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(1,2) - domination in Middle and Central Graph of $K_{1,n}$, C_n and P_n

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Abstract In this paper we discuss domination and (1,2) - domination in middle graph and central graph of $K_{1,n}, C_n$ and P_n.

Keywords Dominating set, Domination Number, (1,2) - dominating Set, (1,2) - domination Number, Middle Graph, Central Graph

1. Introduction

Domination in graphs has become an important area of research in graph theory,as evidenced by the many results contained in the two books by Haynes,Hedetniemi and Slater(1998)[6].Vernold Vivin J(2010) have studied the harmonious coloring of line graph, middle graph, central graphs of certain special graphs[12].Venketakrishnan and Swaminathan (2010)[15] have studied the colorclass domination number of middle graph and central graphs of $K_{1,n}$, C_n and P_n. In this paper we discuss (1,2)-domination in the middle and central graphs of $K_{1,n}$, C_n and P_n.

By a graph $G=(V,E)$ we mean a finite ,undirected graph without loops or multiple edges. A subset D of V is a *dominating set* of G if every vertex of $V-D$ is adjacent to a vertex of D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

A (1,2) - dominating set in a graph G is a set S having the property that for every vertex v in $V-S$ there is atleast one vertex in S at distance 1 from v and a second vertex in S at distance atmost 2 from v. The order of the smallest (1,2) - dominating set of G is called the *(1,2) - domination number* of G denoted by $\gamma_{(1,2)}$.

For a given graph $G=(V,E)$ of order n,the *central graph* $C(G)$ is obtained, by subdividing each edge in E exactly once and joining all the nonadjacent vertices of G. The central graph $C(G)$ of a graph G is an example of a split graph, where a split grah is a graph whose vertex set V can be partitioned into two sets, V_1 and V_2, where every pair of vertices in V_1 are adjacent , and no two vertices in V_2 are adjacent.

The *middle graph* $M(G)$ of a graph G, is the graph whose vertex set is $V(G)\cup E(G)$ where two vertices are adjacent if and only if they are either adjacent edges of G or one is a vertex and the other is an edge incident with it. That is, two vertices x and y in the vertex set of $M(G)$ are adjacent in if x,y are in $E(G)$ and x,y are adjacent in G or x is in $V(G)$,y is in $E(G)$, and x is incident to y in G. The related ideas regarding these graphs can be seen in[3,12,13,14].

2. (1,2) - domination in Middle Graphs of $K_{1,n}$, C_n and P_n.

Theorem 2.1

For any star graph $K_{1,n}$, $\gamma\ [M(K_{1,n})]=n$.

Proof:

Let $V(K_{1,n})=\{v,v_1,v_2,....,v_n\}$ and $E(G)=\{e_1,e_2,......,e_n\}$ .By the definition of middle graph, we have $V[M(K_{1,n})]=\{v\}\cup\{e_i/1\le i\le n\}\cup\{v_i/1\le i\le n\}$ in which the vertices $e_1,e_2,....., e_n,v$ induces a clique of order $n+1$. Hence $\gamma[M(K_{1,n})]\ge n$. But $\{v_1,v_2,.....,v_n\}$ is an independent set and each e_i is adjacent to v_i.

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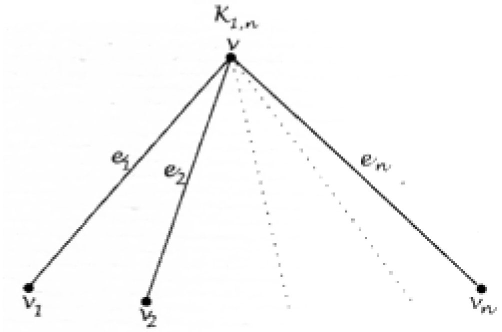
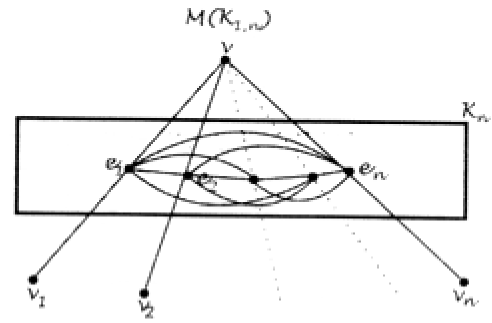
Therefore $\gamma[M(K_{1,n})] \le n$

Hence, γ $[M(K_{1,n})] = n$.



