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HIV/AIDS Prevalence and Transmission Risks from Mother to Child during the Pregnancy Process in Sub-saharan Africa

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Abstract This paper investigates the life dynamics properties of pregnant HIV/AIDS infected women with a joint utility function expressing both the mother and her future baby health states during the pregnancy process in Sub-Saharan Africa where medical-care shortages and delays to learn about the infection exist. Non separability of the utility function makes life dynamics oscillate caused by HIV/AIDS prevalence which generates transmission risks of the pandemic to the baby and presents cycles as well as hopf bifurcation while life fluctuates endogenously over time. Applying those results to the Cobb Douglas utility and production functions we find that optimal life dynamics existence and stability depend on illness gravity and medical care investment interaction.

Keywords HIV/AIDS, Transmission Risk, Infection Gravity, Medical Care, Oscillatory Dynamics, Optimal Path

1. Introduction

This article develops a theoretical model focused on HIV/AIDS acquisition and prevalence in order to establish optimal life dynamics stability when HIV/AIDS transmission risks from mother to child during the pregnancy process exists in Sub-Saharan Africa because of difficulties to fight the pandemic in the area. The analysis deals both with infected and non infected pregnant women where the first deals with optimal growth methods of investigation and the second with oscillatory dynamics methods in order to establish optimal dynamic paths existence and stability since a healthy pregnant woman at her pregnancy beginning time may came out to be infected along her birth gift process. The application of the theory to the Cobb Douglas function highlights a suitable prevention method which shows that optimal life dynamics existence and stability depend on illness gravity and medical care investment interaction.

The scientific contribution of this article holds on a unified study of optimal and oscillatory dynamical systems tools in the same growth model applied to the establishment of a steady state in lives dynamics target. Whereas most of the study based on HIV/AIDS in growth models focus on its impact on GDP, human capital accumulation of orphans and labor productivity, this article looks for lives preservation for population growth stability. Finally, the existence of transmission risk possibility of the pandemic along the birth process may create optimal or oscillatory dynamics in Sub-Saharan Africa, thus cycles and hopf bifurcation may appear because the utility function is no more additively separable and no more ensures optimal path existence and stability.

The human immunodeficiency virus (HIV) pandemic is one of the most serious health crises the world is facing today. AIDS has killed more than 25 million people since 1981 and an estimated 38.6 million people are now living with HIV, about 2.3 million of them are children[1]. Since 1999, primarily as a result of HIV, average life expectancy has declined in 38 countries. In the most severely affected countries, average life expectancy is now 49 years i.e 13 years less than in the absence of AIDS[2]. A disproportionate burden has been placed on women and children, who in many settings continue to experience high rates of new HIV infections and of HIV-related illness and death. In 2005 alone, an estimated 540 000 children were newly infected with HIV, with about 90% of these infections occurring in Sub-Saharan Africa. Most children living with HIV acquire the infection through mother-to-child transmission (MTCT), which can occur during pregnancy, labor and delivery or during breastfeeding. In the absence of any intervention the risk of such transmission is 15–30% in non-breastfeeding populations. Breastfeeding by an infected mother increases the risk by 5–20% to a total of 20–45%[3]. The risk of MTCT can be reduced to under 2% by interventions that include antiretroviral (ARV) prophylaxis given to women during pregnancy and labor and to the infant in the first weeks of life, obstetrical interventions including elective

