))^{\alpha_1})\right)^{\alpha_2}\right) \leq 0 \text{ for all } n \geq N.$$

Then there exists $n \geq n_1 \geq n_2$ such that

$$\Delta z(n) \geq b^{-\frac{1}{\alpha_1}}(n) \left(a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1})\right)^{\alpha_2}\right)^{\frac{1}{\alpha}} \left(\sum_{s=n_2}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}. \quad (2.4)$$

Lemma 2.2. Assume that (2.1) holds. Let $\{x(n)\}$ be an eventually positive solution of equation (1.1). Then for sufficiently large n , there are only two possible cases:

$$(I): \Delta z(n) > 0, \Delta(b(n)(\Delta z(n))^{\alpha_1}) > 0,$$

$$(II): \Delta z(n) < 0, \Delta(b(n)(\Delta z(n))^{\alpha_1}) > 0.$$

Lemma 2.3. Assume that (2.1) holds. Let $\{x(n)\}$ be an eventually positive solution of the equation (1.1) and suppose that Case (II) of Lemma 2.2 holds. If

$$\sum_{v=n_0}^{\infty} \left(b^{-\frac{1}{\alpha_1}}(v) \left(\sum_{u=n}^{\infty} a^{-\frac{1}{\alpha_2}}(u) \left(\sum_{s=n}^{\infty} q(s) \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}} \right) = \infty, \quad (2.5)$$

then $x(n) \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 2.1. Let (2.1) and (2.5) hold. If the first order delay equation

$$\Delta(y_n) + q(n)f\left(y^{\frac{1}{\alpha}}(g(n))\right)f\left(1-p(g(s))\right)f\left(\sum_{s=n_0}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s) \left(\sum_{u=n_0}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}\right) = 0, \quad (2.6)$$

is oscillatory then equation (1.1) is oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n)) > 0$. From the proof of Lemma 2.2 there are two possible cases. Assume that (I) holds. From Lemma (2.1), we have

$$\Delta z(n) \geq b^{-\frac{1}{\alpha_1}}(n) y^{\frac{1}{\alpha}}(n) \left(\sum_{s=n_2}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}.$$

where $y(n) := a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1})\right)^{\alpha_2}$. Summing the above inequality from n_2 to $n-1$, we obtain

$$z(n) \geq \sum_{s=n_2}^{n-1} b^{-\frac{1}{\alpha_1}}(s) y^{\frac{1}{\alpha}}(s) \left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}} \geq y^{\frac{1}{\alpha}}(n) \sum_{s=n_2}^{n-1} b^{-\frac{1}{\alpha_1}}(s) \left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}. \quad (2.7)$$

There exists a $n_3 \geq n_2$ with $(n) \geq n_2$ for all $n \geq n_3$, such that

$$z(g(n)) \geq y^{\frac{1}{\alpha}}(g(n)) \sum_{s=n_2}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s) \left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}, \text{ for all } n \geq n_3. \quad (2.8)$$

Since $\Delta z(n) > 0$ and $\tau(g(n)) \leq g(n)$, then

$$x(g(n)) = z(g(n)) - p(g(n))x(\tau(g(n)))$$

$$\geq z(g(n)) - p(g(n))z(\tau(g(n)))$$

$$\geq z(g(n)) \left(1 - p(g(n))\right). \quad (2.9)$$

From (2.8), we obtain

$$x(g(n)) \geq \left(1 - p(g(n))\right) y^{\frac{1}{\alpha}}(g(n)) \sum_{s=n_2}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s) \left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u) \right)^{\frac{1}{\alpha_1}}.$$

From equation (1.1), (A_5) and the last inequality, we obtain, for $n \geq n_3$,

$$-\Delta y(n) \geq q(n) f\left(1 - p(g(n))\right) f\left(y^{\frac{1}{\alpha}}(g(n))\right) f\left(\sum_{s=n_2}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s) \left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u) \right)^{\frac{1}{\alpha_1}}\right).$$

Summing the last inequality from n to ∞ , we get

$$y(n) \geq \sum_{s=n}^{\infty} q(s) f\left(y^{\frac{1}{\alpha}}(g(s))\right) f\left(1 - p(g(s))\right) f\left(\sum_{v=n_2}^{g(s)-1} b^{-\frac{1}{\alpha_1}}(v) \left(\sum_{u=n_2}^{v-1} a^{-\frac{1}{\alpha_2}}(u) \right)^{\frac{1}{\alpha_1}}\right).$$

The sequence $\{y(n)\}$ is obviously strictly decreasing. Hence, by the discrete analog of Theorem 1 in [14] we conclude that there exists a positive solution $\{y(n)\}$ of equation (2.6) which tends to zero. This contradicts that (2.6) is oscillatory. If the (II) holds, we are then back to the proof of Lemma 2.3 to show that $\lim_{n \rightarrow \infty} x(n) = 0$. The proof is complete.

Theorem 2.2. Assume that (2.2), (2.5) and (2.6) hold. If

$$\sum_{s=n_0}^{\infty} \left\{ a^{-\frac{1}{\alpha_2}}(s) \left(\sum_{r=n_0}^{s-1} q(r) f\left(1 - p(g(r))\right) f\left(\sum_{u=n_0}^{g(r)-1} b^{-\frac{1}{\alpha_1}}(u) f\left(\sum_{v=g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(v) \right)^{\frac{1}{\alpha_1}}\right)^{\frac{1}{\alpha_2}} \right) \right\} = \infty, \quad (2.10)$$

then every solution of equation (1.1) oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. Let $\{x(n)\}$ is eventually positive solution of (1.1) for $n \geq n_1$. Say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n)) > 0$. Based on condition (2.2), there exist three possible cases (I), (II) (as those of Lemma 2.2), and

(III): $\Delta z(n) > 0$, $\Delta(b(n)(\Delta z(n))^{\alpha}) < 0$ for all large n .