Star graph $K_{1,n}$



Middle graph of Star graph $K_{1,n}$

Theorem 2.2

For any star graph $K_{1,n}$, (1,2)- domination number $\gamma_{(1,2)}[M(K_{1,n})] = n$

Proof:

Let $V(K_{1,n}) = \{v, v_1, v_2, ..., v_n\}$ and $E(G) = \{e_1, e_2, ..., e_n\}$ .By the definition of middle graph, we have $V[M(K_{1,n})] = \{v\} \cup \{e_i / 1 \le i \le n\} \cup \{v_i / 1 \le i \le n\}$ in which the vertices $e_1, e_2, ..., e_n, v$ induces a clique of order $n+1$. In $M(K_{1,n})$ the vertex v is adjacent to $\{e_i / 1 \le i \le n\}$ and $\{v_1, v_2, ..., v_n\}$ is an independent set and each e_i is adjacent to v_i. So $\{e_i / 1 \le i \le n\}$ will form a (1,2)- dominating set. That is, $\gamma_{(1,2)}[M(K_{1,n})] \ge n$

Since { $e_1, e_2, ..., e_n, v$ } induces a clique, $\gamma_{(1,2)}$ $[M(K_{1,n})] \le n$.

Therefore $\gamma_{(1,2)}[M(K_{1,n})] = n$

Theorem 2.3

For any cycle C_n, γ [($M(C_n)$]= $\left\lceil \frac{n}{3} \right\rceil + 1$..

Proof:

Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, ..., e_n\}$ where $e_i = v_i v_{i+1} (1 \le i \le n-1), e_n = v_n v_1$. By the definition of middle graph , $M(C_n)$ has the vertex set $V(C_n) \cup E(C_n)$ in which each e_i is adjacent with $e_{i+1} (i = 1,2,....., n-1)$ and e_n is adjacent with v_1. In $M(C_n)$, $v_1, e_1, v_2, e_2, ..., e_{n-1}, v_1$ induces a cycle of length $2n$.

But we know that for $n \ge 3$, γ(C_n)= $\left\lceil \frac{n}{3} \right\rceil$(Theorem 2.1,[5]).

Thus it is clear that , γ [(M(C_n)] = $\left\lceil \frac{n}{3} \right\rceil + 1$..

Theorem 2.4

For any cycle C_n, $\gamma_{(1,2)}$[$M(C_n)$] = $\left\lceil \frac{n}{3} \right\rceil + 1$.

Proof:

Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, ..., e_n\}$ where $e_i = v_i v_{i+1} (1 \le i \le n-1), e_n = v_n v_1$. By the definition of middle graph , $M(C_n)$ has the vertex set $V(C_n) \cup E(C_n)$ in which each e_i is adjacent with $e_{i+1} (i = 1,2,....., n-1)$ and e_n is adjacent with v_1. In $M(C_n)$, $v_1, e_1, v_2, e_2, ..., e_{n-1}, v_1$ induces a cycle of length $2n$

We have by theorem 3.1 in[10], for any cycle Cn, γ(1,2)(Cn) = $\left\lfloor \frac{(n+2)}{3} \right\rfloor$. Hence $\gamma_{(1,2)}$ [$M(C_n)$] = $\left\lceil \frac{n}{3} \right\rceil + 1$.

Theorem 2.5

For any path $P_n, \gamma(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil$.