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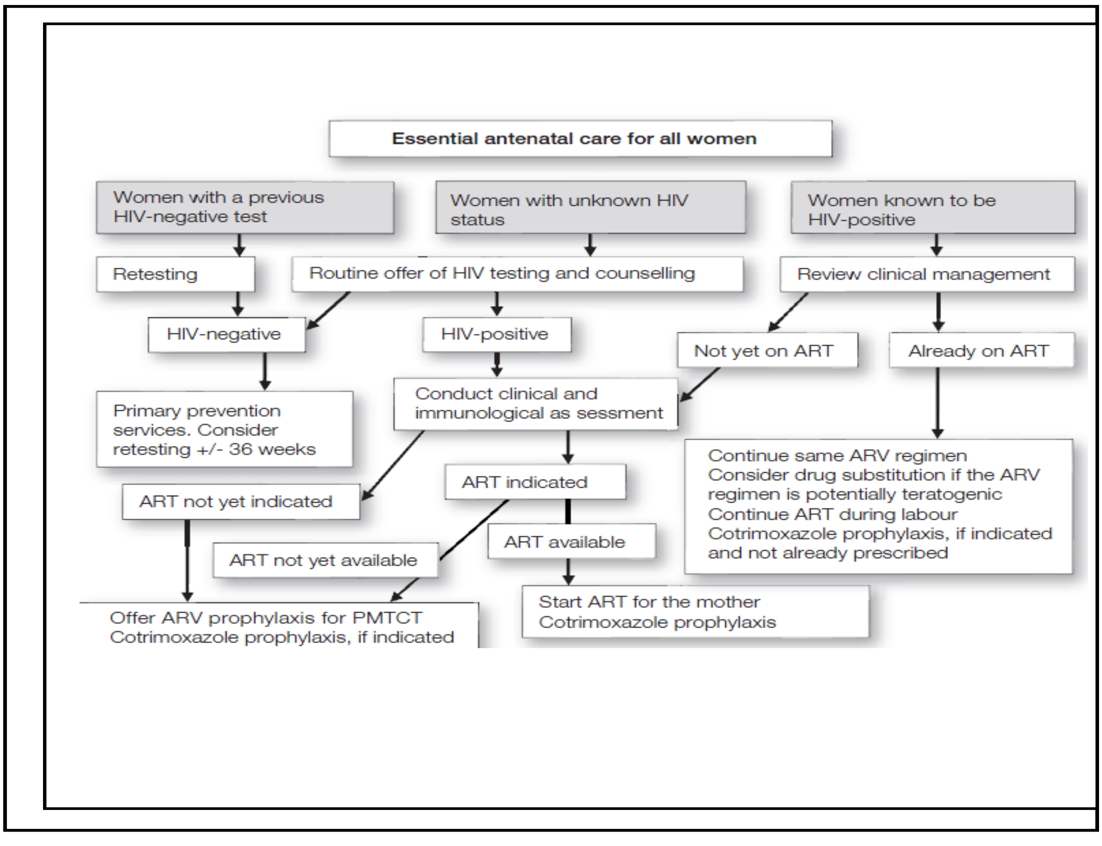
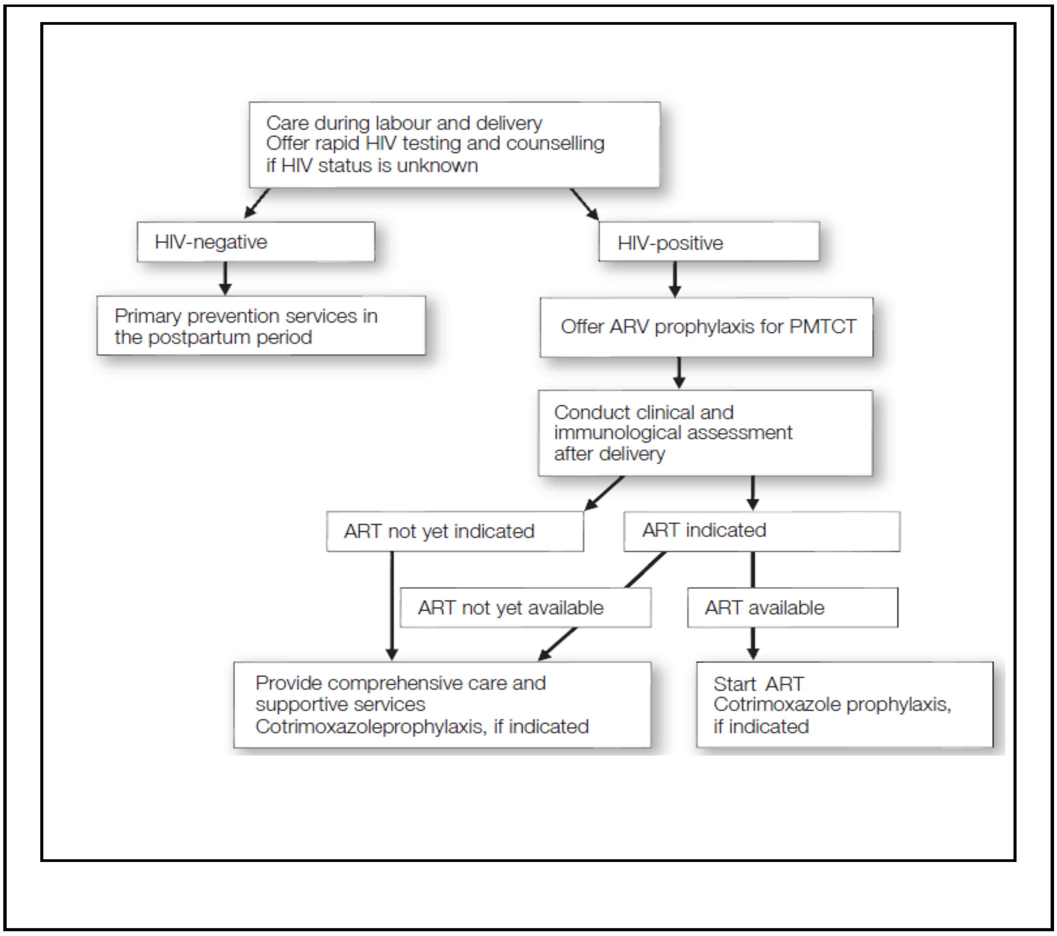
caesarean delivery (prior to the onset of labor and rupture of membranes), and complete avoidance ofbreastfeeding [4]-[6]. With these interventions, new HIV infections in children are becoming increasingly rare in many parts of the world, particularly in high-income countries where research has focused on more complex regimens and has shown that triple-ARV combinations given to women during pregnancy and labor can reduce the risk of transmission to under 2% [7]-[9]. These regimens are discontinued after childbirth for women without indications for ART (antiretroviral therapy). Since around 1998[10], triple-ARV combinations have increasingly been used to prevent MTCT; currently the majority of pregnant women living with HIV in Europe and North America receive such regimens[11]. In these settings and without breastfeeding, HIV infection in infants has been nearly eliminated. In contrast, Sub-Saharan Africa accounted for 69% of all new infections and nearly half of all HIV-related deaths globally. While 8 million people worldwide can access treatment, nearly 7 million additional people in need of access did not have it as of 2011. Moreover, for every one person on treatment, two are infected. Without effective HIV prevention, the number of people requiring treatment will become unsustainable. Despite the increase in AIDS funding during the past decade, financing gaps persist (International assistance declined from $7.6 billion in 2009 to $6.9 billion in 2010.), and the bulk of likely available funds is unpredictable and mainly for treatment. As new infections rise, country and donor investments in prevention are not being sustained. Efforts are actually targeted for prevention of mother-to-child transmission of HIV, in helping countries accelerate progress on maternal and child health, in line with the 2015 Millennium Development Goals (World Bank Weekly update, November 28, 2012). ARV Drugs treats Pregnant Women and Prevent HIV Infection in Infants are consistent and the Call to Action towards an HIV free and AIDS-free Generation explain the need to study the impact of HIV/AIDS for the case of MTCT in order to develop prevention tools in Sub-Saharan Africa. Therefore, this analysis models the situations of ARV Drugs for Treating Pregnant Women and Preventing HIV infection in Infants to support, the Call to Action Towards an HIV free and AIDS-free Generation. Indeed, the classification made applies pregnant women infected by HIV/AIDS virus investigation methods of oscillating optimal paths where the solution may not exist in growth literature. In the concern of pregnant women where infection is free, optimal stable path can be found because of HIV/AIDS absence. When the mother is infected by HIV/AIDS and may transmit it to the future baby, optimal dynamics oscillates and presents both cycles and Hopf bifurcation at critical points. In that case, the social planner uses the utility function which is no more additively separable but jointly expressed explaining more cares on lives preservation since they are no more submitted to natural birth process (see figures 1 and 2).



MTCT: mother to child transmission
ART: antiretroviral therapy

Figure 1. Comprehensive for the prevention of MTCT: women seen during pregnancy



MTCT: mother to child transmission
ART: antiretroviral therapy

Figure 2. Comprehensive for the prevention of MTCT: women seen during labour

Since Ramsey[12], dynamic growth models have been built in order to provide precise conditions which ensure convergence of optimal paths to the long-run equilibrium. Thus, the aggregate growth model analyzed by Cass and Koopmans[13]-[14] form the basis for much of the intuition economists have about long-run growth through precise criteria established using suitable assumptions. In standard one sector dynamic models since the representative agent's utility function is concave, the optimal path is monotone and converges to the steady state for all given production function. But when the central planner takes account of the life cycle of each agent in the economy, preferences have to be additively separable[15]-[16]. Because the main problem due to dynamic optimizing model is related to the existence of a social optimum which may lead to the non finiteness of objective functions when maximizing a discounted sum of integral over an infinite horizon raising the possibility that feasible exogenous variable paths grow so fast that the objective function is not finite and the optimum fail to exist. Conditions that ensure convergence of optimal path to the long run equilibrium generally depend on discount factor and concavity properties of utility and production function [17]-[18] Without the acceptance of the quoted properties conditions, dynamic growth models may oscillate and in some circumstances present cycles[19]-[20]. Indeterminacy theorems provided by Boldrin and Montrucchio[21] and Sorger[22] show that virtually every dynamics is compatible with the usual assumptions of concavity, perfect competition and perfect foresight. When the utility function of the representative agent is not additively separable, optimal paths don't necessarily converge to the steady state and appears that the optimal solution oscillates and presents cycles since the cross derivative of the indirect utility function is negative[23].

