

Solutions of a Nonlocal Elliptic Problem Involving $p(x)$ -Kirchhoff-type Equation

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Abstract The present paper deals with a Kirchhoff problem under homogeneous Dirichlet boundary conditions, set in a bounded smooth domain Ω of $\mathbb{R}^N$. The problem studied is a stationary version of the original Kirchhoff equation, involving the $p(x)$ -Laplacian operator, in the framework of the variable exponent Lebesgue and Sobolev spaces. The question of the existence of weak solutions is treated. Applying the Mountain Pass Theorem of Ambrosetti and Rabinowitz, the existence of a nontrivial weak solution is obtained in the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$.

Keywords Variational Method, $p(x)$ -Laplacian, Nonlocal Problem, Mountain-Pass Theorem, Ambrosetti-Rabinowitz's Condition, Palais-Smale Condition

1. Introduction

In the present paper we are concerned with the following problem

$$-m \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f(x, u) \text{ in } \Omega, \quad (P)$$

$$u = 0 \text{ on } \partial\Omega$$

Where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $p \in C(\overline{\Omega})$ with $1 < p(x) < N$ for any $x \in \overline{\Omega}$, m is a continuous function and f is a Carathéodory function.

Problem (P) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by [12]. To be more precise, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, h, P_0, E, L are constants which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. There are papers [4, 11, 14] in which the authors give the existence of solutions of Kirchhoff-type and p -Kirchhoff-type equations. Moreover, for $p(x)$ -Kirchhoff-type equations see, for example, [2, 5-7].

The $p(x)$ -Laplace operator $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is a natural generalization of the p -Laplacian operator

$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ where $p > 1$ is a real constant. The

main difference between them is that p -Laplacian operator is $(p-1)$ -homogenous, but the $p(x)$ -Laplacian operator, when $p(x)$ is not constant, is not homogeneous. This causes many problems, some classical theories and methods, such as the Lagrange multiplier theorem and the theory of Sobolev spaces, are not applicable. For $p(x)$ -Laplacian operator, we refer the readers to [9, 10, 15, 16] and references there in. Moreover, the nonlinear problems involving the $p(x)$ -Laplacian operator are extremely attractive because they can be used to model dynamical phenomena which arise from the study of electrorheological fluids or elastic mechanics. Problems with variable exponent growth conditions also appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium. The detailed application backgrounds of the $p(x)$ -Laplacian can be found in [1, 3, 17, 19] and the references there in.

2. Preliminaries

First, we recall some basic properties of spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ (for details, see e.g., [8, 14]).

Set

$$C_+(\overline{\Omega}) = \{p : p \in C(\overline{\Omega}), p(x) > 1 \text{ for any } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, denote

$$p^- := \inf_{x \in \Omega} p(x), \quad p^+ := \sup_{x \in \Omega} p(x) < \infty,$$

and define the variable exponent Lebesgue space by

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$L^{p(x)}(\Omega) = \left\{ u \text{ is a measurable real function on } \Omega : \int_{\Omega} |u|^{p(x)} dx < \infty \right\}$.

We define a norm, the so-called Luxemburg norm, on $L^{p(x)}(\Omega)$ by the formula

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

and then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space.

Define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) ; |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

then it can be equipped with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm $\|u\| = |\nabla u|_{p(x)}$, since the well-known Poincaré inequality holds.

Proposition 1. [8,13] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$,

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$

and $v \in L^{p'(x)}(\Omega)$, we have

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) |u|_{p(x)} |v|_{p'(x)}.$$

Proposition 2. [8,13] Denote $\rho(u) = \int_{\Omega} |u|^{p(x)} dx$,

$\forall u, u_n \in L^{p(x)}(\Omega)$, then one has

$$(i) |u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+},$$

$$(ii) |u|_{p(x)} \leq 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-},$$

$$(iii) \lim_{n \rightarrow \infty} |u_n|_{p(x)} = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \rho(u_n) = 0,$$

$$(iii) \lim_{n \rightarrow \infty} |u_n|_{p(x)} \rightarrow \infty \Leftrightarrow \lim_{n \rightarrow \infty} \rho(u_n) \rightarrow \infty.$$