Proof:

Let $P_n : v_1, v_2, v_3, ..., v_{n+1}$ be a path of length n and let $v_i v_{i+1} = e_i$. By the definition of middle graph, $M(P_n)$ has the vertex set $V(P_n) \cup E(P_n) = \{v_i / 1 \le i \le n+1\} \cup \{e_i / 1 \le i \le n\}$ in which each v_i is adjacent to e_i and e_i is adjacent to v_{i+1}. Also e_i is adjacent to e_{i+1}.

Case 1: $|V(P_n)| = 2k$

Then. The vertices $v_1, e_1, v_2, e_2, ..., e_{2k-1}, v_k$ induces a path of length $4k$.

Case 2: $|V(P_n)| = 2k+1$

Then $|V(M(P_n))| = 4k+1$. The vertices $v_1, e_1, v_2, e_2, ..., e_{2k}, v_{2k+1}$ induces a path of length $4k$.

In both the cases, $M(P_n)$ is a path of length $4k$. That is, $n = 4k$.

But we have for $n \ge 3$, $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$ (Theorem 2.1,[5]). Therefore, $\gamma(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2.6

For any path $P_n, \gamma_{(1,2)}(M(P_n)) = \left\lceil \frac{2n}{3} \right\rceil$.

Proof:

Let $P_n : v_1, v_2, v_3, ..., v_{n+1}$ be a path of length n and let $v_i v_{i+1} = e_i$.

By the definition of middle graph $M(P_n)$ has the vertex set $V(P_n) \cup E(P_n) = \{v_i / 1 \le i \le n+1\} \cup \{e_i / 1 \le i \le n\}$ in which each v_i is adjacent to e_i and e_i is adjacent to v_{i+1}. Also e_i is adjacent to e_{i+1}.

Case 1: $|V(P_n)| = 2k$

Then $|V(M(P_n))| = 4k-1$. The vertices $v_1, e_1, v_2, e_2, ..., e_{2k-1}, v_k$ induces a path of length $4k$.

Case 2: $|V(P_n)| = 2k+1$

Then $|V(M(P_n))| = 4k+1$. The vertices $v_1, e_1, v_2, e_2, ..., e_{2k}, v_{2k+1}$ induces a path of length $4k$.

In both the cases, $M(P_n)$ is a path of length $4k$. That is, $n = 4k$.

Then by theorem 2.1 in[10] we have $\gamma_{(1,2)}(M(P_n)) = \left\lceil \frac{2n}{3} \right\rceil$

3. (1,2) - domination in Central Graph of $K_{1,n}$, C_n and P_n.

3.1. (1,2)-domination in $C(K_{1,n})$

Theorem 3.1

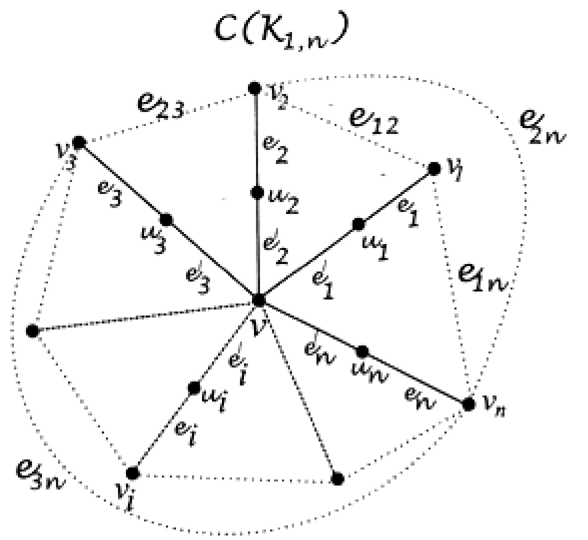
For any star graph $K_{1,n}$, $\gamma(C(K_{1,n})) = 2$

Proof:

Let $V(K_{1,n}) = \{v, v_1, v_2, ..., v_n\}$ where deg $v = n$. By the definition of central graph of $K_{1,n}$ we denote the vertices of subdivision by $u_1, u_2, ..., u_n$. That is, vv_i is subdivided by $u_i (1 \le i \le n)$. Let $e_i = v_i u_i$ and $e_i^{'} = vu_i (1 \le i \le n)$. Therefore $V(C(K_{1,n})) = \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \cup \{v\}$. By the definition of central graph the subgraph induced by the vertex set $\{v_1, v_2, ..., v_n\}$ is K_n and let e_{ij} be the edge of $C(K_{1,n})$, connecting the vertex v_i and v_j $(i < j)$.

$$E(C(K_{1,n})) = \{e_i : 1 \le i \le n\} \cup \{e_i^{'} : 1 \le i \le n\}$$
$$\cup \{e_{ij} : 1 \le i \le n-1, i+1 \le j \le n\}$$

Since in the central graph of a star, the central vertex v together with any one of v_i's form a dominating set. Therefore we have $\gamma(C(K_{1,n})) = 2$.



Central Graph of Star Graph $K_{1,n}$

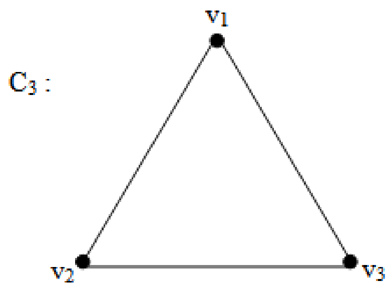
Theorem 3.2

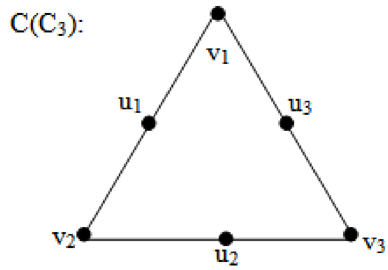
For any star graph $K_{1,n}$, $\gamma_{(1,2)}(C(K_{1,n})) = 2$

Proof:

Let $V(K_{1,n}) = \{v, v_1, v_2, ..., v_n\}$ where deg v = n. By the definition of central graph of $K_{1,n}$ we denote the vertices of subdivision by $u_1, u_2, ..., u_n$. That is, vv_i is subdivided by $u_i (1 \le i \le n)$. Let $e_i = v_i u_i$ and $e_i^{'} = vu_i (1 \le i \le n)$. Therefore $V(C(K_{1,n})) = \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \cup \{v\}$. By the definition of central graph the subgraph induced by the vertex set $\{v_1, v_2, ..., v_n\}$ is Kn and let eij be the edge of $C(K_{1,n})$, connecting the vertex vi and vj $(i < j)$. Then $E(C(K_{1,n})) = \{e_i : 1 \le i \le n\}$ $\cup \{e_i^{'} : 1 \le i \le n\} \cup \{e_{ij} : 1 \le i \le n-1, i+1 \le j \le n\}$. Since in the central graph of a star, the central vertex of the star is adjacent to every vertex in the central graph the vertex v together with any one of the vetices $u_i (1 \le i \le n)$ will form a {(1,2)-dominating set. Hence $\gamma_{(1,2)}(C(K_{1,n})) = 2$.

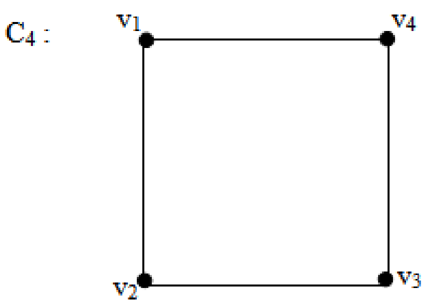
3.2. (1,2)-domination in $C(C_n)$: Consider the Following Examples

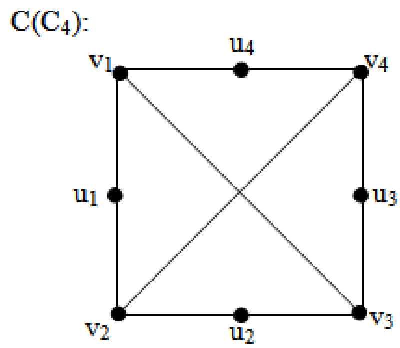




{v1,u2} is a dominating set and also (1,2)-dominating set.