In parallel, the macroeconomics effects of HIV/AIDS on growth can be summarized in three approaches, the ‘first approach doesn’t incorporate the effects of ARV (anti retroviral drugs) provision on life expectancy studies. Those macroeconomic models show little decrease of HIV/AIDS impact on growth[24]-[27] or even an increase in *per capita* GDP growth because HIV-related declines in GDP are offset by increased mortality and population decline[28]. The second approach incorporates the effects of ARV provision on life expectancy and emphasizes different mechanics, that of reduced fertility in response to the epidemic[29]-[38]. However, the main problem with efforts to model the macroeconomic effects of HIV/AIDS in Africa is that projections hinge crucially on assumptions made about micro-level behavioral responses of individuals which leads to the tendency of micro studies of HIV/AIDS mostly provided[39]-[40]. The third approach of HIV/AIDS studies the way it can be assimilated to a perfect foresight dynamics[41]-[47]. Goldstein and al[48] show that pregnant women whose first clinic visit coincides with the nurse's attendance are 58 percentage points more likely to test for HIV and 46 percent more likely to deliver in a hospital. Furthermore, women with high pretest expectations of being HIV positive, whose visit coincides with nurse attendance, are 25 and 7.4 percentage points more likely to deliver in a hospital and receive PMTCT (prevention of mother to child transmission) medication and 9 percentage points less likely to breast-feed than women whose visit coincides with nurse absence. The shortcomings that prevent pregnant women from testing on a subsequent visit are common in sub-Saharan Africa. *In contrast, our article presents a theoretical framework dealing with HIV/AIDS transmission risks from mother to child during the pregnancy process in order to look for required conditions for optimal steady state in life terms*. The article presents theoretical tools first and applies them to the concrete case of infection risks for the baby. Medical care and economic growth expressed by the discount factor conjugate to establish optimal paths. We find that optimal dynamics depend on illness gravity and medical care investment interaction.

The article is based on Michel and Venditti[49] on the one hand and on Groenka and Liu[50] on the other hand. It is organized like follow, section 2 presents the theory, section 3 applies the theory to specific functions and section 4 concludes.

2. The Theoretical Model

This article presents the required tools to study optimal growth and oscillatory dynamics. The first case refers to the healthy future mothers for whom life gift doesn't fluctuate in contrast to the second case where life has uncertainty character in his gift activity because of HIV/AIDS disease and medical care shortages. We want to understand the way the baby's life fluctuates endogenously according to his mother's health state in an economic environment where prevail health care shortages and delays to learn about the infection prevalence in Sub-Saharan Africa.

2.1. The Additively Separable Case: the Non Infected Future Mothers

This first part presents the theory in the case of *per-capita* non infected pregnant woman by HIV/AIDS virus denoted s_t^{mh} whose womb is carrying a baby, s_{t+1}^{ch}. The main purpose of the analysis is to present useful tools which allow the study of how the mother's organism can be prevent from the HIV/AIDS virus infection and its transmission to the baby during the pregnancy process. The control is done in order to know if changes outcome i.e if the mother turns out to be infected at a given time during the pregnancy process, then tools control used change to slow the infection gravity in such a way that the pregnancy process reaches its end and ensure that the new born will be healthy and live the required time which is two periods. The pregnancy process takes place during the first period before the birth of the baby who born at the end of that first period and is young during the second period when his mother dies at the end of the second period.

The instantaneous utility function $u_t^h\left(s_t^{mh}, s_{t+1}^h\right)$ is additively separable in the non HIV/AIDS infection case. The healthy parameters are indexed by h whereas the mother and the child parameters are respectively indexed by m and c.

Assumption1: the instantaneous additively separable utility function $u_t^h\left(s_t^{mh}, s_{t+1}^{ch}\right) = v_t^h\left(s_t^{mh}\right) + w_t^h\left(s_{t+1}^{ch}\right)$ is strictly increasing with respect to each argument i.e $u_1^h \succ 0$ and $u_2^h \succ 0$, concave i.e $u_{11}^h < 0$ and $u_{22}^h < 0$ and C^2 over the interior of the set N_t^h (total stock of non infected pregnant women) where $u_t^h\left(s_t^{mh}, 0\right) = u_t^h\left(0, s_{t+1}^{ch}\right) = \infty$

Optimal strategy for the couple of per-capita mother and her future baby combines the utilities of successive generations in order to guarantee life along the time periods such that:

$$\underset{\left(s_t^{mh}, s_{t+1}^{ch}\right)}{Max} \left\{ \sum_{t=0}^{\infty} \delta^t N_t^h \left[v_t^h\left(s_t^{mh}\right) + w_t^h\left(s_{t+1}^{ch}\right) \right] \right\} \quad (2.1)$$

Where δ Є[0,1] is the discount factor, s_0^{mh} is a given healthy pregnant woman at initial period $t \geq 0$.

The non infected pregnant women stock may vary along the pregnancy process in the way that, they may be infected by their partners before having given birth i.e before the end of the pregnancy process. There thus exists an infection risk parameter $\gamma_t^j \succ 0$ where $j=m,c$ which may make lives vary and alter health through possible increase of infection's gravity and attains the future baby's health and life expectancy because of delays to learn about the illness and the gravity generated.

The concavity property of the additively instantaneous utility function holds on the variables (s_t^{mh}, s_{t+1}^{ch}), per-capita optimal program becomes:

$$\underset{\left(s_t^{mh}, s_{t+1}^{ch}\right)}{Max} \left\{ \sum_{t=0}^{\infty} \delta^t \left[v_t^h\left(s_t^{mh}\right) + w_t^h\left(s_{t+1}^{ch}\right) \right] \right\} \quad (2.2)$$

Note that, since $\gamma^m > 0$, then as we'll see later on that the instantaneous utility function is no more additively separable but turns out to be joint and is thus out of this case study.

Healthcare is ensured by a stock of investment K_{t+1}^h (which may be given by the donors in cooperation with the poor country government). The production of healthcare provided by the social planner is a neoclassical production function $Y_t^h = F\left(K_t^h, N_t^h\right)$ with respect to the following assumption:

Assumption 2: The function $Y_t^h = F\left(K_t^h, N_t^h\right)$ is C^2

and homogenous of degree 1 with respect to capital stock of drugs produced for the use of, the mother and the future baby's health states include in the interior of the set N_t^{h2}. $\subset R^2$ Moreover, $F_1(K_t^h, N_t^h) \succ 0$, $F_{11}(K_t^h, N_t^h) < 0$ and capital fully depreciates on use along the pregnancy care provision.

Assuming that capital fully depreciates on use, *per-capita* financial constraint in association to life process in the economy is expressed such that: $f(k_t^h) = k_{t+1}^h + s_t^{mh} + s_{t+1}^{ch}$ and per-capita stationary capital is determinate by $f'(k_t^{h*}) = 1 + r_t$ is the return and cost equilibrium relationship i.e $k_t^{h*} = f'^{-1}(1 + r_t)$. The expected lives care production function can be expressed in *per-capita* terms i.e $F(k_t^h, 1) = f(k_t^h)$ is *per-capita* healthcare production provided by *per-capita* investment $k_t^h = K_t^h / N_t^h$ devoted to both the mother and the forthcoming child health states. Thus growth in life expectancy gain is such that

$$g^h = \frac{s^{ch*}}{s^{m\,h*}} - 1 \tag{2.3}$$

Proposition1: Let γ^m be the mother's HIV/AIDS infection risk parameter and γ^c be the HIV/AIDS transmission risk from the mother to the child parameter, if $\gamma^m > 0$ then the additively separable utility function $u_t^h(s_t^{mh}, s_{t+1}^{ch})$ converges to the non separable utility function $u_t^i(s_t^{mi}, s_{t+1}^{ci})$

