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A Proof Method on Labelings of Graph

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Abstract Codial labeling and total product cordial labeling of pyramid graph were discussed, and graph decomposition method was introduced. It is proved that pyramid graphs are cordial and total product cordial.

Keywords Cordial Graph, Total Product Cordial Graph, Pyramid Graph

1. Introduction and Definitions

In1987, Cahit[1] introduced the notion of cordial labeling. Sundaram, Ponraj and Somasundar-am[6] introduced the notions of product cordial labeling and total product cordial labeling in 2006. As for the detailed results on cordial graph, product cordial graph and total product cordial graph, the readers can be referred to[2]. Lee and Wang[4] defined pyramid graph, and proved that pyramid graph is graceful. Most graph labeling methods trace their original one introduced by Rosa[5] in 1967, or one given by Graham and Sloane[3] in 1980. In this paper, graph decomposition method was introduced, it is proved that pyramid graphs are cordial and total product cordial.

Definition 1.1.Cordial Graph: a cordial labeling of a graph G with vertex set V is a function f from V to {0, 1} such that if each edge uv assigned the label $g(uv) = |f(u) - f(v)|$, the number of vertices labeled with 1 and the number of vertices labeled with 0 differ by at most 1, the number of edges labeled with 1 and the number of edges labeled with 0 differ by at most 1. A graph $G(V, E)$ is called cordial graph if it admits a cordial labeling.

Definition 1.2. total product cordial graph: a total product cordial labeling of a graph G with vertex set V is a function f from V to {0,1} such that if each edge uv is assigned the label $g(uv) = f(u)f(v)$, the number of vertices and edges labeled with 1 and the number of vertices and edges labeled with 0 differ by at most 1. A graph is called total product cordial graph if it admits a total product cordial labeling.

Definition 1.3. pyramid graph: a pyramid graph obtained by arranging vertices into a finite number of lines with i vertices in the i th line and every line the j th vertex in that line is joined to the j th vertex and the (j +1)th vertex of the next line. A pyramid graph that has $\frac{n(n+1)}{2}$ vertices is denoted by J_n is illustrated in Fig.1.

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2. Results

First, we let $v_i = |\{v \in V(G) \mid f(v) = i\}|$, $e_i = |\{uv \in E(G) \mid g(uv) = i\}|$.

Theorem 2.1. J_n is cordial graph.

In the follow discussion, J_n is divided into some subgraphs, and defined the vertex labelings in these subgraphs, to discuss J_n.

Proof the vertices of J_n are divided into groups every four lines successively and the vertices in the rest lines (if exists) form one group, in each group, there are odd number of vertices in the first line and the third line, there are even number of vertices in the second line and the fourth line. The rules of vertices in each group are defined as follows:

1. the vertex labelings in the same line are alternating 1, 0;
2. the first vertex labeling in the first line and the fourth line are 0;
3. the first vertex labeling in the second line and the third line are 1.

Based on the previous rules, in every group, the number of vertices labeled with 0 is one more than the number of vertices labeled with 1 in the first line, the number of vertices labeled with 1 is one more than the number of vertices labeled with 0 in the third line, the number of vertices labeled with 0 is the same as the number of vertices labeled with 1 in the second and fourth lines, therefore

when $n \equiv 0 (\bmod 4)$, $|v_0 - v_1| = 0$ in J_n;

when $n \equiv 1 (\bmod 4)$, there is only one line in the last group, so $|v_0 - v_1| = 1$ in J_n;

