

The Limiting Probabilities in the Process of Servicing

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Abstract Using z - transformation in Markov processes connected with the fact that the probabilities of transition for the n steps in them are geometric progressions, so that with the help of z -transformation expression for these probabilities can be obtained in analytical form.

Keywords limiting probability, Probability of state, Markov Processes, Z-transform

1. Introduction

Markov chain with a countable state space is an interesting part of Markov processes. Markov processes have found wide application in the application of probability theory and statistics. Some of these applications require more specific information. Other applications require great community. The first type consists of finite chains. The second type includes various models of queuing. Markov process in the study of complex systems is the most convenient. The concepts of the system and the transitions from one state to another are attractive. Since AA Markov introduced the concepts of this theory; many of the leading mathematicians of the world as Richard A. Howard, John Kemeny and others continued to develop the theory of Markov chains.

Application of z -transformation for determining the limit probability of markovian processes is shown in the article. In the problem of maintenance of machines can also use the z -transformation. Using z -transformation the expression for the probability can be obtained in an analytical form. And also with z -transformation can find exact analytical expressions of probability states.

2. The Use of z - transforms the Problem of Maintenance of Machines

z - Transformation can be defined as follows:

$$f(z) = \sum_{n=0}^{\infty} f(n)z^n \quad (1)$$

Consider a discrete set of integers n as time, then the function $f(n)$ takes arbitrary values of $f(0)$, $f(1)$,

$f(2)$, ... for positive time n , equal to zero for negative.

Function $f(n)$ and its transformation $f(z)$ are related to-one correspondence, for example, the unit step function

$f(n)(=1)$ corresponds to its z - transformation $\frac{1}{1-z}$,

and $n \cdot \alpha^n$ matches $\frac{1 \cdot z}{(1 - \alpha \cdot z)^2}$. Consider the problem

of maintenance of machines [2]. The machine can be serviced periodically, e.g once per hour. At every moment, there are two states. One working, and another - failing (state 2). If a machine fails, it can be restored to full working condition. The probability to remain in state 1 is equal to 0.7, while the probability to move to state 2 is equal to 0.3. And if the machine is in failing state, then the probability of transition to state 1 is equal to 0.6, and stay in failing state is 0.4. The transition matrix in this case is represented as:

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

We obtain:

$$I - z \cdot P = \begin{bmatrix} 1 - \frac{7}{10} \cdot z & -\frac{3}{10} \cdot z \\ -\frac{6}{10} \cdot z & 1 - \frac{4}{10} \cdot z \end{bmatrix} \quad (2)$$

In this expression I is the identity matrix. Next:

$$(I - z \cdot P)^{-1} = \frac{1}{(1 - \frac{4}{10} \cdot z) \cdot (1 - \frac{7}{10} \cdot z) - \frac{18}{100} \cdot z^2} \cdot \begin{bmatrix} 1 - \frac{4}{10} \cdot z & \frac{3}{10} \cdot z \\ \frac{6}{10} \cdot z & 1 - \frac{7}{10} \cdot z \end{bmatrix}$$

So as

$$\left(1 - \frac{4}{10} \cdot z\right) \cdot \left(1 - \frac{7}{10} \cdot z\right) - \frac{18}{100} \cdot z^2 =$$

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$$\begin{aligned}
& 1 - \frac{z}{10} \cdot z - \frac{4}{10} \cdot z + \frac{28}{100} \cdot z^2 - \\
& - \frac{18}{100} \cdot z^2 = 1 - \frac{11z}{10} + \frac{10 \cdot z^2}{100} = \\
& 1 - \frac{1}{10} \cdot z - z + \frac{1}{10} \cdot z^2 = 1 - \frac{1}{10} \cdot z - \\
& - z \cdot \left(1 - \frac{1}{10} \cdot z \right) = \left(1 - \frac{1}{10} \cdot z \right) \cdot (1 - z)
\end{aligned}$$

Then we obtain:

$$(I - z \cdot P)^{-1} = \begin{bmatrix} \frac{1 - \frac{4}{10} \cdot z}{(1 - \frac{1}{10} z)(1 - z)} & \frac{\frac{3}{10} \cdot z}{(1 - \frac{1}{10} z)(1 - z)} \\ \frac{\frac{6}{10} \cdot z}{(1 - \frac{1}{10} z)(1 - z)} & \frac{1 - \frac{7}{10} \cdot z}{(1 - \frac{1}{10} z)(1 - z)} \end{bmatrix}$$

The latter matrix can be represented as the sum of two terms, but until then we make the following:

$$\frac{1 - \frac{4}{10} \cdot z}{\left(1 - \frac{1}{10} \cdot z \right) \cdot (1 - z)} = \frac{A}{1 - \frac{1}{10} \cdot z} + \frac{B}{1 - z}$$

From this we obtain: $A = \frac{1}{3}$, $B = \frac{2}{3}$.

$$\text{Then: } \frac{1 - \frac{4}{10} \cdot z}{\left(1 - \frac{1}{10} \cdot z \right) \cdot (1 - z)} = \frac{\frac{1}{3}}{1 - \frac{1}{10} \cdot z} + \frac{\frac{2}{3}}{1 - z}$$

For the following element of matrix we obtain:

$$\frac{\frac{3}{10} \cdot z}{\left(1 - \frac{1}{10} \cdot z \right) \cdot (1 - z)} = \frac{C}{1 - \frac{1}{10} \cdot z} + \frac{D}{1 - z}$$