PROOF: if $\gamma^m > 0$ for a given s_t^{mh} then there exist $\gamma^c \geq 0$ associated to s_{t+1}^{ch} such that $s_t^{mh} = \rho(\gamma^m)$ and $s_{t+1}^{ch} = \rho(\gamma^c) s_t^{mh}$ therefore, $(s_t^{mh}, s_{t+1}^{ch}) = s_t^{mh}(1, \rho(\gamma^m))$. Indeed, it can't be find $(\lambda_i)_{i=1,2}$ such that $\lambda_1 s_t^{mh} + \lambda_2 s_t^{ch} = 0$ lead to $(\lambda_i)_{i=1,2} = 0$ for all i. Thus $u(\lambda_1 s_t^{mh} + \lambda_2 s_t^{ch}) \leq \lambda_1 u(s_t^{mh}) + \lambda_2 u(s_{t+1}^{ch}) = (\lambda_1 + \lambda_2 \rho(\gamma^c)) u(s_t^{mh}) \neq v(s_t^{mh}) + w(s_{t+1}^{ch})$. Indeed since $\gamma^m > 0$ for a given s_t^{mh}, then $t \geq 0$ we have: $\sum[v(s_t^{mh}) + w(s_{t+1}^{ch}).] \neq \sum[u(s_t^{mh}, s_t^{ch})]$ which yields to $u^h(s_t^{mh}, s_{t+1}^{ch})$ converges *to* $u^i(s_t^{mi}, s_{t+1}^{ci})$ since $\gamma^m > 0$ for a given s_t^{mh}

2.2. The Non-separable Case

In this case, *per-capita* pregnant woman is infected by the virus HIV/AIDS at the beginning time or during her pregnancy process and thus the deal becomes *first* avoid her death occurrence before the end of the pregnancy time. *Second*, because her health alteration degree may be higher than expected, medical affords are conducted in a joint way both for the mother and her baby knowing that the mother's health alteration impact on the future baby may be greater than expected due to HIV/AIDS prevalence and transmission susceptibility since medical care requirement is under the needs. Indeed, this case is studied using a non separable utility function and optimal steady state may not exist since the mother's health alteration caused by the pandemic is too great to be saved on time i.e has crossed critical bound allowing the human organism survive until birth outcomes. Thus life dynamics oscillates i.e is uncertain and presents cycles when the virus transmission to the baby is avoid or eventually be chaotic when it is not possible anymore to converge to the steady state and death occurrence is non reversible. Therefore, the goal is to look for necessary existence conditions of a stable equilibrium in life dynamics prevention both for the mother and the baby. In contrast to the previous case, now infections risks for the baby are greater without prevention and treatment. Both the analytical problem and the tools used are more complex and deal with oscillatory dynamics in order to establish optimality conditions at each step.

The departure point is financial support equation which remains almost the same than before but now variables are indexed with *i* (infection) instead of *h* (healthy). The mother and the child beneficiate from medical treatment supported by a non significant investment i.e $f(k_t^i) = k_{t+1}^i + s_t^{mi} + s_{t+1}^{ci}$ where $k_{t+1}^i \geq k_{t+1}^h$ because of more health state damages done by the pandemic inside the body's organism, indeed we have $-s_t^{mi} + a_t^i = k_{t+1}^i$ where $a_t^{ci} = -s_{t+1}^{ci} + f(k_t^i)$ is per-capita future baby inside the womb and means that financial funds support the baby and the mother's health, $f(k_t^i)$ is healthcare provision. Therefore in the next period, the baby's health evolution follows, $a_{+t+1}^{ci} = -s_{t+1}^{ci} + f(k_{t+1}^i) = f(-s_t^{mi} + a_t^i) - s_{t+1}^{ci}$ which means that the baby's stock grows since healthcare quality access is free. In our case, his life depends on his mother's health state in accordance to the virus transmission possibility during the pregnancy. The economic problem can be expressed such that

$$\underset{(s_t^{m\,i}, s_{t+1}^{cii})}{Max} \left\{ \sum_{t=0}^{\infty} \delta^t N_t^i \left[u_t^i \left(s_t^{mi}, s_{t+1}^{ci} \right) \right] \right\} \tag{2.4}$$

The instantaneous utility function $u_t^i(s_t^{mi}, s_{t+1}^{ci}) = u_t^i(s_t^{mi}(\gamma^m), s_{t+1}^{ci})(\gamma^c))$ is no more additively separable like in the previous case where the study made before determinate a stable optimal path. If $\delta = 1$, the objective is not properly defined because the sum of the infinite may not converge and Ramsey cannot applied.

Now it is easier to formulate the problem through the use of the indirect utility function i.e $V(a^i, b^i) = Max\{u^i(s^{mi}, s^{ci})\}$ constrained to $b^i = f(-s^{mi} + a^i) - s^{ci}$ where s^{ci} is per-capita future baby carried by a pregnant woman which medical care tries to protect in order to make it becoming a baby at the end of the period, $s_t^{ci} = s_{t+1}^{ci}$ is child alive at the beginning of the second period. Using the change of variable such that $x^i = a^i - s^{mi}$ it yields $f(x^i) - s^{ci} = b^i$, the indirect utility function becomes:

$$V(a^i, b^i) = Max\{u^i(-x^i + a^i, -b^i + f(x^i))\} \tag{2.5}$$

We can also consider the reduced form of the optimal growth model in defining a smaller life production possibility set. Let consider the non empty set E^i to be a subset of total pregnant HIV/AIDS infected women N^i such that $E^i=\{(a^i, b^i) / a^i\geq 0$ and $0\leq b^i\leq f(x^i)\}$. Since f is a continuous and a strictly concave function, E^i is a closed convex set. If $(a^i, b^i) \subset E^i$, then a^i is the input and b^i is the feasible output per young. There exist a level x^* such that if $0<x^i\leq x^*$ then $x^*\geq f(x^i)\geq x^i$. Under those assumptions, let us consider the set: $R^2 \supset E_1{}^i=\{(a^i, b^i) \in E^i / \; 0 \leq x^i \leq x^*\} \subset N^{i2}$, thus $E_1{}^i$ is a compact convex set.

Proposition2: If the instantaneous utility function $u^i(s^{mi}, s^{ci})$ and the production function $f(k^i)$ are continuous, increasing and concave, then the indirect utility function $V(a^i, b^i)$ is concave over $E_1{}^i$

The proof if given in [47]

Consider the indirect utility function, under the assumptions 1 and 2, the maximum is necessarily interior and is characterized by the first order condition i.e $U_1(x^{i*}, a^i, b^i)=0$ in the other words we have:

$$U_1^i\left(x^i,a^i,b^i\right)=-u_1^i\left(-x^i+a^i,-b^i+f\left(x^i\right)\right)$$
$$+f'\left(x^i\right)u_2^i\left(-x^i+a^i,-b^i+f\left(x^i\right)\right)$$
$$=0$$

To prove uniqueness of x^{i*} we need to study the concavity with respect to x^{i*} of the function $u^i(-x^i+a^i, b^i+f(x^i))$ whose derivative with respect to x^i is $U_1(x^i, a^i, b^i)$

The second derivative with respect to x^i is therefore:

$$U_{11}^i = u_{11}^i - 2f'u_{21}^i + \left(f'\right)^2 u_{22} + f''u_2^i = -A + f''u_2^i < 0$$

Where: $A = -(u_{11}^i - 2f'u_{21}^i + \left(f'\right)^2 u_{22}) > 0$

Because of the concavity of u and f then $U_{11}{}^i>0$ meaning that $u^i(-x^i+a^i, -b^i+f(x^i))$ function is not strictly concave with respect to x^i therefore, using the implicit function theorem through which we can express x^* as a function of a^i, b^i i.e $x^{i*}(a^i, b^i)$ such that:

$$\frac{\partial x^{i*}}{\partial a^i} = -\frac{U_{12}\left(x^{i*},a^i,b^i\right)}{U_{11}\left(x^{i*},a^i,b^i\right)} \text{ and } \frac{\partial x^{i*}}{\partial b^i} = -\frac{U_{13}\left(x^{i*},a^i,b^i\right)}{U_{11}\left(x^{i*},a^i,b^i\right)}$$

