Applied Mathematics 2012, 2(5): 157-165
DOI: 10.5923/j.am.20120205.02

A Study of a Generalization of Ramanujan's Third Order and Sixth Order Mock Theta Functions

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Abstract A generalization of sixth order and third order mock theta functions is given and shown that these generalized functions belong to the class of F_q-functions. Multibasic expansion and q-integral representation of these generalized functions is also given.

Keywords Basic Hypergeometric Series, Mock Theta Functions, q-Integral, Multibasic Expansions

1. Introduction

S. Ramanujan in his last letter to G.H. Hardy[9, pp 354-355] introduced seventeen functions whom he called mock theta functions, as they were not theta functions. He stated two conditions for a function to be a mock theta function:

(0) For every root of unity ζ, there is a θ-function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.

(1) There is no single θ-function which works for all ζ i.e., for every θ-function $\theta(q)$ there is some root of unity ζ for which difference $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

Of the seventeen mock theta functions, four were of third order, ten of fifth order in two groups with five functions in each group and three of seventh order. Ramanujan did not specify what he meant by the order of a mock theta function. Later Watson[15] added three more third order mock theta functions, making the four third order mock theta functions to seven.

G.E. Andrews[1] while visiting Trinity College Cambridge University discovered some notebooks of Ramanujan, and called it the "Lost" Notebook. In the Notebook Andrews found seven more mock theta functions and some identities and Andrews and Hickerson[2] called them of sixth order.

The sixth order mock theta functions of Ramanujan are

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q)_{2n}},$$

$$\Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(-q)_{2n+1}},$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-q)_n}{(q;q^2)_{n+1}},$$

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}} (-q)_n}{(q;q^2)_{n+1}},$$

2000 Mathematics Subject Classification.33D15

Key words and phrases. Basic Hypergeometric series, mock theta functions, q-integral, multibasic expansions.

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q)_n},$$

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q)_n}$$

and $$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2} (q)_n}{(q^3;q^3)_n}.$$

The third order mock theta functions of Ramanujan are

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n},$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n},$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)\ldots\ldots(1-q^n+q^{2n})},$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2},$$

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Published online at http://journal.sapub.org/am

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}$$

We give a generalization of the sixth order and the third order mock theta functions. The generalized sixth order mock theta functions are

$$\text{and } \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)\ldots\ldots(1+q^{2n+1}+q^{4n+2})}.$$

$$\Phi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}, \tag{1.1}$$

$$\Psi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n (t)_n q^{n(n-1)+n\alpha} z^{2n+1} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n+1}}, \tag{1.2}$$

$$\rho(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{\frac{n(n-3)}{2}+n\alpha} z^n (-z;q)_n}{(z^2/q;q^2)_{n+1}}, \tag{1.3}$$

$$\sigma(t,\alpha,z;q) = \frac{1}{2(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{\frac{n(n-1)}{2}+n\alpha} z^{n+1} (-z/q;q)_{n+1}}{(z^2/q;q^2)_{n+1}}, \tag{1.4}$$

$$\lambda(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n (-1)^n q^{n\alpha} (q^3/z^2;q^2)_n}{(-q^2/z;q)_n}, \tag{1.5}$$

$$\mu(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n (-1)^n q^{n(\alpha-1)} (q^3/z^2;q^2)_n}{(-q^2/z;q)_n} \tag{1.6}$$

and

$$\gamma(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n(n-3)+n\alpha} z^{2n}}{(v^2 z;q)_n (v^4 z;q)_n}. \tag{1.7}$$

For $t = 0, \alpha = 1$, we have the generalized functions of Choi[4]. For $z = q^l$ and using induction, these functions satisfy the Ramanujan's requirement for a mock theta function.

The generalized third order mock theta functions are

$$f(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-4n+n\beta} \alpha^n z^{2n}}{(-z;q)_n (-\alpha z/q;q)_n}, \tag{1.8}$$

$$\phi(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-3n+n\beta} z^{2n}}{(-\alpha z^2/q;q^2)_n}, \tag{1.9}$$

$$\psi(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-n+n\beta} z^{2n+1}}{(\alpha z^2/q^2;q^2)_{n+1}}, \tag{1.10}$$

$$\nu(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-2n+n\beta} z^{2n}}{(-\alpha^2 z^2/q^3;q^2)_{n+1}}, \tag{1.11}$$

$$\omega(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{2n^2-5n-4+n\beta} \alpha^{2n} z^{4(n+1)}}{(z^2/q;q^2)_{n+1}(\alpha^2 z^2/q^3;q^2)_{n+1}}, \tag{1.12}$$

$$\chi(t,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-3n+n\beta} z^{2n}}{(vz;q)_n (-v^2 z;q)_n}, \tag{1.13}$$

and

$$\rho(t,\beta,z;q) = \frac{z^4}{q^4}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{2n^2-3n+n\beta} z^{4n}}{(v^2 z^2/q;q^2)_{n+1}(v^{-2} z^2/q;q^2)_{n+1}}. \tag{1.14}$$

where

$$v = e^{\frac{\pi i}{3}}.$$

For $\beta = 1$ and $z = q$ we have generalized five third order mock theta functions namely f, ϕ, ψ, ν, ω of Andrews[1]. For $t = 0, \beta = 1$, $\alpha = q$ and $z = q$ the generalized functions f, ϕ, ψ and χ reduce to the third order mock theta functions of Ramanujan and ω, ν and ρ to the third order mock theta functions of Watson[15].

In this study we will show that these generalized functions are F_q-functions. This is done in section 3.