From this we find: $C = -\frac{1}{3}$, $D = \frac{1}{3}$.

$$\text{Then: } \frac{\frac{3}{10} \cdot z}{\left(1 - \frac{1}{10} \cdot z \right) \cdot (1 - z)} = \frac{-\frac{1}{3}}{1 - \frac{1}{10} \cdot z} + \frac{\frac{1}{3}}{1 - z}$$

For the next element of transient matrix we obtain:

$$\frac{\frac{6}{10} \cdot z}{\left(1 - \frac{1}{10} \cdot z \right) \cdot (1 - z)} = \frac{K}{1 - \frac{1}{10} \cdot z} + \frac{L}{1 - z}$$

After defining K and L $\left(K = -\frac{2}{3}, L = \frac{2}{3} \right)$ we obtain:

$$\frac{\frac{6}{10} \cdot z}{\left(1 - \frac{1}{10} \cdot z \right) \cdot (1 - z)} = \frac{-\frac{2}{3}}{1 - \frac{1}{10} \cdot z} + \frac{\frac{2}{3}}{1 - z}$$

Now let's calculate the last element:

$$\frac{1 - \frac{7}{10} z}{(1 - \frac{1}{10} z)(1 - z)} = \frac{T}{1 - \frac{1}{10} z} + \frac{M}{1 - z}$$

After finding T and M , $T = \frac{2}{3}$, $M = \frac{1}{3}$, we get the following expression:

$$\frac{1 - \frac{7}{10} z}{(1 - \frac{1}{10} z)(1 - z)} = \frac{\frac{2}{3}}{1 - \frac{1}{10} z} + \frac{\frac{1}{3}}{1 - z}$$

As a result transient matrix is expressed as the sum of two terms:

$$(I - z \cdot P)^{-1} = \begin{bmatrix} \frac{\frac{1}{3}}{1 - \frac{1}{10} z} + \frac{\frac{2}{3}}{1 - z} & \frac{-\frac{1}{3}}{1 - \frac{1}{10} z} + \frac{\frac{1}{3}}{1 - z} \\ \frac{-\frac{2}{3}}{1 - \frac{1}{10} z} + \frac{\frac{2}{3}}{1 - z} & \frac{\frac{2}{3}}{1 - \frac{1}{10} z} + \frac{\frac{1}{3}}{1 - z} \end{bmatrix}$$

Or in the following way:

$$(I - z \cdot P)^{-1} = \frac{1}{1 - \frac{1}{10} \cdot z} \cdot \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} + \frac{1}{1-z} \cdot \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The inverse transform of this matrix, denoted in the form

$F^{-1}(n)$ will be:

$$F^{-1}(n) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} + \frac{1}{10^n} \cdot \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

If the machine is in working condition (state 1), ie $\pi(0) = (1 \ 0)$, then

$$\pi(n) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} + \frac{1}{10^n} \cdot \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

or

$$\pi_1(n) = \frac{2}{3} + \frac{1}{3} \cdot \left(\frac{1}{10}\right)^n, \quad \pi_2(n) = \frac{1}{3} - \frac{1}{3} \cdot \left(\frac{1}{10}\right)^n$$

Moreover $\pi_1(n)$ and $\pi_2(n)$ are the exact analytical expressions of the probabilities of states.

If n increases indefinitely, then $\pi_1(n)$ tends to $\frac{2}{3}$,

and $\pi_2(n)$ tends to $\frac{1}{3}$, that is, they are approaching to the limit state probabilities of the process. If the machine is in state 2, or $\pi(0) = (0 \ 1)$, then

$$\pi(n) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} + \frac{1}{10^n} \cdot \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}, \quad \text{or}$$

$$\pi_1(n) = \frac{2}{3} + \left(\frac{1}{10}\right)^n \cdot \left(-\frac{2}{3}\right), \quad \pi_2(n) = \frac{1}{3} + \left(\frac{1}{10}\right)^n \cdot \frac{2}{3}.$$

You can see that with the growth of n probabilities of the states are approaching to the limit probabilities of the process:

$$\pi_1(n) = \frac{2}{3}, \quad \pi_2(n) = \frac{1}{3}.$$

Thus we see that the marginal probabilities do not depend on the values of the probabilities of the initial state of the process. This is the ergodicity of these processes.

3. Conclusions

With the help of z-transformation we obtain exact analytical expressions of probability. Thus we see that the marginal probabilities in the problem of maintenance of machines do not depend on the values of the probabilities of the initial state of the process. This is the ergodicity of these processes. The following works on Markov processes [1], [6], [10] may be mentioned as good. The problem of maintenance of machinery has important meaning for the industry, as this problem leads to the problem of replacing the equipment. If there is a task of replacing equipments, you will see the following problem - which car to buy? Particularly the problem of maintenance of machinery concerned residents, owners of cars in countries, which do not specialize in the production of machines and equipment for cars. For ergodic chains sufficiently large period of operation comes stationary mode, in which the probabilities of the system states are independent of time and independent of the probability distribution at the initial time. Ergodic dynamical systems are convenient because they can be described by statistical methods with enough observation time. The system passes every state with definite probability. Therefore, this matrix can be constructed from statistics dates. Abstract matrix is not appropriate to be applied in such cases. If the matrix is stochastic, the sum of elements of any line is equal to 1. And any abstract matrix is not required to satisfy this condition.

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