Which expresses analytically such that:

$$\frac{\partial x^{i*}}{\partial a^i} = \frac{u_{11}^i - f'u_{21}^i}{A - f''u_2^i} = \frac{B}{A - f''u_2^i}$$
$$\text{where } B = -\left(u_{11}^i - f'u_{21}^i\right) \geq 0$$
$$\frac{\partial x^{i*}}{\partial b^i} = \frac{-u_{12}^i + f'u_{21}^i}{A - f''u_2^i} = \frac{C}{A - f''u_2^i}$$
$$\text{where } C = -\left(-u_{12}^i + f'u_{21}^i\right) \geq 0$$

To fully determinate the concavity of the indirect utility function $V\left(a^i,b^i\right) = u^i\left(-x^i+a^i,-b^i+f\left(x^i\right)\right)$ we may compute its second derivatives and use the *Shepard lemma* through which we obtain $V_1 = u_1^i\left(-x^i+a^i,-b^i+f\left(x^i\right)\right)$ and

$$V_2 = -u_2^i\left(-x^i+a^i,-b^i+f\left(x^i\right)\right)$$

therfore V is a C^2 function and satisfies:

$$V_{11} = u_{11}^i + \frac{B^2}{A - f''u_2^i} \quad ; V_{12} = -u_{12}^i + \frac{BC}{A - f''u_2^i}$$

$$V_{22} = u_{22}^i + \frac{C^2}{A - f''u_2^i}$$

Concavity of V implies $V_{11}\leq 0$ and $V_{22}\leq 0$ and is verified

Therefore we clarified using the implicit function theorem that a unique x^{i*} exist and can be written such that $x^{i*}=x^{i*}(a^{i*},b^{i*})$, now we need to establish its stability.

2.3. Study of Local Stability of the Steady State in a Non separable Case

As in the standard one sector optimal growth models, the stationary capital stock is given by the modified golden rule and doesn't depend on the agents' utility function. Therefore steady state can be expressed such that:

$$1 + r^* = f'\left(k^{i*}\right) = \delta^{-1} \quad (2.5)$$

Because capital fully depreciates in one period and δ is the discount factor, stationary mother and baby born stocks are expressed by the following conditions:

$$s^{mi} + s^{ci} = -k^{i*} + f\left(k^{i*}\right) \quad (2.6)$$

$$u_1^i\left(s^{mi},s^{ci}\right) = \left(1+r^*\right)u_2^i\left(s^{mi},s^{ci}\right) \quad (2.7)$$

Depending on the sign of the indirect utility function's cross derivative V_{12} the sustainable dynamics may be monotone or oscillating.

Proposition3: let $\left\{\left(a_t^i\right)_{t\geq 0}\right\}$ be an optimal sustainable path and let $\left(a_t^i, a_{t+1}^i\right) \in \mathring{E}_1$ where a_{t+1}^i is per-capita baby or new born

If $V\left(a^i,b^i\right) \geq 0 \;\; \forall \left(a^i,b^i\right) \in \mathring{E}_1$, then the optimal dynamics is monotone

If $V\left(a^i,b^i\right) < 0 \;\; \forall \left(a^i,b^i\right) \in \mathring{E}_1$, then the optimal dynamics is oscillating

The sign of V_{12} must be evaluated around the steady state neighborhood (a^{i*}, a^{i*}). In our framework, we obtain:

Lemma1: $V\left(a^i,b^i\right) \geq 0 \;\; \forall \left(a^i,b^i\right) \in \mathring{E}_1$ if and only if $-f'H^i - f''u_2^i u_{12}^i \leq 0$ where $H^i = u_{22}^i u_{11}^{-} - \left(u_{12}^i\right)^2 \geq 0$ is the determinant of the Hessian matrix of u^i evaluated at (s^{mi}, s^{ci})

PROOF

$$V_{12} \geq 0 \Leftrightarrow -u_{12}^i + \frac{BC}{A - f''u_2^i} \geq 0$$
$$\Leftrightarrow -u_{12}^i\left(A - f''u_2^i\right) + BC \geq 0$$

$$u_{12}^i\left[\left(-u_{11}^i+2f'u_{12}^i-(f')^2u_{22}^i-\right)f''u_2^i\right]$$
$$\leq\left(-u_{11}^i+f'u_{12}^i\right)\left(u_{12}^i-f''u_2^i\right)$$
$$\Leftrightarrow -f'\left(u_{11}^iu_{22}^i-\left(u_{12}^i\right)^2\right)-f''u_2^iu_{12}^i\leq 0$$
$$\text{where } H^i=u_{11}^iu_{22}^i-\left(u_{12}^i\right)^2\geq 0$$

is the determinant of the Hessian matrix of u^i evaluated at (s^{mi}, s^{ci}) and $f'=\delta^{-1}$

The sign of the instantaneous utility function's cross derivative u_{12}^i plays a fundamental role. Indeed, if $u_{12}^i\leq 0$ (or $u_{12}^i\succ 0$ but not too strong) then $V_{12}\geqslant 0$ (or $V_{12}<0$) and the dynamics is monotone (or oscillating). We then encompass the separable case, $u_{12}i=0$ and extend the result to the cases where marginal utility of the babies is a decreasing function i.e death occurrence is coupled with a weakly increasing function of the mothers health state in particular if $V_{12}\leq 0$. Finally, if the curvature of the production function is given, an adequate increase of the utility function's degree of concavity may lead to a positive cross derivative V_{12} and the dynamics is monotone. Thus, there exists a trade-off between the degrees of concavity of f and u. Oscillating paths may be obtained if the production function is more concave than the instantaneous utility function specifically when medical treatment efficiency is less strong than the virus action in health alteration.

A sustainable growth in lives path: $\left(a_t^i,a_{t+1}^i\right)\in\mathring{E}_1$ necessarily satisfies the Euler-Lagrange equation i.e

$$V_2\left(a_t^i,a_{t+1}^i\right)+\delta V_1\left(a_{t+1}^i,a_{t+2}i\right)=0\qquad\forall t\geq 0\quad(2.8)$$

Let $a^{i*}=f(k^{i*})-s^{ci*}$ be the sustainable steady state associated to the indirect optimal function, $V_{ij}^*=V_{ij}^*(a^*,a^*)_{Ij=1,2}$. The roots of the Jacobian matrix associated with the linearization of the previous equation evaluated at (a^{i*}, a^{i*}) will be the solution of the characteristic polynomial

$$\delta V_{12}^*\lambda^2+\delta\left(V_{11}^*+V_{22}^*\right)\lambda+V_{21}^*=0\qquad\forall t\geq 0\,(2.9)$$

To study local stability of the steady state, we need the following definition:

Definition1: An optimal steady state (a^{i*}, a^{i*}) of the Euler Lagrange equation reaches the sustainable path if the two roots λ of the previous characteristic polynomial are such that one is strictly inside (-1,1) and the other is strictly outside.