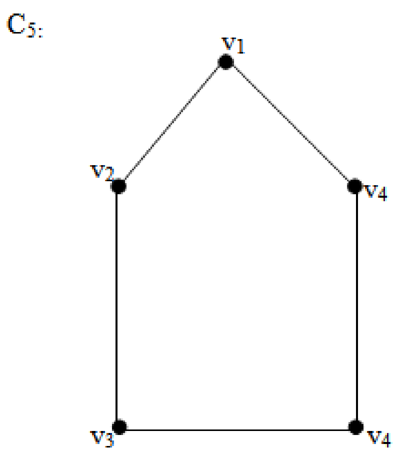
$\gamma(C(C_3)) = 2 = \gamma_{(1,2)}(C(C_3)$

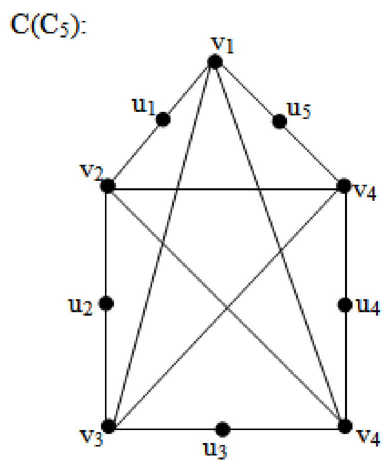




{v1,u2,u3} is a (1,2)-dominating set.

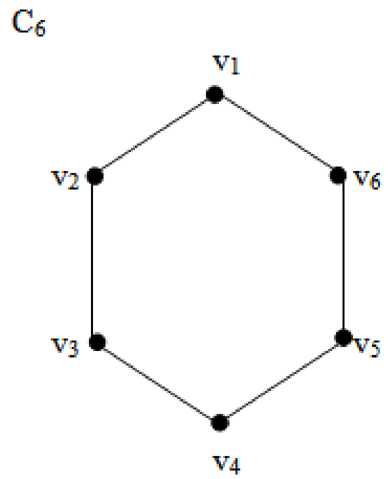
$\gamma(C(C_4)) = 3 = \gamma_{(1,2)}(C(C_4)$

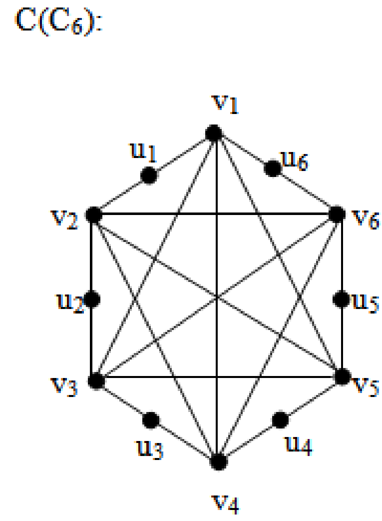




{v1,u2,u3,u4} is a (1,2)-dominating set.

$\gamma(C(C_5)) = 4 = \gamma_{(1,2)}(C(C_5)$





$\{v_1,u_2,u_3,u_4,u_5\}$ is a (1,2)-dominating set.

$$\gamma(C(C_6)) = 5 = \gamma_{(1,2)}(C(C_6))$$

Theorem 3.3

For any cycle C_n, $\gamma(C(C_n)) = n-1$

Proof:

Let C_n be any cycle of length n and let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, ..., e_n\}$.