Assume that (I) holds. Then we are back to the proof of Theorem 2.1 to get contradiction to (2.6). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n \rightarrow \infty} x(n) = 0$. Assume that (III) holds. Then, we have

$$\begin{aligned} z(n) - z(n_3) &= \sum_{s=n_3}^{n-1} \Delta z(s) = \sum_{s=n_3}^{n-1} b^{-\frac{1}{\alpha_1}}(s) \left(b(s) (\Delta z(s))^{\alpha_1} \right)^{\frac{1}{\alpha_1}} \\ &\geq \left(b(n) (\Delta z(n))^{\alpha_1} \right)^{\frac{1}{\alpha_1}} \sum_{s=n_3}^{n-1} b^{-\frac{1}{\alpha_1}}(s), \text{ for } n \geq n_3, \end{aligned}$$

There exists a $n_4 \geq n_3$ with $g(n) \geq n_3$ for all $n \geq n_4$, such that

$$z(g(n)) \geq \left(b(g(n)) (\Delta z(g(n)))^{\alpha_1} \right)^{\frac{1}{\alpha_1}} \sum_{s=n_3}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s), \text{ for } n \geq n_4.$$

From equation (1.1), (2.9), (A_5) and the last inequality, we obtain, for $n \geq n_4$

$$0 \geq \Delta \left(a(n) (\Delta v(n))^{\alpha_2} \right) + q(n) f\left(1 - p(g(n))\right) f\left(v^{\frac{1}{\alpha_1}}(g(n))\right) f\left(\sum_{s=n_3}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s)\right), \quad (2.11)$$

where $v(n) := b(n) (\Delta z(n))^{\alpha_1}$. It is clear that $v(n) > 0$ and $\Delta v(n) < 0$. It follows that

$$-\Delta v(n) \geq -\frac{a^{\frac{1}{\alpha_2}}(n_4) \Delta v(n_4)}{a^{\frac{1}{\alpha_2}}(n)} \text{ for } n \geq n_4.$$

Summing the last inequality from n to ∞ , we obtain

$$v(n) \geq -a^{\frac{1}{\alpha_2}}(n_4) \Delta v(n_4) \sum_{s=n}^{\infty} a^{-\frac{1}{\alpha_2}}(s) = K_1 \sum_{s=n}^{\infty} a^{-\frac{1}{\alpha_2}}(s), \text{ for } n \geq n_4$$

where $K_1 := -a^{\frac{1}{\alpha_2}}(n_4) \Delta v(n_4) > 0$. There exists a $n_5 \geq n_4$ with $g(n) \geq n_4$ for all $n \geq n_5$, such that

$$v(g(n)) \geq K_1 \sum_{s=g(n)}^{\infty} a^{-\frac{1}{\alpha_2}}(s), \text{ for } n \geq n_5.$$

Summing (2.11) from n_5 to $n-1$ and using the above inequality, we find

$$\begin{aligned} \sum_{r=n_5}^{n-1} q(r) f(1-p(g(r))) f\left(\sum_{s=n_3}^{g(r)-1} b^{-\frac{1}{\alpha_1}}(s)\right) f\left(K_1 \sum_{k=g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\right)^{\frac{1}{\alpha_1}} \\ \leq a(n_5) (\Delta v(n_5))^{\alpha_2} - a(n) (\Delta v(n))^{\alpha_2}. \end{aligned}$$

In view of (A_5) , we see that

$$\left(\frac{L}{a(n)} \sum_{r=n_5}^{n-1} q(r) f(1-p(g(r))) f\left(\sum_{s=n_3}^{g(r)-1} b^{-\frac{1}{\alpha_1}}(s)\right)\right)^{\frac{1}{\alpha_2}} \left(f\left(\sum_{k=g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\right)^{\frac{1}{\alpha_1}}\right)^{\frac{1}{\alpha_2}} \leq -\Delta v(n),$$

where $L := f\left(K_1^{\frac{1}{\alpha_1}}\right)$. Summing the above inequality from n_5 to ∞ , we obtain

$$L^{\frac{1}{\alpha_2}} \sum_{s=n_5}^{\infty} \left\{ a^{-\frac{1}{\alpha_2}}(s) \left(\sum_{r=n_5}^{s-1} q(r) f(1-p(g(r))) f\left(\sum_{v=n_3}^{g(r)-1} b^{-\frac{1}{\alpha_1}}(v)\right) f\left(\sum_{k=g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\right)^{\frac{1}{\alpha_1}} \right)^{\frac{1}{\alpha_2}} \right\}$$

$\leq v(n_5) < \infty$,

which contradicts the condition (2.10). The proof is complete.

Theorem 2.3. Assume that (2.3), (2.5), (2.6) and (2.10) hold. If

$$\sum_{l=n_0}^{\infty} \left\{ b^{-\frac{1}{\alpha_1}}(l) \left(\sum_{k=n_0}^{l-1} a^{-\frac{1}{\alpha_2}}(k) \left(\sum_{s=n_0}^{k-1} q(s) f(1-p(g(s))) f\left(\sum_{r=g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right) \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}} \right\} = \infty, \quad (2.12)$$

then every solution of equation (1.1) oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. Let $\{x(n)\}$ is eventually positive solution of (1.1) for $n \geq n_1$. Say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n)) > 0$. By (2.3), there exist four possible cases: (I), (II), (III) (as those of Theorem 2.2) and

$$(IV) \Delta z(n) < 0, \quad \Delta(b(n)(\Delta z(n))^{\alpha}) < 0 \text{ for all large } n.$$