This definition allows us to sue the saddle point stability which is usually employed in the optimal growth literature. In this model, the existence of a root strictly inside ensures the existence of a one dimensional stable manifold. Following Scheinkman lemma, under the assumption of strict concavity, the projection of the stable manifold on the space (a_t^i, a_{t+1}^i) is a local diffeomorphism. Then for each initial value a_0^i close to a^{i*}, there exists an unique a_1 such that (a_0^i, a_1^i) is on the stable manifold and the optimal path converges to the steady state. Our model is symmetric i.e $V_{12}=V_{21}$ and thus all roots provided by the characteristic polynomial are real. From Benhabib-Nishimura, we have the following theorem.

Theorem1: the steady state (a^{i*}, a^{i*}) is the optimal sustainable path if and only if:

$$\delta V_{11}^*\lambda^2+V_{22}^*+(1+\delta)\left|V_{21}\right|^*<0\qquad(2.10)$$

Note that: because V is strictly concave, if $\delta=1$ the previous condition holds and the saddle point exists. Otherwise if $\delta<1$ then the indirect utility function strict concavity is no more guarantee and the dynamics oscillates.

Proposition4: If the instantaneous utility function u^i is such that

$u_{12}(s^{mi}, s^{ci})<0$, then the steady state (a^{i*},a^{i*}) is a regular saddle point if and only if:

$$\delta\left(u_{11}^i-u_{12}^i\right)+\left(u_{22}^i-u_{12}^i\right)<0\qquad(2.11)$$

PROOF: if $u_{12}(s^{mi}, s^{ci})<0$ then $V_{21}^*>0$ indeed the condition of theorem1 becomes

$$\delta\left(u_{11}^i+\frac{B^2}{A-f''u_2^i}\right)+\left(u_{22}^i+\frac{C^2}{A-f''u_2^i}\right)$$
$$+\left(1+\delta\right)\left(-u_{12}^i+\frac{BC}{A-f''u_2^i}\right)<0\qquad\forall t\geq 0$$

Replacing A,B,C by their respective values, the previous equation becomes

$$(1-f')(\delta f'-1)H^i-f''u_2\left[\delta\left(u_{11}^i-u_{12}^i\right)+\left(u_{22}^i-u_{12}^i\right)\right]<0$$

where $H^i\geq 0$

Let us consider now the case of oscillating dynamics i.e when $V_{12}=-u_{12}^i+BC/(A-f'u_2^i)<0$ the production function needs to be concave with respect to the instantaneous utility function. We already know that $u_{12}(s^{mi}, s^{ci})\succ 0$ is a necessary condition for $V_{21}<0$. Moreover, lemma 1 shows that there exists a trade-off between the degrees of concavity of f and u^i. Therefore $V_{21}<0$ is obtained when the production function is concave with respect to the instantaneous utility function. Therefore we can announce a necessary and sufficient condition for the saddle point property of the steady state.

Proposition5: assuming that the instantaneous utility function $u_{12}(s^{mi},s^{ci})\succ 0$ and satisfies with the production function condition i.e $-f'H^i-f''u_2{}^iu_{22}{}^i\succ 0$ where $H^i\geq 0$ is the determinant of the Hessian matrix of u^i evaluated at (s^{mi},s^{ci}), then the steady state (a^{i*},a^{i*}) is a regular saddle point if and only if:

$$-\frac{2(1+\delta)}{\delta}(\delta f'-1)H^i$$
$$-f''u_2\left[\delta\left(u_{11}^i-u_{12}^i\right)+\left(u_{22}^i-u_{12}^i\right)\right]<0$$

PROOF: under the assumption of the proposition5 i.e $u_{12}(s^{mi}, s^{ci})\succ 0$ *we have* $V_{12}<0$, the condition for the saddle point existence is:

$$-\frac{2(1+\delta)}{\delta}H^i-f''u_2^{i*}\left[\delta\left(u_{11}^i+u_{12}^i\right)+\left(u_{22}^i+u_{12}^i\right)\right]<0$$

under the assumptions of the proposition we have $V_{12}<0$, the condition of theorem 1 is $\delta V_{11}^*\lambda^2+V_{22}^*+(1+\delta)\left|V_{21}\right|^*<0$ thus,

$$\delta\left(u_{11}^i+\frac{B^2}{A-f''u_2^i}\right)+\left(u_{22}^i+\frac{C^2}{A-f''u_2^i}\right)$$
$$+(1+\delta)\left(-u_{12}^i+\frac{BC}{A-f''u_2^i}\right)<0$$

given the values of B, C, A the previous expression is equivalent to

$$(1+f')(\delta f'+1)H^i-$$
$$f''u_2\left[\delta\left(u_{11}^i+u_{12}^i\right)+\left(u_{22}^i+u_{12}^i\right)\right]<0$$

and Since $f'=\delta^{-1}$ and $H^i \geq 0$, proposition 5 is verified. Note that if $H^i=0$, the stability condition becomes $\delta\left(u_{11}^i+u_{12}^i\right)+\left(u_{22}^i+u_{12}^i\right)<0$

2.4. Flip bifurcation and Competitive Cycles in a Non Separable Case

Under the assumption of $V_{12}<0$ we know that optimal paths are oscillating. If the condition of theorem 1 doesn't hold, then there exists a bifurcation value δ^* which generates two period cycles. Consequently, if $V_{12}<0$ we have a flip bifurcation which generates two period cycles.

Definition2: δ^* is a flip bifurcation value if locally on one side there exist two period cycles and not on the other side. According to assumptions 1 and 2, the saddle point stability provided by theorem 1 is satisfied when $\delta=1$ and leads to the following theorem of Cartigny and Venditti

Theorem2: If $V_{12}(a^i,b^i)$ is strictly negative, then there exist a flip bifurcation value $\delta^* \varepsilon (0,1)$ if and only if the sign of $\delta V_{11}^*+V_{22}^*-(1+\delta)V_{21}$ changes when δ crosses δ^*

Proposition6: *Assume that the instantaneous utility function u^i is such that $u^i(s^{mi}, s^{ci}) \succ 0$ and satisfies with the production function $-f'H^i - f''u_2^i u_{22}^i \succ 0$ when $H^i \geq 0$ is the determinant of the Hessian matrix evaluated at (s^{mi}, s^{ci}) then there exist a flip bifurcation value $\delta^* \varepsilon(0,1)$ if and only if the sign of:*

$$-\frac{2(1+\delta)}{\delta}H^i-f''u_2\left[\delta\left(u_{11}^i+u_{12}^i\right)+\left(u_{22}^i+u_{12}^i\right)\right]$$

changes when δ crosses δ^*.

Note that if the instantaneous utility function is not too concave in the neighborhood of the steady state *(i.e H^{i*} is close to zero)* whereas the production function is highly concave *(i.e $|f''|$ is high enough)*, the condition of proposition 5 may be satisfied especially if $H^{i*}=0$, the condition becomes, the sign of $\delta\left(u_{11}^i+u_{12}^i\right)+\left(u_{22}^i+u_{12}^i\right)$ *changes when δ crosses δ^**

3. Application of the Theory

3.1. The Additively Separable Case

We assume that the non infected pregnant women have an additively separable utility function expressed such that:

$$u\left(s_t^{mh}, s_t^{ch}\right)=\ln\left(s_t^{mh}\right)+\beta\ln\left(s_{t+1}^{ch}\right) \quad (3.1)$$

where $0<\beta<1$ is the elasticity of the baby

The production function is expressed such that

$$F(k,1)=k^{\theta} \text{ [1]} \quad (3.2)$$

where $\theta \in (0,1)$.