By the definition of central graph $C(C_n)$ has the vertex set $V(C_n) \cup \{u_i : 1 \le i \le n\}$ where u_i is a vertex of subdivision of the edge v_iv_{i+1} $(1 \le i \le n-1)$ and u_n is a vertex of subdivision of the edge v_nv_1. In $C(C_n)$ we can note that the vertex v_i is adjacent with all vertices except the vertices v_{i+1} and v_{i-1} for $1 \le i \le n-1$. v_1 is adjacent with all vertices except v_{n-1} and v_1. The total number of edges incident with v_i is $(n-1)$ for every

i = 1,2,…,n and $\{u_i, (1 \le i \le n)\}$ is an independent set. So $\{v_i\} \cup \{u_i / 2 \le i \le n-1\}$

will be a dominating set. So the dominating set will consist of (n-1) vertices.

Hence $\gamma(C(C_n)) = n-1$

Theorem 3.4

For any cycle C_n, $\gamma_{(1,2)}(C(C_n)) = n-1$

Proof:

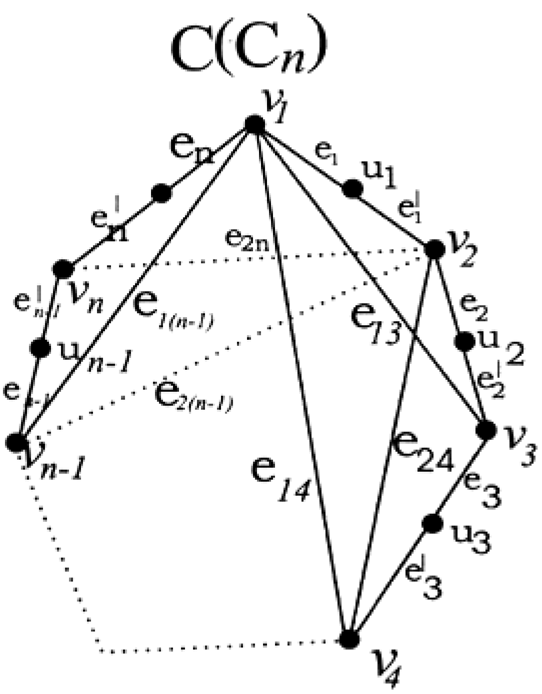
Let C_n be any cycle of length n and let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, ..., e_n\}$.

By the definition of central graph $C(C_n)$ has the vertex set $V(C_n) \cup \{u_i : 1 \le i \le n\}$ where u_i is a vertex of subdivision of the edge v_iv_{i+1} $(1 \le i \le n-1)$ and u_n is a vertex of subdivision of the edge v_nv_1. In $C(C_n)$ we can note that the vertex v_i is adjacent with all vertices except the vertices v_{i+1} and v_{i-1} for $1 \le i \le n-1$. v_1 is adjacent with all vertices except v_{n-1} and v_1. The total number of edges incident with v_i is $(n-1)$ for every

i = 1,2,…,n and $\{u_i, (1 \le i \le n)\}$ is an independent set. So $\{v_i\} \cup \{u_i / 2 \le i \le n-1\}$

will be a dominating set. But every minimum cardinality dominating set is also a (1,2)-dominating set in $C(C_n)$

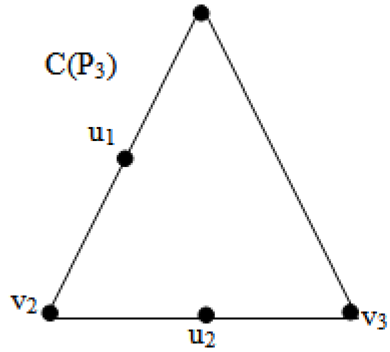
Hence $\gamma_{(1,2)}(C(C_n)) = n-1$



3.3. (1,2)-domination in $C(P_n)$

P_3 :