Assume that (I) holds. Then we are back to the proof of Theorem 2.1 to get contradiction to (2.6). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n \rightarrow \infty} x(n) = 0$. Assume that (III) holds. Then we are back to the proof of Theorem 2.2 to get contradiction to (2.10). Assume that (IV) holds. We one can choose $n_3 \geq n_2$ with $g(n) \geq n_2$ for all $n \geq n_3$, such that

$$z(g(n)) \geq -\left(b(g(n))(\Delta z(g(n)))^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \sum_{r=g(n)}^{\infty} b^{-\frac{1}{\alpha_1}}(r) = K_2 \sum_{r=g(n)}^{\infty} b^{-\frac{1}{\alpha_1}}(r) \text{ for } n \geq n_3,$$

where $K_2 := -\left(b(g(n))(\Delta z(g(n)))^{\alpha_1}\right)^{\frac{1}{\alpha_1}} > 0$. Thus equation (1.1), (2.9) and (A_5) yield

$$\Delta \left(a(n) \left(\Delta \left(b(n) (\Delta z(n))^{\alpha_1} \right) \right)^{\alpha_2} \right) \leq Lq(n)f \left(1 - p(g(n)) \right) f \left(\sum_{r=g(n)}^{\infty} b^{-\frac{1}{\alpha_1}}(r) \right).$$

where $L := -f(K_2)$. Summing the above inequality from n_3 to $n-1$, we find

$$\Delta \left(b(n) (\Delta z(n))^{\alpha_1} \right) \leq L^{\frac{1}{\alpha_2}} a^{-\frac{1}{\alpha_2}}(n) \left(\sum_{s=n_3}^{n-1} q(s)f \left(1 - p(g(s)) \right) f \left(\sum_{r=g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r) \right) \right)^{\frac{1}{\alpha_2}}.$$

Again summing the above inequality from n_3 to $n-1$, we find

$$\Delta z(n) \leq K_3 b^{-\frac{1}{\alpha_1}}(n) \left(\sum_{s=n_3}^{n-1} a^{-\frac{1}{\alpha_2}}(s) \left(\sum_{u=n_3}^{s-1} q(u)f \left(1 - p(g(u)) \right) f \left(\sum_{r=g(u)}^{\infty} b^{-\frac{1}{\alpha_1}}(r) \right) \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}}.$$

where $K_3 := L^{\frac{1}{\alpha}}$. Finally, summing the above inequality from n_3 to $n-1$, we have

$$z(n) \leq z(n_3) - K_3 \sum_{s=n_3}^{n-1} \left\{ b^{-\frac{1}{\alpha_1}}(s) \left(\sum_{u=n_3}^{s-1} a^{-\frac{1}{\alpha_2}}(u) \right)^{\frac{1}{\alpha_1}} \times \left(\left(\sum_{v=n_3}^{u-1} q(v)f \left(1 - p(g(v)) \right) f \left(\sum_{r=g(v)}^{\infty} b^{-\frac{1}{\alpha_1}}(r) \right) \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}} \right\}.$$

From condition (2.12), we get $\lim_{n \rightarrow \infty} z(n) = -\infty$. Since $0 < x(n) \leq z(n)$ then, $x(n) \rightarrow -\infty$ as $n \rightarrow \infty$, which contradicts the fact that $x(n)$ is a positive solution of (1.1). The proof is complete.

Theorem 2.4. Let (S_1) , (2.1) and (2.5) hold. Further, assume that there exists a positive nondecreasing sequence $\{\rho(n)\}$, such that

$$\limsup_{m \rightarrow \infty} \frac{1}{H(m, 0)} \sum_{n=0}^{m-1} \left\{ K\rho(n)q(n)H(m, n) \left(1 - p(g(n)) \right)^{\alpha} - \frac{\vartheta^2(m, n)H(m, n)}{2^{3-\alpha}\varphi(n)} \right\} = \infty. \quad (2.13)$$

Then every solution of equation (1.1) oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n)) > 0$. From the proof of Lemma 2.2 there are two possible cases. Assume that (I) holds. Define the sequence $\omega(n)$ by

$$\omega(n) := \rho(n) \frac{a(n) \left(\Delta \left(b(n) (\Delta z(n))^{\alpha_1} \right) \right)^{\alpha_2}}{z^{\alpha}(g(n))}. \quad (2.14)$$

Then $\omega(n) > 0$. From (2.14), we have

$$\begin{aligned} \Delta \omega(n) &= \Delta \rho(n) \frac{a(n+1) \left(\Delta \left(b(n+1) (\Delta z(n+1))^{\alpha_1} \right) \right)^{\alpha_2}}{z^{\alpha}(g(n+1))} + \rho(n) \frac{\Delta \left(a(n) \left(\Delta \left(b(n) (\Delta z(n))^{\alpha_1} \right) \right)^{\alpha_2} \right)}{z^{\alpha}(g(n))} \\ &\quad - \frac{\rho(n)a(n+1) \left(\Delta \left(b(n+1) (\Delta z(n+1))^{\alpha_1} \right) \right)^{\alpha_2} \Delta \left(z^{\alpha}(g(n)) \right)}{z^{\alpha}(g(n+1))z^{\alpha}(g(n))}. \end{aligned}$$