Proposition7: the additively separable utility, (3.1) and the production functions (3.2) define the stable optimal solution composed of the vector of variables $\left(s^{mh*}, s^{ch*}, k^*, y^*\right)$ expressed by (3)-(6) i.e

$$k^*=(\theta\delta)^{1/1-\lambda} \quad (3.3)$$

$$s^{ch*}=k^*\frac{\beta(\theta\delta-1)}{\delta+\beta} \quad (3.4)$$

$$s^{ch*}=k^*\frac{\delta(\theta\delta-1)}{\delta+\beta} \quad (3.5)$$

$$y^*=(\theta\delta-2)k^* \quad (3.6)$$

Proof: we know that, on the one hand, u is concave because $u_{11}<0$,$u_{22}<0$ and $u_{12}=u_{21}=0$. On the other hand, f is concave because $f'>0$ and $f''<0$, therefore the optimal solution exist and is unique.

According to the theory, we have presented above, $u_1=(1+r)u_2$ and $s^{mh}+s^{mc}=f(k)-k$; $1+r=f'=1/\delta$ which yield stationary values of production, capital stocks and consumptions expressed by (3)-(6)

Growth in life expectancy gain is thus expressed by equation (3.7) which yields

$$g^h=\frac{\beta-\delta}{\delta} \quad (3.7)$$

Life is expressed by the difference between young and old rates such that population growth increases when the elasticity of the babies is higher than that of the discount rate or the economic growth rate.

In order to study the cases where the mothers are infected while waiting for new born, we need to introduce additional assumptions.

Assuming γ^m to be the mother's *HIV/AIDS* infection risk parameter and γ^c be the *HIV/AIDS* transmission risk from the mother to the child parameter, *if* $\gamma^m \succ 0$ then $s_t^{mh}=s_t^{mh}\left(\gamma^m\right)$ i.e the mother's health state is conditioned on the pandemic prevalence. Therefore we have $s_t^{ch}=s_{t+1}^{ch}\left(\gamma^c, s_t^{mh}\right)$ i.e the baby's health state depends on his mother's infection gravity. Indeed, $\left(s_t^{mh}, s_{t+1}^{ch}\right)=\left(s_t^{mh}\left(\gamma^m\right), s_{t+1}^{ch}\left(\gamma^c, s_t^{mh}\right)\right)=s_t^{mh}\left(\gamma^m\right)\left(1,\gamma^c\right)$, it is no more possible to separate functionally the mother and the baby variables because for two given scalars λ_1 and λ_2 , such that $\lambda_1 s_t^{mh}+\lambda_2 s_{t+1}^{ck}=0$ we cannot have $\lambda_1=\lambda_2=0$ for those variables to be expressed freely and taking the utility function we have $u\left(\lambda_1 s_t^{mh}+\lambda_2 s_{t+1}^{ck}\right)\neq\lambda_1 u(s_t^{mh})+\lambda_2 u(s_{t+1}^{ck})$. Indeed

1 This function is used by Michel-Venditti [49]

$u(s_t^{mh}, s_{t+1}^{ck}) = u^i(s_t^{mh}, s_{t+1}^{ck})$, the additively separable utility function $u_t^h(s_t^{mh}, s_{t+1}^{ch})$ converges to the non separable utility function $u_t^i(s_t^{mi}, s_{t+1}^{ci})$ which oscillates in life duration over time.

3.2. The Oscillatory Dynamics

The utility function is now of Cobb Douglas i.e it is no more additively separable and expressed such that equation (8) i.e:

$$u\left(s_t^{mh}, s_{t+1}^{ch}\right) = \frac{\left(s_t^{mh}\right)^{\gamma^m}\left(s_{t+1}^{ch}\right)^{\gamma^c}}{\gamma^m + \gamma^c} \quad (3.8)$$

[2]

The production function is almost the same as before i.e

$$F(k,1) = k^\theta \quad (3.9)$$

Where $0 < \theta < 1$. Using the same formulas as before with $f'(k^*) = \gamma(k^*)^{\gamma-1} = \delta^{-1}$, the stationary values of production, capital stocks and consumptions are expressed such that:

$$k^* = (\theta\delta)^{1/1-\theta}; \quad (3.10)$$

$$s^{mi*} = k^* \frac{\gamma^m(1-\delta\theta)}{\theta(\gamma^m\delta + \gamma^c)} \quad (3.11)$$

$$s^{ci**} = k^* \frac{\gamma^c(1-\delta\theta)}{\theta\delta(\gamma^m\delta + \gamma^c)} \quad (3.12)$$

$$y^* = k^* \frac{\gamma^m + \theta\gamma^c}{\theta(\gamma^m\delta + \gamma^c)} \quad (3.13)$$

Population growth is given by

$$g = \frac{\gamma^m - \delta\gamma^c}{\delta\gamma^c} \quad (3.14)$$

Compare to the previous case, population growth depends now on the pandemic prevalence essentially in Africa, specifically on the mother's infection gravity. If the pandemic is transmitted, then the economic growth represented by the discount factor is higher. The aim of the following discussion is to reduce the uncertainty in life gift and HIV/AIDS prevalence in establishing conditions for which optimal growth in life terms exist.

3.2.1. Study of the Sign of the Cross Derivative of *u*

$$u_{12} = \frac{\gamma^m\gamma^c}{\gamma^m + \gamma^c}\left(s_t^{ci}\right)^{\gamma^c - 1}\left(s_{t+1}^{mi}\right)^{\gamma^m - 1} \quad (3.15)$$

If $\gamma^c<0$ and $\gamma^m<0$ we have $u_{12}<0$, then the steady state is reached. Otherwise if $\gamma^c>0$ and $\gamma^m>0$, we have $u_{12}>0$.

From lemma 1, if $\gamma^c<0$ and $\gamma^m<0$, then optimal paths are monotone because $u_{12}<0$. Otherwise, if $\gamma^c+\gamma^m=1$ and $\gamma^c>0$, $\gamma^m>0$, we have $H^i = 0$ and the optimal dynamics is oscillating. Finally if $\gamma^c+\gamma^m<1$, we have:

$$f'H^i - f''u_2u_{12} < 0$$
$$\Leftrightarrow -f'\left(1-\gamma^c-\gamma^m\right) - f''s_t^{mi}\gamma^c < 0$$

. Therefore, it is necessary to study the sign of V_{12} in the neighborhood of the steady state. It follows that

$$-f''s_t^{mi} = \frac{(1-\theta)(1-\delta\theta)\gamma^m}{\delta\theta(\gamma^c + \delta\gamma^m)} > 0 \quad (3.16)$$