Proposition 3. [8,13] If $u, u_n \in L^{p(x)}(\Omega)$,

then the following statements are equivalent :

$$(i) \lim_{n \rightarrow \infty} |u_n - u|_{p(x)} = 0,$$

$$(ii) \lim_{n \rightarrow \infty} \rho(u_n - u) = 0,$$

$$(iii) u_n \rightarrow u \text{ in measure } \Omega \text{ and } \lim_{n \rightarrow \infty} \rho(u_n) = \rho(u).$$

Proposition 4. [8,13]

(i) If $1 < p^- \leq p^+ < \infty$, then the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$

and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces,

(ii) If $p_1, p_2 \in C_+(\overline{\Omega})$ and $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then

the embedding $L^{p_1(x)}(\Omega) \rightarrow L^{p_2(x)}(\Omega)$ is continuous,

(iii) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then

the embedding $W_0^{1,p(x)}(\Omega) \rightarrow L^{q(x)}(\Omega)$ is continuous and compact,

where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Definition 5. Let X be a Banach space and

$I : X \rightarrow \mathbb{R}$ a C^1 -functional.

We say that a functional I satisfies the

Palais-Smale (PS) for short condition

if any sequence $\{u_n\}$ in X such that $\{I(u_n)\}$ is bounded

and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$, admits a convergent

subsequence.

Proposition 6. [8,14] Let X be a Banach space and

define the functional

$$\Lambda = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx. \text{ Then } \Lambda : X \rightarrow \mathbb{R} \text{ is convex.}$$

The mapping $\Lambda' : X \rightarrow X^*$

is a strictly monotone, bounded homeomorphism,

and of (S_+) type, namely

$$u_n \rightarrow u \text{ (weakly) and } \overline{\lim}_{n \rightarrow \infty} \langle \Lambda'(u_n), u_n - u \rangle \leq 0$$

implies $u_n \rightarrow u$ (strongly).

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (P) if

$$m \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx = \int_{\Omega} f(x, u) \varphi,$$

where $\varphi \in W_0^{1,p(x)}(\Omega)$.

We associate to the problem (P) the energy functional, defined as $I : W_0^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$,

$$I(u) = M \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) - \int_{\Omega} F(x, u),$$

where $M(t) = \int_0^t m(s) ds$ and $F(x, u) = \int_0^u f(x, s) ds$.

We know that from (m0) and (f0) (see Section 3) I is

well defined and in a standard way we can prove that $I \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and that the critical points of I are solutions of (P). Moreover, the derivative of I is given by

$$\begin{aligned} \langle I'(u), \varphi \rangle &= m \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx \\ &\quad - \int_{\Omega} f(x, u) \varphi, \end{aligned}$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$.

3. The Main Results

Now, we are ready to set and prove the main result of the present paper.

Theorem 7. Assume that the following assumptions hold :

(m0) $m : (0, +\infty) \rightarrow (0, +\infty)$ is a continuous function such that,

$$(1-\eta)t^{\alpha(x)-1} \leq m(t) \leq (1+\eta)t^{\alpha(x)-1} \quad \forall t > 0, 1 > \eta \geq 0$$

and $\alpha(x) \geq 1 \quad \forall x \in \Omega$;

(f0) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function and satisfies the growth condition $|f(x, t)| \leq c(1 + |t|^{\beta(x)-1}) \quad \forall (x, t) \in \Omega \times \mathbb{R}$,

where $\beta \in C_+(\overline{\Omega})$ and $\beta(x) < p^*(x)$ for all $x \in \Omega$;

(f1) $f(x, t) = o(|t|^{p^+-1})$, $t \rightarrow 0$ uniformly for $x \in \Omega$,

where $\frac{\beta^-}{\alpha^+} > p^+$;

(AR) Ambrosetti – Rabinowitz's condition holds, i.e.,

$$\exists K > 0, \theta > \left(\frac{1+\eta}{1-\eta} \right) \frac{\alpha^+(p^+)^{\alpha^+}}{(p^-)^{\alpha^- - 1}} \text{ such that}$$

$$0 < \theta F(x, t) \leq f(x, t)t, \quad |t| > K, \quad a.e. x \in \Omega.$$

Then (P) has at least one nontrivial weak solution.