Now, by using the inequality

$$x^{\alpha} - y^{\alpha} \geq 2^{1-\alpha}(x-y)^{\alpha} \text{ for all } x \geq y > 0 \text{ and } \alpha \geq 1, \quad (2.15)$$

then, we have

$$\begin{aligned}
\Delta(z^\alpha(g(n))) &= z^\alpha(g(n+1)) - z^\alpha(g(n)) \\
&\geq 2^{1-\alpha} (z(g(n+1)) - z(g(n)))^\alpha \\
&= 2^{1-\alpha} (\Delta z(g(n)))^\alpha, \alpha \geq 1.
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta\omega(n) \leq \Delta\rho(n) \frac{a(n+1) \left(\Delta(b(n+1)(\Delta z(n+1))^{\alpha_1}) \right)^{\alpha_2}}{z^\alpha(g(n+1))} + \rho(n) \frac{\Delta \left(a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \right)}{z^\alpha(g(n))} \\
- \frac{2^{1-\alpha} \rho(n) a(n+1) \left(\Delta(b(n+1)(\Delta z(n+1))^{\alpha_1}) \right)^{\alpha_2} (\Delta z(g(n)))^\alpha}{z^\alpha(g(n+1)) z^\alpha(g(n))}. \quad (2.16)
\end{aligned}$$

From Lemma 2.1, there exists $n_3 \geq n_2$ with $g(n) \geq n_2$ for all $n \geq n_3$ such that

$$(\Delta z(g(n)))^\alpha \geq \left(a(g(n)) \left(\Delta(b(g(n))(\Delta z(g(n)))^{\alpha_1}) \right)^{\alpha_2} \right) b^{-\alpha_2}(g(n)) \left(\sum_{s=n_2}^{g(n)-1} a^{\frac{1}{\alpha_2}}(s) \right)^{\alpha_2}. \quad (2.17)$$

Since $\Delta \left(a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \right) \leq 0$, $g(n) < n$, we get

$$\begin{aligned}
a(n+1) \left(\Delta(b(n+1)(\Delta z(n+1))^{\alpha_1}) \right)^{\alpha_2} &\leq a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \\
&\leq a(g(n)) \left(\Delta(b(g(n))(\Delta z(g(n)))^{\alpha_1}) \right)^{\alpha_2}. \quad (2.18)
\end{aligned}$$

From (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned}
\Delta\omega(n) \leq \frac{\Delta\rho(n)}{\rho(n+1)} \omega(n+1) + \rho(n) \frac{\Delta \left(a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \right)}{z^\alpha(g(n))} \\
- \frac{2^{1-\alpha} \rho(n)}{\rho^2(n+1) b^{\alpha_2}(g(n))} \left(\sum_{s=n_2}^{g(n)-1} a^{\frac{1}{\alpha_2}}(s) \right)^{\alpha_2} \omega^2(n+1).
\end{aligned}$$

From (2.9), we obtain

$$\begin{aligned}
&\sum_{n=k}^{m-1} K\rho(n)q(n)H(m,n) \left(1 - p(g(n)) \right)^\alpha \\
&\leq - \sum_{n=k}^{m-1} H(m,n) \Delta\omega(n) + \sum_{n=k}^{m-1} \frac{H(m,n)\Delta\rho(n)}{\rho(n+1)} \omega(n+1) - \sum_{n=k}^{m-1} 2^{1-\alpha} H(m,n) \varphi(n) \omega^2(n+1).
\end{aligned}$$

Which yields after summing by parts

$$\begin{aligned}
\sum_{n=k}^{m-1} K\rho(n)q(n)H(m,n) \left(1 - p(g(n)) \right)^\alpha &\leq H(m,k) \omega(k) + \sum_{n=k}^{m-1} \vartheta(m,n) H(m,n) \omega(n+1) \\
&\quad - \sum_{n=k}^{m-1} 2^{1-\alpha} H(m,n) \varphi(n) \omega^2(n+1).
\end{aligned}$$

Using the inequality $Bu - Au^2 \leq \frac{B^2}{4A}$, $A > 0$, we have

$$\sum_{n=k}^{m-1} K\rho(n)q(n)H(m,n) \left(1 - p(g(n))\right)^\alpha \leq H(m,k)\omega(k) + \sum_{n=k}^{m-1} \frac{\vartheta^2(m,n)H(m,n)}{2^{3-\alpha}\varphi(n)}. \quad (2.19)$$

Then,

$$\sum_{n=k}^{m-1} \left\{ K\rho(n)q(n)H(m,n) \left(1 - p(g(n))\right)^\alpha - \frac{\vartheta^2(m,n)H(m,n)}{2^{3-\alpha}\varphi(n)} \right\} \leq H(m,k)\omega(k) \leq H(m,0)|\omega(k)|.$$

Hence,

$$\begin{aligned} \sum_{n=0}^{m-1} \left\{ K\rho(n)q(n)H(m,n) \left(1 - p(g(n))\right)^\alpha - \frac{\vartheta^2(m,n)H(m,n)}{2^{3-\alpha}\varphi(n)} \right\} \\ \leq H(m,0) \left\{ \sum_{n=0}^{k-1} \left| K\rho(n) \left(1 - p(g(n))\right)^\alpha q(n) \right| + |\omega(k)| \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{1}{H(m,0)} \sum_{n=0}^{m-1} \left\{ K\rho(n)q(n)H(m,n) \left(1 - p(g(n))\right)^\alpha - \frac{\vartheta^2(m,n)H(m,n)}{2^{3-\alpha}\varphi(n)} \right\} \\ \leq \sum_{n=0}^{k-1} \left| K\rho(n) \left(1 - p(g(n))\right)^\alpha q(n) \right| + |\omega(k)| < \infty, \end{aligned}$$

which is contrary to (2.13). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n \rightarrow \infty} x(n) = 0$. This completes the proof of Theorem 2.4.