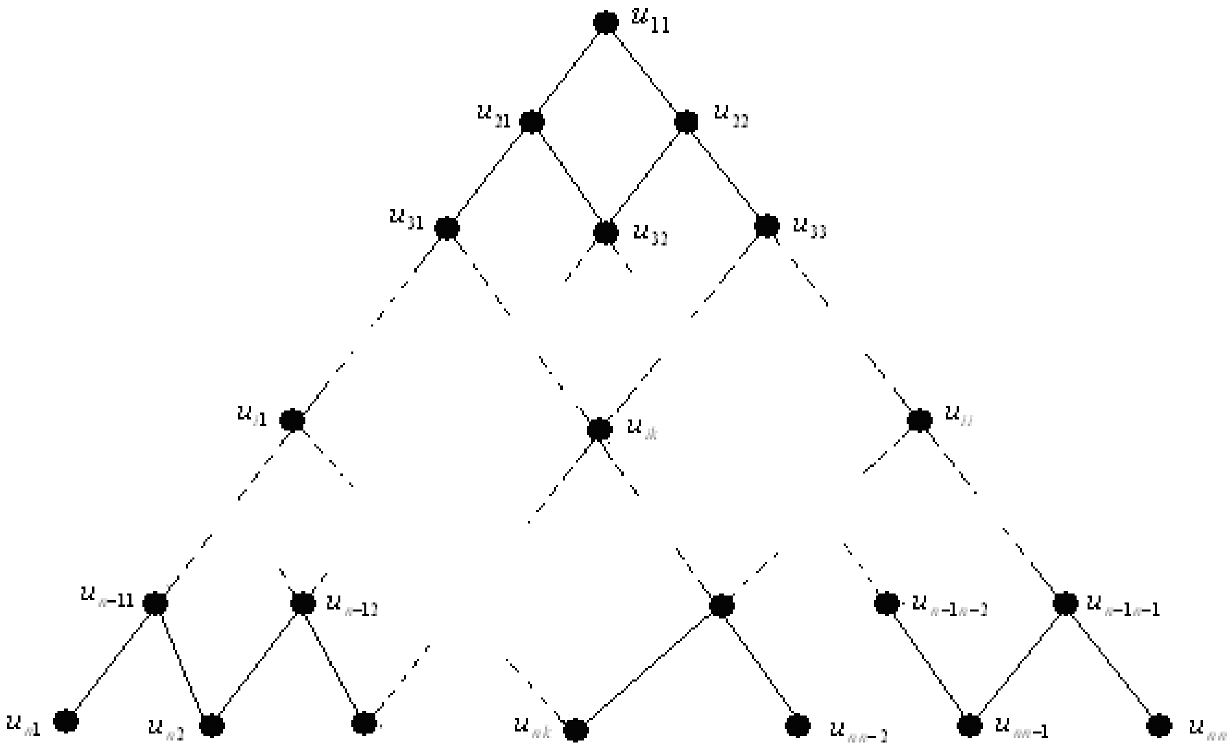

Figure 1. pyramid J_n

when $n \equiv 2(\mathrm{mod}\,4)$, there are only two lines in the last group, so $|v_0 - v_1| = 1$ in J_n;

when $n \equiv 3(\mathrm{mod}\,4)$, there are three lines in the last group, $|v_0 - v_1| = 0$ in J_n.

Except the vertices in the last line, each vertex in other lines is adjacent to two vertices in the next line in J_n, based on the previous rules and definition 1.1, it is obtained that edge labeled with 0 and 1 which are got between each vertex in each line and adjacent vertices in the next line appear in pairs, so $|e_0 - e_1| = 0$, therefore, J_n is cordial.

Theorem 2.2. J_n is total product cordial graph.

In the follow discussion, J_n is divided into two part, one is J_{n-2}, another is the last two lines in J_n, used the vertex labelings in the last two lines and the total product cordial property of J_{n-2}, to discuss J_n.

Lemma 2.1. if n is even, then J_n is total product cordial graph.

Proof First, the rules are defined as follows:

1. the vertex labelings of the first $(n-2)$ lines in J_n are the vertex labelings in J_{n-2};

2. in J_n, the vertex labelings in the odd lines are $0, 1, 0, 1, \cdots, 0, 1, 0$;

3. when $n \geq 10$, successive six graphs from one group;

4. in $J_{2(6k-1)}$, the vertex labelings in the last line are $1, 1, \cdots, 1, 0, 0, \cdots, 0$, the number of vertices labeled with 1 is $(10k-1)$ and the number of vertices labeled with 0 is $(2k-1)$ in the last line;

5. in $J_{2(6k)}$, the vertex labelings in the last line are $0, 1, 1, \cdots, 1, 0, 0, \cdots, 0$, the number of vertices labeled with 1 is $(10k-1)$ and the number of vertices labeled with 0 is $(2k+1)$ in the last line;

6. in $J_{2(6k+1)}$, the vertex labelings in the last line are $0, 1, 1, \cdots, 1, 0, 0, \cdots, 0$, the number of vertices labeled with 1 is $(10k+2)$ and the number of vertices labeled with 0 is $(2k)$ in the last line;