The proof of Theorem 7 follows from the following two Lemmas.

Lemma 8. Suppose (m0), (AR) and (f0) hold.

Then I satisfies (PS) condition.

Proof. Let us assume that there exists a sequence $\{u_n\}$ in $W_0^{1,p(x)}(\Omega)$ such that

$$|I(u_n)| \leq c \text{ and } I'(u_n) \rightarrow 0.$$

Therefore,

$$\begin{aligned} c + \|u_n\| &\geq I(u_n) - \frac{1}{\theta} \langle I'(u_n), u_n \rangle \\ &\geq \frac{(1-\eta)}{\alpha^+(p^+)^{\alpha^+}} \left(\int_{\Omega} |\nabla u_n|^{p(x)} dx \right)^{\alpha(x)} - \frac{1+\eta}{\theta(p^-)^{\alpha^- - 1}} \left(\int_{\Omega} |\nabla u_n|^{p(x)} dx \right)^{\alpha(x)-1} \\ &\quad \times \left(\int_{\Omega} |\nabla u_n|^{p(x)} dx \right) + \int_{\Omega} \left(\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n) \right) dx \\ &\geq \left(\frac{(1-\eta)}{\alpha^+(p^+)^{\alpha^+}} - \frac{1+\eta}{\theta(p^-)^{\alpha^- - 1}} \right) \left(\int_{\Omega} |\nabla u_n|^{p(x)} dx \right)^{\alpha(x)} + c \\ &\geq \left(\frac{(1-\eta)}{\alpha^+(p^+)^{\alpha^+}} - \frac{1+\eta}{\theta(p^-)^{\alpha^- - 1}} \right) \|u_n\|^{\alpha^- p^-} + c. \end{aligned}$$

Therefore, $\{\|u_n\|\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. From this bound estimate, going to a subsequence if necessary, there exists a u in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightarrow u$ (weakly) in $W_0^{1,p(x)}(\Omega)$. Thanks to the compact embedding $W_0^{1,p(x)}(\Omega) \rightarrow L^{\beta(x)}(\Omega)$, we get

$$\begin{aligned} u_n &\rightarrow u \text{ (strongly) in } L^{\beta(x)}(\Omega), \\ u_n &\rightarrow u \text{ a.e. } x \in \Omega. \end{aligned}$$

Since, $\langle I'(u_n), u_n - u \rangle \rightarrow 0$, we have

$$\begin{aligned} &\langle I'(u_n), u_n - u \rangle \\ &= m \left(\int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \\ &\quad - \int_{\Omega} f(x, u_n) (u_n - u) dx \rightarrow 0. \end{aligned}$$

From (f0) and Proposition 1, it follows

$$\begin{aligned} &\left| \int_{\Omega} f(x, u_n) (u_n - u) dx \right| \leq \\ &c \left| u_n \right|^{\beta(x)-1} \Big|_{\beta'(x)} |u_n - u|_{\beta(x)} + c \int_{\Omega} |u_n - u| dx. \end{aligned}$$

By the embeddings, we get

$$\int_{\Omega} f(x, u_n) (u_n - u) dx \rightarrow 0.$$

Hence,

$$m \left(\int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \rightarrow 0.$$

From (m0), it follows

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \rightarrow 0. \quad (3.1)$$

Since the functional (3.1) is of type (S_+) , we get $u_n \rightarrow u$ (strongly) in $W_0^{1,p(x)}(\Omega)$. I satisfies (PS) condition.

Lemma 9. Suppose (m_0) , (AR) , (f_0) and (f_1) hold.