$C(P_3)$

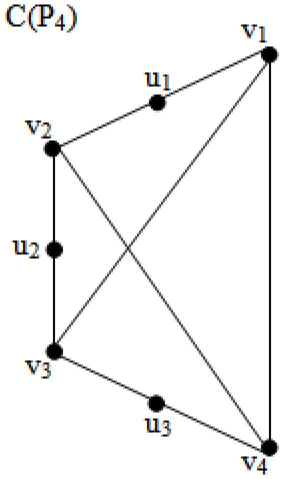


$v_1,u_2\}$ is a dominating set and also (1,2)-dominating set.

$$\gamma(C(P_3)) = 2 = \gamma_{(1,2)}(C(P_3)).$$

P_4 :

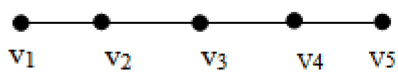
$C(P_4)$

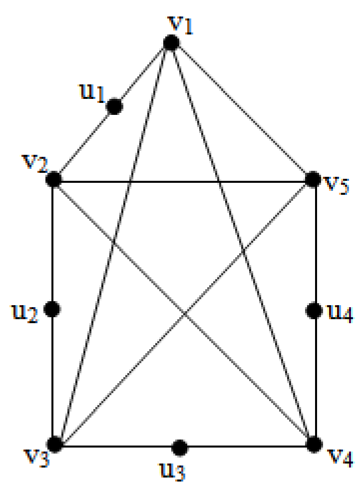


$\{v_1,u_2,u_3\}$ is a (1,2)-dominating set.

$$\gamma(C(P_4)) = 3 = \gamma_{(1,2)}(C(P_4))$$

P_5 :





$\{v_1,u_2,u_3,u_4\}$ is a (1,2)-dominating set.

$$\gamma(C(P_5)) = 4 = \gamma_{(1,2)}(C(P_5))$$

Theorem 3.5

For any path P_n $\gamma(C(P_n)) = n-1$

Proof.

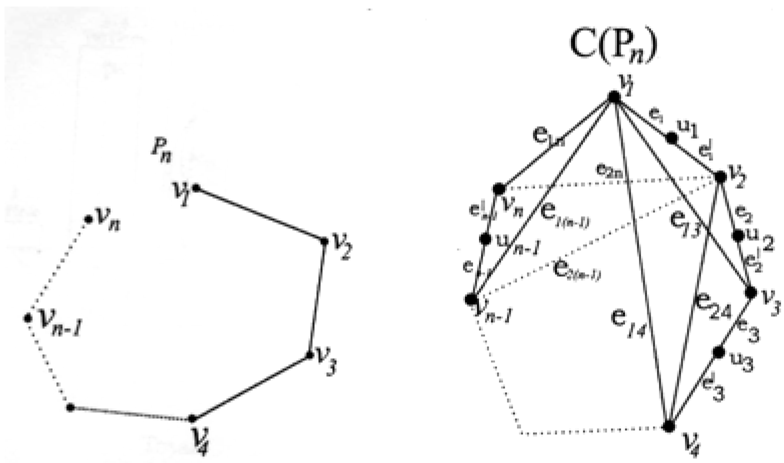
Let P_n be any path of length $n-1$ with vertices $v_1,v_2,..........,v_n$. On the process of centralisation of P_n, let u_i be the vertex of subdivision of the edges v_iv_{i-1} $(1 \le i \le n)$.

Also let $v_iu_i=e_i$ and $u_iv_{i+1}= e_i^{'}$ $(1 \le i \le n\text{-}1)$.

By the definition of central graph the non-adjacent vertices v_i *and* v_j *of* P_n are adjacent in $C(P_n)$ by the edge e_{ij}. Therefore $V(C(P_n)) = \{v_i/1 \le i \le n\} \cup \{u_i/1 \le i \le n\text{-}1\}$ and $E(C(P_n))=\{e_i :1\le i \le n\text{-}1\}\cup\{ e_i^{'} :1\le i\le n\text{-}1\} \cup\{e_{ij} : 1 \le i \le n -2, i+2 \le j \le n\}$. In $C(P_n)$ we can see that the vertex v_i is adjacent with all vertices except the vertices v_{i+1} and v_{i-1} for $1 \le i \le n\text{-}1$. v_1 is adjacent with all vertices except v_2. $\{ u_i, (1 \le i \le n)\}$ is an independent set So $\{ v_i\} \cup \{u_i / 2 \le i \le n-1\}$. So $\{ v_i\} \cup \{u_i / 2 \le i \le n-1\}$

will be a dominating set.