7. in $J_{2(6k+2)}$, the vertex labelings in the last line are $1, 1, \cdots, 1, 0, 0, \cdots, 0$, the number of vertices labeled with 1 is $(10k+3)$ and the number of vertices labeled with 0 is $(2k+1)$ in the last line;

8. in $J_{2(6k+3)}$, the vertex labelings in the last line are $0, 1, 1, \cdots, 1, 0, 0, \cdots, 0$, the number of vertices labeled with 1 is $(10k+5)$ and the number of vertices labeled with 0 is $(2k+1)$ in the last line;

9. in $J_{2(6k+4)}$, the vertex labelings in the last line are

$0, 1, 1, \cdots, 1, 0, 0, \cdots, 0$, the number of vertices labeled with 1 is $(10k+6)$ and the number of vertices labeled with 0 is $(2k+2)$ in the last line.

The vertex labelings in J_8 are illustrated in Fig.2, according to the rule 1, we obtain that J_2, J_4, J_6, J_8 are total product cordial too.

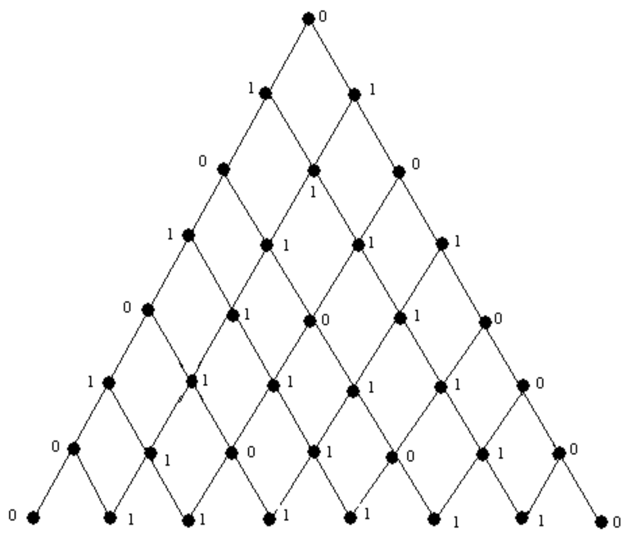

Figure 2. The vertex labelings in J_8

J_{10}, J_{12}, J_{14}, J_{16}, J_{18}, J_{20} of the first group are discussed as follows. The followings are got by calculation:

in J_{10}, $v_1 \cup e_1 = 73$, $v_0 \cup e_0 = 72$; in J_{12}, $v_1 \cup e_1 = v_0 \cup e_0 = 105$;

in J_{14}, $v_1 \cup e_1 = 144$, $v_0 \cup e_0 = 143$; in J_{16}, $v_1 \cup e_1 = v_0 \cup e_0 = 188$;

in J_{18}, $v_1 \cup e_1 = 239$, $v_0 \cup e_0 = 238$; in J_{20}, $v_1 \cup e_1 = v_0 \cup e_0 = 295$,

so J_{10}, J_{12}, J_{14}, J_{16}, J_{18}, J_{20} are total product cordial,

Suppose, $J_{2(6k-1)}$, $J_{2(6k)}$, $J_{2(6k+1)}$, $J_{2(6k+2)}$, $J_{2(6k+3)}$, $J_{2(6k+4)}$ in the (k)th $(k>1)$ group are total product cordial, the labelings of every graph in the $(k+1)$ th group are discussed.

In $J_{2(6k+5)}$, according to rule 4, the number of vertices labeled with 1 is $(10k+9)$ and the number of vertices labeled with 0 is $(2k+1)$ in the last line; according to rule 2, the number of vertices labeled with 1 is $(6k+4)$ and the number of vertices labeled with 0 is $(6k+5)$ in the second line from the bottom. The numbers of edges labeled with 1 and with 0 in the last two lines are discussed, because the vertex labelings are alternating 0 and 1in in the second line from the bottom, the vertex labelings in the third line from the bottom are the vertex labelings in the last line in $J_{2(6k+4)}$, in edges that between the vertices in the third line from the bottom and the vertices in the second line from the bottom, the number of edges labeled with 1 is $10k+6$, the number of edges labeled with 0 is $10k+6+2(2k+2)=14k+10$; in edges that between the vertices in the second line from the bottom and the vertices in the last line, the number of edges labeled with 1 is $(10k+8)$, the number of edges labeled with 0 is $1+10k+8+2(2k+1)-1=14k+10$, so the number of vertices and edges labeled with 1 in the last two lines of $J_{2(6k+5)}$ is