Then the following statements hold:

(i) There exist two positive real numbers γ and a such that

$$I(u) \geq a > 0, \quad u \in W_0^{1,p(x)}(\Omega) \text{ with } \|u\| = \gamma;$$

(ii) There exists a u in $W_0^{1,p(x)}(\Omega)$ such that $\|u\| > \gamma$, $I(u) < 0$.

Proof.

(i) Let us assume $\|u\| < 1$. Then by (m_0) , we have

$$I(u) \geq \frac{(1-\eta)}{\alpha^+(p^+)} \|u_n\|^{\alpha^+ p^+} - \int_{\Omega} F(x, u).$$

By the continuous embeddings

$$W_0^{1,p(x)}(\Omega) \rightarrow L^{\beta(x)}(\Omega), \text{ and}$$

$$W_0^{1,p(x)}(\Omega) \rightarrow L^{p^+}(\Omega),$$

there exist some positive constants c such

that, $\forall u \in W_0^{1,p(x)}(\Omega)$,

$$\|u\|_{\beta(x)} \leq c \|u\| \quad \text{and} \quad \|u\|_{p^+} \leq c \|u\|, \quad (3.2)$$

Further, using (f_0) and (f_1) , we get

$$F(x, t) \leq \varepsilon |t|^{\gamma p^+} + c_{\varepsilon} |t|^{\beta(x)}, \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$

Therefore,

$$\begin{aligned} I(u) &\geq \frac{(1-\eta)}{\alpha^+(p^+)^{\alpha^+}} \|u\|^{\alpha^+ p^+} - \varepsilon \int_{\Omega} |u|^{p^+} dx - c_{\varepsilon} \int_{\Omega} |u|^{\beta(x)} dx \\ &\geq \left(\frac{(1-\eta)}{\alpha^+(p^+)^{\alpha^+}} \|u\|^{\alpha^+ p^+ - p^+} - \varepsilon c^{p^+} - c_{\varepsilon} \|u\|^{\beta^- - p^+} \right) \|u\|^{p^+}. \end{aligned}$$

Let us define the function

$$\Phi(t) = \frac{(1-\eta)}{\alpha^+(p^+)^{\alpha^+}} t^{\alpha^+ p^+ - p^+} - c_{\varepsilon} t^{\beta^- - p^+}.$$

Since $\frac{\beta^-}{\alpha^+} > p^+$, it is clear that there exists a γ such that

$$\max_{t \geq 0} \Phi(t) = \Phi(\gamma) > 0. \quad \text{Hence, for a fixed}$$

$\varepsilon \in \left(0, \frac{\Phi(\gamma)}{c^{p^+}}\right)$ small enough, there exist two positive real numbers γ and a such that

$$I(u) \geq a > 0, \quad u \in W_0^{1,p(x)}(\Omega) \text{ with } \|u\| = \gamma \in (0, 1).$$

(ii) From (AR) , we have

$$F(x, tu) \geq t^{\theta} F(x, u), \quad \forall t \geq 1, \text{ a.e. } x \in \Omega.$$

Hence, for $\omega \in W_0^{1,p(x)}(\Omega)$, $\omega \neq 0$, and $t > 1$ we have

$$\begin{aligned} I(t\omega) &= M \left(\int_{\Omega} \frac{|\nabla t\omega|^{p(x)}}{p(x)} dx \right) - \int_{\Omega} F(x, t\omega) \\ &\leq \frac{(1+\eta)}{\alpha^-(p^-)^{\alpha^-}} t^{\alpha^+ p^+} \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\alpha^+} - t^{\theta} \int_{\Omega} F(x, \omega). \end{aligned}$$

From (AR) , it can be obtained that $\theta > \alpha^+ p^+$. Therefore,

$$I(t\omega) \rightarrow -\infty \quad (t \rightarrow +\infty).$$

Proof of Theorem 7. From Lemma 8, Lemma 9 and the fact that $I(0) = 0$, I satisfies the Mountain Pass theorem (see e.g., [18]). Therefore, I has at least one nontrivial critical point, i.e., (P) has a nontrivial weak solution. We are done.

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