Corollary 2.1. Assume that all the assumptions of Theorem 2.4 hold, except the condition (2.13) is replaced by

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} K\rho(n)q(n)H(m,n) \left(1 - p(g(n))\right)^\alpha = \infty, \\ \limsup_{m \rightarrow \infty} \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} \frac{\vartheta^2(m,n)H(m,n)}{2^{3-\alpha}\varphi(n)} < \infty. \end{aligned}$$

Then equation (1.1) is oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

When $H(m,n) = 1$, we obtain the following result

Corollary 2.2. Assume that all the assumptions of Theorem 2.4 hold, except the condition (2.13) is replaced by

$$\limsup_{n \rightarrow \infty} \sum_{s=n_0}^{n-1} \left\{ K\rho(s)q(s)(1 - p(g(n)))^\alpha - \frac{1}{2^{3-\alpha}\rho(s)} \frac{(\Delta\rho(s))^2 b^{\alpha_2}(g(s))}{\left(\sum_{u=n_2}^{g(s)-1} a^{-\frac{1}{\alpha_2}}(u) \right)^{\alpha_2}} \right\} = \infty. \quad (2.20)$$

Then equation (1.1) is oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Theorem 2.5. Assume that (S_1) , (2.2), (2.5), and (2.13) hold. If

$$\limsup_{n \rightarrow \infty} \sum_{u=n_0}^{n-1} \left\{ b^{-\frac{1}{\alpha_1}}(u) \left(\sum_{s=n_0}^{u-1} a^{-\frac{1}{\alpha_2}}(s) \left(\sum_{t=n_0}^{s-1} \Psi(t) \left(\sum_{\tau=g(t)}^{\infty} a^{-1}(\tau) \right)^{\alpha_2} \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}} \right\} = \infty, \quad (2.21)$$

then every solution of equation (1.1) oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. Let $\{x(n)\}$ is eventually positive solution of (1.1) for $n \geq n_1$. Say $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n)) > 0$. Based on condition (2.2), there exist three possible cases. Assume that (I) holds. Then we are back to the proof of Theorem 2.1 to get contradiction to (2.13). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that

$\lim_{n \rightarrow \infty} x(n) = 0$. Assume that (III) holds. Then, there exists $n_2 \geq n_1$ such that $\Delta z(n) > 0$, $\Delta(b(n)(\Delta z(n))^{\alpha_1}) < 0$ for all $n \geq n_2$. Then, we have

$$\Delta z(n) = \frac{(b(n)(\Delta z(n))^{\alpha_1})^{\frac{1}{\alpha_1}}}{(b(n))^{\frac{1}{\alpha_1}}}.$$

Summing the above inequality from n_2 to $n - 1$, we obtain

$$z(n) - z(n_2) = \sum_{s=n_2}^{n-1} \frac{(b(s)(\Delta z(s))^{\alpha_1})^{\frac{1}{\alpha_1}}}{(b(s))^{\frac{1}{\alpha_1}}} \geq (b(n)(\Delta z(n))^{\alpha_1})^{\frac{1}{\alpha_1}} \sum_{s=n_2}^{n-1} \frac{1}{(b(s))^{\frac{1}{\alpha_1}}}. \quad (2.22)$$

Hence there exists a $n_3 \geq n_2$ such that

$$z(g(n)) \geq (b(g(n))(\Delta z(g(n))^{\alpha_1})^{\frac{1}{\alpha_1}} \sum_{s=n_2}^{g(n)-1} \frac{1}{(b(s))^{\frac{1}{\alpha_1}}}, \text{ for } n \geq n_3.$$

From equation (1.1), (S_1) , (2.9) and the last inequality, we obtain

$$\Delta(a(n)(\Delta v(n))^{\alpha_2}) + K(1 - p(g(n)))^{\alpha} q(n) (b(g(n))(\Delta z(g(n))^{\alpha_1})^{\alpha_2} \left(\sum_{s=n_2}^{g(n)-1} \frac{1}{(b(s))^{\frac{1}{\alpha_1}}} \right)^{\alpha} \leq 0, \quad (2.23)$$

where $v(n) := b(n)(\Delta z(n))^{\alpha_1}$. It is clear that $v(n) > 0$ and $\Delta v(n) < 0$. It follows that

$$\Delta(a(n)(\Delta v(n))^{\alpha_2}) + \Psi(n)v^{\alpha_2}(g(n)) \leq 0, \quad \text{for } n \geq n_3. \quad (2.24)$$

Since $g(n) \rightarrow \infty$ as $n \rightarrow \infty$, we can choose $n_4 \geq n_3$ such that $g(n) \geq n_4$ for $n \geq n_4$ and thus

$$\begin{aligned} \sum_{s=g(n)}^{\infty} \Delta v(s) &= v(\infty) - v(g(n)) = \sum_{s=g(n)}^{\infty} a(s)\Delta v(s) \frac{1}{a(s)} \\ &< a(g(n))\Delta v(g(n)) \sum_{s=g(n)}^{\infty} \frac{1}{a(s)} < a(n_4)\Delta v(n_4) \sum_{s=g(n)}^{\infty} \frac{1}{a(s)}. \end{aligned}$$

Thus

$$-v(g(n)) < a(n_4)\Delta v(n_4) \sum_{s=g(n)}^{\infty} \frac{1}{a(s)}.$$

Substituting back in (2.24), we have

$$\Delta(a(n)(\Delta v(n))^{\alpha_2}) < L^{\alpha_2} \Psi(n) \left(\sum_{s=g(n)}^{\infty} \frac{1}{a(s)} \right)^{\alpha_2}, \quad \text{for } n \geq n_4, \quad (2.25)$$

where $L := a(n_4)\Delta v(n_4) < 0$. Summing this inequality from n_4 to $n - 1$, we see that

$$a(n)(\Delta v(n))^{\alpha_2} < a(n)(\Delta v(n))^{\alpha_2} - a(n_4)(\Delta v(n_4))^{\alpha_2} < L^{\alpha_2} \sum_{s=n_4}^{n-1} \Psi(s) \left(\sum_{\tau=g(s)}^{\infty} \frac{1}{a(\tau)} \right)^{\alpha_2}.$$