$\gamma(C(P_n))= n\text{-}1$.



Path P_n Central Graph of Path P_n

Theorem 3.6

For any path P_n, $\gamma_{(1,2)}((C(P_n))= n\text{-}1$.

Proof:

Let P_n be any path of length n-1 with vertices $v_1,v_2,\ldots\ldots,v_n$. On the process of centralisation of P_n, let u_i be the vertex of subdivision of the edges v_iv_{i-1} $(1 \le i \le n)$.

Also let $v_iu_i=e_i$ and $u_iv_{i+1}= e_i^{'}$ $(1 \le i \le n\text{-}1)$.

By the definition of central graph the non-adjacent vertices v_i and v_j of P_n are adjacent in $C(P_n)$ by the edge e_{ij}. Therefore $V(C(P_n)) = \{v_i/1 \le i \le n\} \cup \{u_i/1 \le i \le n\text{-}1\}$ and $E(C(P_n))=\{e_i :1\le i\le n\text{-}1\}\cup\{ e_i^{'} :1\le i\le n\text{-}1\} \cup \{e_{ij} : 1 \le i \le n -2, i+2 \le j \le n\}$. In $C(P_n)$ we can see that the vertex v_i is adjacent with all vertices except the vertices v_{i+1} and v_{i-1} for $1 \le i \le n\text{-}1$. v_1 is adjacent with all vertices except v_2. $\{ u_i, (1 \le i \le n)\}$ is an independent set So $\{ v_i\} \cup \{u_i / 2 \le i \le n-1\}$. So $\{ v_i\} \cup \{u_i / 2 \le i \le n-1\}$

will be a dominating set.

$\gamma(C(P_n))= n\text{-}1$.

But every minimum cardinality dominating set of the central graph of a path is also a (1,2)- dominating set. Hence $\gamma_{(1,2)}(C(P_n)) = n\text{-}1$.

4. Relation between Domination Number and (1,2)-domination Number in Middle and Central Graph of Stars, Cycles and Paths

Theorem 4.1

In the middle graph of a star, $[M(K_{1,n})]$, the domination number equals the (1,2)-domination number.

Proof:

This result is obvious from theorem 2.1 and 2.2.

Theorem 4.2

In the middle graph of cycles, $[M(C_n)]$, the domination number equals the (1,2)-domination number.

Proof:

This result is obtained by theorem 2.3 and theorem 2.4.

Theorem 4.3

In the middle graph of paths, $M(P_n)$, the domination number is less than or equal to the (1,2)- domination number.

Proof:

This result is obtained by theorem 2.5 and theorem 2.6.

Theorem 4.4

In the central graph of any star, $C(K_{1,n})$, the domination number equals the (1,2)-domination number.

Proof:

This is clear from theorem 3.1 and 3.2.

Theorem 4.5

In the central graph of cycles, $C(C_n)$, the domination number equals the (1,2)- domination number.

Proof:

This result is due to the theorem 3.3 and 3.4.

Theorem 4.6

In the central graph of paths, $C(P_n)$, the domination number equals the (1,2)- domination number.

Proof:

This result is obtained from theorem 3.5 and 3.6.

5. Conclusions

In this paper we have extended (1,2)- domination to the middle graph and central graph of stars, cycles, and paths and discussed both domination and (1,2)- domination number of these graphs. In all cases it is important to see that the domination number is less than or equal to the (1,2)- domination which coincides the result established in[8].

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