$$6k+4+10k+9+10k+6+10k+8=36k+27,$$

the number of vertices and edges labeled with 0 is

$$6k+5+2k+1+14k+10+14k+10=36k+26,$$

because $v_1 \cup e_1 = v_0 \cup e_0$ in $J_{2(6k+4)}$, so $v_1 \cup e_1 - v_0 \cup e_0 = 1$ in $J_{2(6k+5)}$.

In $J_{2(6k+6)}$, according to rule 5, the number of vertices labeled with 1 is $(10k+9)$ and the number of vertices labeled with 0 is $(2k+3)$ in the last line; according to rule 2, the number of vertices labeled with 1 is $(6k+5)$ and the number of vertices labeled with 0 is $(6k+6)$ in the second line from the bottom, the vertex labelings in the third line from the bottom are the vertex labelings in the last line in $J_{2(6k+5)}$, in edges that between the vertices in the third line from the bottom and the vertices in the second line from the bottom, the number of edges labeled with 1 is $10k+9$, the number of edges labeled with 0 is $10k+9+2(2k+1)=14k+11$, similarly, the number of the edges in the last two lines labeled with 1 is $(10k+9)$, the number of edges in the last two lines labeled with 0 is $10k+9+1+2(2k+1)+1=14k+13$, so the number of vertices and edges labeled with 1 in the last two lines of $J_{2(6k+6)}$ is $16k+14+10k+9+10k+9=36k+32$, the number of vertices and edges labeled with 0 in the last two lines is $8k+9+14k+11+14k+13=36k+33$, because $v_1 \cup e_1 -$ $v_0 \cup e_0 = 1$ in $J_{2(6k+5)}$, so $v_1 \cup e_1 = v_0 \cup e_0$ in $J_{2(6k+6)}$.

The discussions of labeling number in the next part are similar to the above ones, therefore the number of the labelings of vertices and edges in the last two lines are given as follows.

In $J_{2(6k+7)}$, according to rule 2 and 6, the number of vertices labeled with 1 is $(16k+18)$, the number of vertices labeled with 0 is $(8k+9)$, the number of edges labeled with 1 is $(20k+21)$, the number of edges labeled with 0 is $(28k+29)$;

In $J_{2(6k+8)}$, according to rule 2 and 7, the number of vertices labeled with 1 is $(16k+20)$, the number of vertices labeled with 0 is $(8k+11)$, the number of edges labeled with 1 is $(20k+24)$, the number of edges labeled with 0 is $(28k+34)$;

In $J_{2(6k+9)}$, according to rule 2 and 8, the number of vertices labeled with 1 is $(16k+23)$, the number of vertices labeled with 0 is $(8k+12)$, the number of edges labeled with 1 is $(20k+28)$, the number of edges labeled with 0 is $(28k+38)$;

In $J_{2(6k+10)}$, according to rule 2 and 9, the number of vertices labeled with 1 is $(16k+25)$, the number of vertices labeled with 0 is $(8k+14)$, the number of edges labeled with 1 is $(20k+31)$, the number of edges labeled with 0 is $(28k+43)$.

Lemma 2.2. if n is odd, then J_n is total product cordial graph.

In the proof of lemma 2.2, the discussion is similar to that in lemma2.1, therefore only the rules and some of results in different cases are given.