where $\Delta v(n) < 0$. Summing again from n_5 to $n - 1$, we have

$$v(n) < L \sum_{s=n_5}^{n-1} a^{-\frac{1}{\alpha_2}}(s) \left(\sum_{t=n_4}^{s-1} \Psi(t) \left(\sum_{\tau=g(t)}^{\infty} \frac{1}{a(\tau)} \right)^{\alpha_2} \right)^{\frac{1}{\alpha_2}}$$

or equivalently

$$\Delta z(n) < \left(\frac{L}{b(n)}\right)^{\frac{1}{\alpha_1}} \left(\sum_{s=n_5}^{n-1} a^{-\frac{1}{\alpha_2}}(s) \left(\sum_{t=n_4}^{s-1} \Psi(t) \left(\sum_{\tau=g(t)}^{\infty} \frac{1}{a(\tau)} \right)^{\alpha_2} \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}}.$$

Summing from n_6 to $n-1$, we have

$$z(n) < z(n_6) + L^{\frac{1}{\alpha_1}} \sum_{u=n_6}^{n-1} \left\{ b^{-\frac{1}{\alpha_1}}(u) \times \left(\sum_{s=n_5}^{u-1} a^{-\frac{1}{\alpha_2}}(s) \left(\sum_{t=n_4}^{s-1} \Psi(t) \left(\sum_{\tau=g(t)}^{\infty} \frac{1}{a(\tau)} \right)^{\alpha_2} \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}} \right\}$$

By condition (2.21), we have $\lim_{n \rightarrow \infty} z(n) = -\infty$. Since $0 < x(n) \leq z(n)$ then, $x(n) \rightarrow -\infty$ as $n \rightarrow \infty$ which contradicts the fact that $x(n) > 0$. The proof is complete.

Theorem 2.6. Assume that (S_1) , (2.3), (2.5), (2.13), and (2.21) hold. If

$$\sum_{u=n_0}^{\infty} \left(b^{-\frac{1}{\alpha_1}}(u) \left(\sum_{s=n_0}^{u-1} a^{-\frac{1}{\alpha_2}}(s) \right)^{\frac{1}{\alpha_1}} \right) = \infty, \quad (2.26)$$

then every solution of equation (1.1) oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. Let $\{x(n)\}$ is eventually positive solution of (1.1) for $n \geq n_1$. Based on condition (2.3), there exist four possible cases. Assume that (I) holds. Then we are back to the proof of Theorem 2.4 to get contradiction to (2.13). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n \rightarrow \infty} x(n) = 0$. Assume that (III) holds. Then we are back to the proof of Theorem 2.5 to get contradiction to (2.21). Assume that (IV) holds. Since $a(n)\Delta(b(n)(\Delta z(n))^{\alpha_1})$ is non-increasing sequence there exists a negative constant K_4 and $n_2 \geq n_1$ such that

$$a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \leq K_4 \text{ for } n \geq n_2.$$

Dividing by $a(n)$ and summing the last inequality from n_1 to $n-1$, we obtain

$$\Delta z(n) \leq b^{-\frac{1}{\alpha_1}}(n) K_4^{\frac{1}{\alpha_2}} \left(\sum_{s=n_1}^{n-1} a^{-\frac{1}{\alpha_2}}(s) \right)^{\frac{1}{\alpha_1}}.$$

Summing the last inequality from n_1 to $n-1$, we obtain

$$z(n) \leq z(n_1) + K_4^{\frac{1}{\alpha_2}} \sum_{u=n_1}^{n-1} \left(b^{-\frac{1}{\alpha_1}}(u) \left(\sum_{s=n_1}^{u-1} a^{-\frac{1}{\alpha_2}}(s) \right)^{\frac{1}{\alpha_1}} \right).$$

From condition (2.26), we get $\lim_{n \rightarrow \infty} z(n) = -\infty$. Since $0 < x(n) \leq z(n)$ then, $x(n) \rightarrow -\infty$ as $n \rightarrow \infty$, which contradicts the fact that $x(n)$ is a positive solution of (1.1). The proof is complete.

Theorem 2.7. Let (S_2) , (2.1) and (2.5) hold. Further, assume that there exists a positive nondecreasing sequence $\{\rho(n)\}$, such that

$$\limsup_{m \rightarrow \infty} \frac{1}{H(m, 0)} \sum_{n=0}^{m-1} \left\{ H(m, n) \rho(n) q(n) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{g^{\alpha+1}(m, n) H(m, n)}{(\Theta(n))^{\alpha}} \right\} = \infty. \quad (2.27)$$

Then every solution of equation (1.1) oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n)) > 0$. From the proof of Lemma 2.2 there are two possible cases. Assume that (I) holds. Define the sequence $\omega(n)$ by

$$\omega(n) := \rho(n) \frac{a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2}}{f(x(g(n)))}. \quad (2.28)$$