Since $f'(k^*)=\delta^{-1}$, the previous inequality is equivalent to

$$-\theta\left(1-\gamma^c-\gamma^m\right)\left(\gamma^c+\delta\gamma^m\right) + \gamma^c\gamma^m(1-\theta)(1-\theta\delta) < 0$$
$$\Leftrightarrow -\delta\gamma^m\theta\left(1-\gamma^c-\gamma^m\theta\right) + \gamma^c\left[\gamma^m - \theta(1-\gamma^c)\right] \equiv h(\delta) < 0$$

is a monotone decreasing function. Thus depending on the values of γ^c, γ^m and θ, if $h(1)<0$ then $V_{12}>0$ for all values of the discount factor on (0,1]. Specifically, if $\theta>\gamma^m/1-\gamma^c$ i.e medical care is high, the optimal dynamics is monotone for all values of δ or the economic growth rate, this condition is satisfied only if γ^c and γ^m are small enough i.e the utility function is highly concave. Otherwise if medical care is insufficient i.e $\theta<\gamma^m/1-\gamma^c$, there must be established an optimal economic growth rate δ^* such that $h(\delta^*)=0$ to ensure population growth stability i.e:

$$\delta^* = \frac{\gamma^c\left[\gamma^m - \theta(1-\gamma^c)\right]}{\theta\gamma^c\left(1-\gamma^c-\gamma^m\theta\right)} \quad (3.17)$$

But optimal paths oscillate for all $\delta<\delta^*$

If $\delta^*>1$ i.e the economic stability level is too high enough meaning:

if $\gamma^c\gamma^m \geq \theta\left[\gamma^c(1-\gamma^c) + \gamma^m(1-\gamma^c-\gamma^m\theta)\right]$ then $V_{12}<0$ for all values of the discount factor in (0,1] because the system is increasing in per-capita income while population growth is reducing. This condition is also satisfied when $\gamma^c\gamma^m$ is too high. Otherwise if $\delta^* \in (0,1)$, the optimal dynamics is monotone only if $\delta \in (\delta^*, 1]$. Then, if the curvature of the production function or health care provision is given, an increase of the utility function's degree of concavity leads to a positive cross derivative V_{12}. The existence of oscillating paths requires that the production function is concave enough with respect to the instantaneous utility function.

3.2.2. Study of Local Stability of the Steady State

If $\gamma^c\gamma^m<0$, optimal paths are monotone because we have $u_{12}<0$, it yields:

$$\delta\left(u_{11}^* - u_{12}^*\right) + u_{22}^* - u_{12}^* =$$

$$s^{mi\gamma^m-1}s^{ci\gamma^c-1}\left[\delta\left(\frac{\gamma^m\left(\gamma^m-1\right)s^{ci*}}{\left(\gamma^c+\gamma^m\right)s^{ci*}} - \frac{\gamma^c\gamma^m}{\gamma^c+\gamma^m}\right) + \frac{\gamma^c\left(\gamma^c-1\right)s^{ci*}}{\left(\gamma^c+\gamma^m\right)s^{ci*}} - \frac{\gamma^c\gamma^m}{\gamma^c+\gamma^m}\right]$$

Because $\frac{s^{ci*}}{s^{mi*}} = \frac{\gamma^c}{\gamma^m\delta}$ we have

2 The function is taken on Michelle-Venditti (1994)

$$\delta\left(u_{11}^{*}-u_{12}^{*}\right)+u_{22}^{*}-u_{12}^{*}= s^{mi\gamma^{m}-1}s^{ci\gamma^{c}-1}\left[-\frac{\left(\gamma^{c}-\delta\gamma^{m}\right)}{\left(\gamma^{c}+\gamma^{m}\right)}\right]<0$$

for all $\delta \in (0,1]$. Then, the steady state is a regular saddle point for all values of the discount factor $\delta \in (0,1]$

Otherwise, if $\gamma^c\gamma^m>0$ and $\gamma^c+\gamma^m \le 1$, the optimal path is oscillating i.e we have $u_{12}>0$ and it depends on the concavity properties of f and u. The required condition given in proposition 5 is that

$$\frac{s^{mi*2\left(\gamma^{c}-1\right)}s^{ci*2\left(\gamma^{c}-1\right)}}{\left(\gamma^{c}+\gamma^{m}\right)^{2}} \gamma^{c}\left[\frac{2(1+\delta)}{\delta}\gamma^{m}\left(\gamma^{c}+\gamma^{m}-1\right)-f''s^{mi*}\left(2\gamma^{c}-1\right)+\delta\gamma^{c}\left(2\gamma^{c}-1\right)\right]<0$$

Since $\gamma^c+\gamma^m-1\le 0$ we conclude that if

$$\gamma^{c}\left(2\gamma^{m}-1\right)+\delta\gamma^{c}\left(2\gamma^{c}-1\right)<0 \qquad (3.18)$$

The optimality condition is satisfied.

Let $S=\gamma^c+\gamma^m\le 1$, it follows $\gamma^{c}\gamma^{m}=\gamma^{m}\left(S-\gamma^{c}\right)\le S^{2}/4$ and we obtain

$$\gamma^{c}+\gamma^{m}-4\gamma^{c}\gamma^{m}\ge S-S^{2}\ge 0 \qquad (3.19)$$

Several conclusions may be given from the above two inequalities

-If $\gamma^c<1/2$ and $\gamma^m\le 1/2$, the saddle point stability is satisfied for all values of the discount factor $\delta \in (0,1]$

-If $\gamma^c>1/2$ and $\gamma^m\le 1/2$ with $\gamma^c+\gamma^m\le 1$, the stability condition is also satisfied for all values of the discount factor $\delta \in (0,1]$. Indeed the sufficient condition (3.18) becomes $\delta<\gamma^{c}\left(2\gamma^{c}-1\right)/\gamma^{c}\left(2\gamma^{m}-1\right)=\delta^*$ and it follows from (3.19) that $\delta^*\ge 1$

- If $\gamma^c<1/2$ and $\gamma^m \ge 1/2$ with $\gamma^c+\gamma^m\le 1$, the sufficient condition becomes $\delta>\delta^*$ i.e economic growth is higher than the equilibrium rate. Indeed the stability condition is satisfied for all $\delta \in (\delta^*,1]$ such that $\delta^*<1$

In oscillating optimal paths in lives terms, the condition of existence of two periods cycles is given by:

$$\left[\frac{2(1+\delta)}{\delta}\gamma^{m}\left(\gamma^{c}+\gamma^{m}-1\right)-f''s^{mi*}\left(2\gamma^{c}-1\right)+\delta\gamma^{c}\left(2\gamma^{c}-1\right)\right]\ge 0 \qquad (3.20)$$

If $\gamma^c+\gamma^m<1$, the instantaneous utility function is more and more concave, the two period cycles exist for more and more extreme values of the discount factor

If $\gamma^c+\gamma^m=1$, the determinant of the hessian matrix is zero i.e $H^i=0$, the above condition becomes

$$\left(2\gamma^{m}-1\right)\left(1-\gamma^{m}-\delta\gamma^{m}\right)\ge 0 \qquad (3.21)$$

If $\gamma^m>1/2$, we obtain $\delta^*=\left(1-\gamma^{c}\right)/\gamma^{c}$, in contrast to the previous case, now if the utility function is homogenous, cycles may exist for δ close to 1. Indeed, it is sufficient to adequately modify the values of γ^m toward ½ and then δ^* tends toward 1.

4. Conclusions

In this paper, we have first proved that in an overlapping generation model with one production sector, if the utility function of health state summarized by per-capita mother and baby is non separable, then optimal steady state in lives terms is not necessarily reached. Depending on the HIV/AIDS gravity and transmission risk and it may appear cycles and hopf bifurcation at critical locus. Applying a specific Cobb Douglas utility function, the event is measured through the comparison between the discount factor which is assimilated to the economic growth rate and population growth of life occurrences view through the HIV/AIDS parameters fluctuations during the pregnancy process. We found that optimal life dynamics existence and stability depend on illness gravity and medical care investment interaction.

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