Proof First, the rules are given as follows:

1. the vertex labelings in the first $(n-2)$ lines in J_n are the vertex labelings in J_{n-2};
2. in J_n, the vertex labelings in the even lines are $1, 0, 1, 0, \cdots, 0, 1, 0$;
3. when $n \geq 15$, successive six graphs from one group;
4. in $J_{2(6k+1)+1}$, the vertex labelings in the last line are $0, 0, \cdots, 0, 1, 1, \cdots, 1$, the number of vertices labeled with 1 is $(10k+2)$ and the number of vertices labeled with 0 is $(2k+1)$ in the last line;
5. in $J_{2(2k+2)+1}$, the vertex labelings in the last line are $0, 0, \cdots, 0, 1, 1, \cdots, 1$, the number of vertices labeled with 1 is $(10k+4)$ and the number of vertices labeled with 0 is $(2k+1)$ in the last line;
6. in $J_{2(6k+3)+1}$, the vertex labelings in the last line are $0, 0, \cdots, 0, 1, 1, \cdots, 1$, the number of vertices labeled with 1 is $(10k+6)$ and the number of vertices labeled with 0 is $(2k+1)$ in the last line;
7. in $J_{2(6k+4)+1}$, the vertex labelings in the last line are $0, 0, \cdots, 0, 1, 1, \cdots, 1, 0$, the number of vertices labeled with 1 is $(10k+7)$ and the number of vertices labeled with 0 is $(2k+2)$ in the last line;
8. in $J_{2(6k+5)+1}$, the vertex labelings in the last line are $0, 0, \cdots, 0, 1, 1, \cdots, 1$, the number of vertices labeled with 1 is $(10k+9)$ and the number of vertices labeled with 0 is $(2k+2)$ in the last line;
9. in $J_{2(6k+6)+1}$, the vertex labelings in the last line are $0, 0, \cdots, 0, 1, 1, \cdots, 1$, the number of vertices labeled with 1 is $(10k+11)$ and the number of vertices labeled with 0 is $(2k+2)$ in the last line.

The vertex labelings in J_{13} are illustrated in Fig.3. According to rule 1, $J_3, J_5, J_7, J_9, J_{11}$, J_{13} are total product cordial.

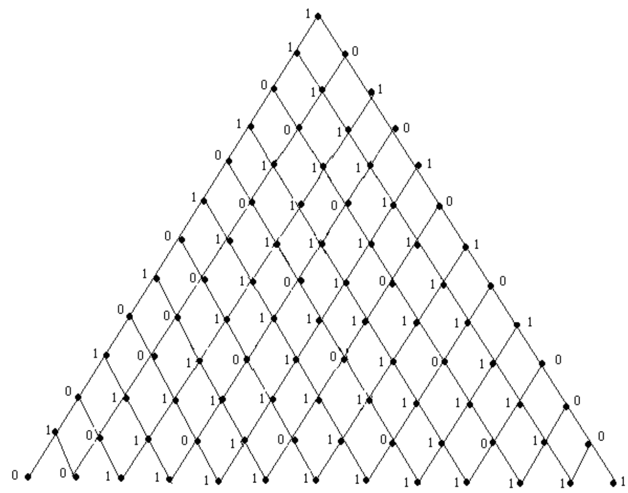

Figure 3. The vertex labelings in J_{13}

J_{15}, J_{17}, J_{19}, J_{21}, J_{23}, J_{25} of the first group are discussed as follows. The followings are got by calculation:

in J_{15}, $v_1 \cup e_1 = v_0 \cup e_0 = 165$; in J_{17}, $v_0 \cup e_0 = 213$, $v_1 \cup e_1 = 212$;

in J_{19}, $v_1 \cup e_1 = v_0 \cup e_0 = 266$; in J_{21}, $v_1 \cup e_1 = 326$, $v_0 \cup e_0 = 325$;

in J_{23}, $v_1 \cup e_1 = v_0 \cup e_0 = 391$; in J_{25}, $v_1 \cup e_1 = 463$, $v_0 \cup e_0 = 462$。

In $J_{2(6k+1)+1}$, $v_1 \cup e_1 = v_0 \cup e_0$; in $J_{2(6k+2)+1}$,

$v_0 \cup e_0 - v_1 \cup e_1 = 1$;

in $J_{2(6k+3)+1}$, $v_1 \cup e_1 = v_0 \cup e_0$; in $J_{2(6k+4)+1}$, $v_1 \cup e_1 - v_0 \cup e_0 = 1$; in $J_{2(6k+5)+1}$, $v_1 \cup e_1 = v_0 \cup e_0$; in $J_{2(6k+6)+1}$, $v_1 \cup e_1 - v_0 \cup e_0 = 1$.

Overall, J_n is total product cordial.

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