Then $\omega(n) > 0$. From (2.28) and (S_2) , we have

$$\begin{aligned} \Delta\omega(n) &= \Delta\rho(n) \frac{a(n+1) \left(\Delta(b(n+1)(\Delta z(n+1))^{\alpha_1}) \right)^{\alpha_2}}{f(x(g(n+1)))} + \rho(n) \frac{\Delta \left(a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \right)}{f(x(g(n)))} \\ &\quad - \frac{\rho(n) a(n+1) \Delta(b(n+1)(\Delta z(n+1))^{\alpha_1})^{\alpha_2} \Delta(f(x(g(n))))}{f(x(g(n+1))) f(x(g(n)))}. \\ &= \Delta\rho(n) \frac{a(n+1) \left(\Delta(b(n+1)(\Delta z(n+1))^{\alpha_1}) \right)^{\alpha_2}}{f(x(g(n+1)))} + \rho(n) \frac{\Delta \left(a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \right)}{f(x(g(n)))} \\ &\quad - \rho(n) \frac{a(n+1) \Delta(b(n+1)(\Delta z(n+1))^{\alpha_1})^{\alpha_2}}{f(x(g(n+1))) f(x(g(n)))} \times B(x(g(n+1)), x(g(n))) \\ &\quad \times [x(g(n+1)) + p(g(n+1))x(g(n+1-\tau))] - [x(g(n)) + p(g(n))x(g(n-\tau))] \\ &= \Delta\rho(n) \frac{a(n+1) \left(\Delta(b(n+1)(\Delta z(n+1))^{\alpha_1}) \right)^{\alpha_2}}{f(x(g(n+1)))} + \rho(n) \frac{\Delta \left(a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \right)}{f(x(g(n)))} \\ &\quad - \rho(n) \frac{a(n+1) \Delta(b(n+1)(\Delta z(n+1))^{\alpha_1})^{\alpha_2}}{f(x(g(n+1))) f(x(g(n)))} B(x(g(n+1)), x(g(n))) \Delta(z(g(n))). \quad (2.29) \end{aligned}$$

From Lemma 2.1, there exists $n_3 \geq n_2$ with $g(n) \geq n_2$ for all $n \geq n_3$ such that

$$\Delta z(g(n)) \geq \left(a(g(n)) \left(\Delta(b(g(n))(\Delta z(g(n)))^{\alpha_1}) \right)^{\alpha_2} \right)^{\frac{1}{\alpha}} b^{-\frac{1}{\alpha_1}}(g(n)) \left(\sum_{s=n_2}^{g(n)-1} a^{-\frac{1}{\alpha_2}}(s) \right)^{\frac{1}{\alpha_1}}. \quad (2.30)$$

From (2.18) and (2.29), we have

$$\begin{aligned} \Delta\omega(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)} \omega(n+1) + \rho(n) \frac{\Delta \left(a(n) \left(\Delta(b(n)(\Delta z(n))^{\alpha_1}) \right)^{\alpha_2} \right)}{f(x(g(n)))} \\ &\quad - \rho(n) \frac{a(n+1) \Delta(b(n+1)(\Delta z(n+1))^{\alpha_1})^{\alpha_2}}{f(x(g(n+1))) f(x(g(n)))} B(x(g(n+1)), x(g(n))) \\ &\quad \times \left(a(n+1) \left(\Delta(b(n+1)(\Delta z(n+1))^{\alpha_1}) \right)^{\alpha_2} \right)^{\frac{1}{\alpha}} b^{-\frac{1}{\alpha_1}}(g(n)) \delta^{\frac{1}{\alpha_1}}(g(n), n_2). \quad (2.31) \end{aligned}$$

From (1.1), (2.28), (S_2) and (2.31), we have

$$\Delta\omega(n) \leq -\rho(n)q(n) + \frac{\Delta\rho(n)}{\rho(n+1)} \omega(n+1) - \mu\rho(n) \frac{\delta^{\frac{1}{\alpha_1}}(g(n), n_2) b^{-\frac{1}{\alpha_1}}(g(n))}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)} \omega^{\frac{\alpha+1}{\alpha}}(n+1). \quad (2.32)$$

Therefore, we have

$$\begin{aligned} \sum_{n=k}^{m-1} H(m, n) \rho(n) q(n) &\leq - \sum_{n=k}^{m-1} H(m, n) \Delta \omega(n) \\ &\quad + \sum_{n=k}^{m-1} H(m, n) \frac{\Delta \rho(n)}{\rho(n+1)} \omega(n+1) - \sum_{n=k}^{m-1} H(m, n) \theta(n) \omega^{\frac{\alpha+1}{\alpha}}(n+1), \end{aligned}$$

which yields after summing by parts

$$\begin{aligned} \sum_{n=k}^{m-1} H(m, n) \rho(n) q(n) &\leq H(m, k) \omega(k) \\ &\quad + \sum_{n=k}^{m-1} \vartheta(m, n) H(m, n) \omega(n+1) - \sum_{n=k}^{m-1} H(m, n) \theta(n) \omega^{\frac{\alpha+1}{\alpha}}(n+1). \end{aligned}$$

Using the inequality $Bu - Au^{\alpha+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$, $A > 0$, we have

$$\sum_{n=k}^{m-1} H(m, n) \rho(n) q(n) \leq H(m, k) \omega(k) + \sum_{n=k}^{m-1} \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\vartheta^{\alpha+1}(m, n) H(m, n)}{(\theta(n))^\alpha}.$$

Then,

$$\begin{aligned} &\sum_{n=k}^{m-1} \left\{ H(m, n) \rho(n) q(n) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\vartheta^{\alpha+1}(m, n) H(m, n)}{(\theta(n))^\alpha} \right\} \\ &\leq H(m, k) \omega(k) \leq H(m, 0) |\omega(k)|. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{n=0}^{m-1} \left\{ H(m, n) \rho(n) q(n) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\vartheta^{\alpha+1}(m, n) H(m, n)}{(\theta(n))^\alpha} \right\} \\ &\leq H(m, 0) \left\{ \sum_{n=0}^{k-1} |\rho(n) q(n)| + |\omega(k)| \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} &\limsup_{m \rightarrow \infty} \frac{1}{H(m, 0)} \sum_{n=0}^{m-1} \left\{ H(m, n) \rho(n) q(n) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\vartheta^{\alpha+1}(m, n) H(m, n)}{(\theta(n))^\alpha} \right\} \\ &\leq \sum_{n=0}^{k-1} |\rho(n) q(n)| + |\omega(k)| < \infty, \end{aligned}$$

which is contrary to (2.27). Assume that (II) holds. Then we are back to the proof of Lemma 2.3 to show that $\lim_{n \rightarrow \infty} x(n) = 0$. This completes the proof of Theorem 2.7.

From Theorem 2.7, if $H(m, n) = 1$, we get the following result

Corollary 2.3. Assume that all the assumptions of Theorem 2.7 hold, except the condition (2.27) is replaced by

$$\limsup_{n \rightarrow \infty} \sum_{s=n_0}^{n-1} \left(\rho(s)q(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{(\Delta \rho(s))^{\alpha+1} b^{\alpha_2}(g(s))}{\left(\mu \rho(s) \delta^{\frac{1}{\alpha_1}}(g(s), n_2) \right)^\alpha} \right) = \infty.$$

Then equation (1.1) is oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Theorem 2.8. Assume that (2.2), (2.5), (2.27) and (2.10) hold. Then every solution of equation (1.1) oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. The proof is similar to that of Theorem 2.2, Theorem 2.7 and hence the details are omitted.

Theorem 2.9. Assume that (2.3), (2.5), (2.27), (2.10) and (2.12) hold. Then every solution of equation (1.1) oscillatory or $\lim_{n \rightarrow \infty} x(n) = 0$.

Proof. The proof is similar to that of Theorem 2.3, Theorem 2.7 and hence the details are omitted.

3. Conclusions

In this paper, we established some new sufficient conditions which insure that every solution of this equation either oscillates or converges to zero. Our results improved and expanded some known results, see e.g. the following results :

Remark 3.1. If $b(n) \equiv 1, \alpha_1 \equiv 1, p(n) \equiv 0$. Then Theorem 2.3 reduced to a special case Theorem 2.1 in [4].

Remark 3.2. If $\alpha_2 \equiv 1, p(n) \equiv 0, g(n) \equiv n - \sigma$. Then Theorem 2.4 extended and improved Theorem 6 in [6].

Remark 3.3. $b(n) \equiv 1, \alpha_1 \equiv \alpha_2 \equiv 1, p(n) \equiv 0$ and $H(m, n) \equiv 1$. Then Theorem 2.4 reduced to a special case of Theorem 1 in [5].

Remark 3.4. If $\alpha_1 \equiv \alpha_2 \equiv 1, p(n) \equiv 0, g(n) \equiv n + l, H(m, n) \equiv 1$. Then Theorem 2.4 extended and improved Theorem 3 in [3].

Remark 3.5. If $\alpha_1 \equiv \alpha_2 \equiv 1, p(n) \equiv 0, g(n) \equiv n - m + 1, H(m, n) \equiv (m - n)^r$. Then Theorem 2.4 reduced to a special case of Theorem 1 in [1].

Remark 3.6. If $b(n) \equiv 1, \alpha_1 \equiv 1, f(x) \equiv x^\alpha$ and $H(m, n) \equiv 1$. Then Theorem 2.4 extended and improved Theorem 2.7 and Theorem 2.8 in [7].

Remark 3.7. If $\alpha_1 \equiv \alpha_2 \equiv 1, n \equiv 1, p(n) \equiv 0, g(n) \equiv n - m + 1$. Then Corollary 2.2 reduced to a special case of Theorem 2 in [1].

Remark 3.8. If $b(n) \equiv 1, f(x) \equiv x^\alpha, H(m, n) \equiv 1$. Then Theorem 2.7 extended and improved Theorem 2.5 and Theorem 2.6 in [7].

Remark 3.9. If $\alpha_2 \equiv 1, p(n) \equiv 0, g(n) \equiv n - \sigma$. Then Theorem 2.9 extended and improved Theorem 15 in [6].

Remark 3.10. If $p(n) \equiv 0$. Then we extended and improved Theorems in [2].

Remark 3.11. If $\alpha_1 \equiv \alpha_2 \equiv 1, q(n) \equiv q(n + 1), g(n) \equiv n + 1 - l$. Then we reduced to Theorems in [8].

Remark 3.12. If $b(n) \equiv 1, \alpha_1 \equiv \alpha_2 \equiv 1, g(n) \equiv n + 1$. Then we reduced to Theorems in [15].

4. Examples

In this section we will show the applications of our oscillation criteria by three examples. We will see that the equations in the example is oscillatory or tend to zero based on the results in Section 2.

Example 4.1. Consider the difference equation

$$\Delta^3 \left(x(n) + \frac{1}{3} x(n - \lambda_1) \right) + \left(\frac{27}{32} \right) x(n - \lambda_2) = 0. \quad (4.1)$$

If we take $\rho(n) = 1$ and $H(m, n) = m - n$, then all conditions of theorem 2.4 are also satisfied. Hence every solution of (4.1) is oscillatory or satisfies $\lim_{n \rightarrow \infty} x(n) = 0$.

Example 4.2. Consider the third order nonlinear neutral difference equation

$$\Delta \left(\frac{1}{n} \left(\Delta^2 \left(x_n + \frac{3}{4} x(n - 2) \right) \right) \right) + n^2 x^3(n - 1) (1 + x^2(n - 1)) = 0, n \geq 1. \quad (4.2)$$

If we take $\rho(n) = n$, then, Corollary 2.2 asserts that every solution of (4.2) is oscillatory or tend to zero.

Example 4.3. Consider the third order difference equation

$$\Delta^3 \left(x(n) + \frac{1}{3} x(n - 2) \right) + \frac{\lambda}{n^2} x(n - 2) = 0, n \geq 1. \quad (4.3)$$

By choosing $\rho(n) = n, \mu = 1$ and $H(m, n) = 1$. Then, by Theorem 2.7, we conclude that every solution of (4.3) either oscillates or tend to zero.

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