Using the difference equation we derive relations between generalized sixth order mock theta functions and relation between generalized third order mock theta functions. This we do in section 4. In section 5 we give a q-integral representation and in section 6, we give multibasic expansion for these generalized functions.

2. Notations

We shall use the following usual basic hypergeometric notations:

For $|q^k| < 1$,

$$(a;q^k)_n = (1-a)(1-aq^k)\cdots(1-aq^{k(n-1)}), \quad n \geq 1$$

$$(a;q^k)_0 = 1, \quad (a;q^k)_\infty = \prod_{j=0}^{\infty}(1-aq^{kj}),$$

$$\phi\begin{bmatrix} a_1,\cdots,a_r : c_{1,1},\cdots,c_{1,r_1} : \cdots : c_{m,1},\cdots,c_{m,r_m} \\ b_1,\cdots,b_s : e_{1,1},\cdots,e_{1,s_1} : \cdots : e_{m,1},\cdots,e_{m,s_m} \end{bmatrix}; q, q_1,\cdots,q_m; z$$

$$= \sum_{n=0}^{\infty}\left\{\begin{array}{l} \dfrac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n} z^n \left[(-1)^n q^{\frac{n^2-n}{2}}\right]^{1+s-r} \\ \displaystyle\prod_{j=1}^{m} \frac{\left(c_{j,1},\cdots,c_{j,r_j};q_j\right)_n}{\left(e_{j,1},\cdots,e_{j,s_j};q_j\right)_n}\left[(-1)^n q_j^{\frac{n^2-n}{2}}\right]^{s_j-r_j} \end{array}\right\}.$$

$${}_A\phi_{A-1}[a_1,a_2\cdots,a_A;b_1,b_2\cdots,b_{A-1};q_1,z]$$

$$= \sum_{n=0}^{\infty}\frac{(a_1;q_1)_n\cdots(a_A;q_1)_n z^n}{(b_1;q_1)_n\cdots(b_{A-1};q_1)_n(q_1;q_1)_n}, \quad |z| < 1.$$

3. The Generalized Functions are F_q-Functions

We show these generalized functions are F_q - functions.

Theorem 1

The generalized functions $\Phi(t,\alpha,z;q), \psi(t,\alpha,z;q), \rho(t,\alpha,z;q), \gamma(t,\alpha,z;q), \sigma(t,\alpha,z;q)$ and the generalized functions $f(t,\alpha,\beta,z;q), \phi(t,\alpha,\beta,z;q), \psi(t,\alpha,\beta,z;q), \nu(t,\alpha,\beta,z;q)$, $\chi(t,\beta,z;q)$, $\rho(t,\beta,z;q)$, $\omega(t,\alpha,\beta,z;q)$ are F_q - functions.

Proofs

We shall give the proof for $\Phi(t,\alpha,z;q)$ only. The proofs for the other functions are similar, hence omitted.

Applying the difference operator $D_{q,t}$ to $\Phi(t,\alpha,z;q)$,we have

$$tD_{q,t}\Phi(t,\alpha,z;q) = \Phi(t,\alpha,z;q) - \Phi(tq,\alpha,z;q)$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-3)+n\alpha}z^{2n}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}$$

$$-\frac{1}{(tq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(tq)_n q^{n(n-3)+n\alpha}z^{2n}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-3)+n\alpha}z^{2n}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}$$

$$-\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-3)+n\alpha}z^{2n}(z^2/q;q^2)_n(1-tq^n)}{(-z^2/q;q)_{2n}}$$

$$= \frac{t}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-3)+n(\alpha+1)}z^{2n}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}$$

$$= t\Phi(t,\alpha+1,z;q).$$

So

$$D_{q,t}\Phi(t,\alpha,z;q) = \Phi(t,\alpha+1,z;q).$$

Hence $\Phi(t,\alpha,z;q)$ is a F_q - function.

As stated earlier the proofs for other functions are similar, so omitted.

4. Relations between the Generalized Functions of Sixth Order Mock Theta Functions and Relations between Generalized Functions of Third Order Mock Theta Functions

Theorem 2

(i) $\Phi(t,\alpha,z;q) = \dfrac{1}{(t)_\infty}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-3)+n\alpha}z^{2n}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n+1}} + \frac{z}{q}\Psi(t,\alpha,z;q).$

(ii) $\sigma(t,\alpha,z;q) = \dfrac{z}{2}\left(1+\dfrac{z}{q}\right)D_{q,t}\rho(t,\alpha,z;q).$

(iii) $D_{q,t}\,\phi(t,\alpha^2,\beta,z;q) = \left(1+\dfrac{\alpha^2z^2}{q^3}\right)\nu(t,\alpha,\beta,z;q).$

(iv) $\psi\left(t,\dfrac{-\alpha^2}{q},\beta,z;q\right) = zD_{q,t}\,\nu(t,\alpha,\beta,z;q).$

Proof of (i)

$$\Phi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-3)+n\alpha}z^{2n}(z^2/q;q^2)_n(1+z^2q^{2n-1})}{(-z^2/q;q)_{2n+1}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-3)+n\alpha}z^{2n}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n+1}}$$

$$+\frac{z}{q}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-1)+n\alpha}z^{2n+1}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n+1}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n(n-3)+n\alpha}z^{2n}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n+1}} + \frac{z}{q}\Psi(t,\alpha,z;q),$$

which proves Theorem 2 (i).

Proof of (ii)

$$\sigma(t,\alpha,z;q) = \frac{z\left(1+\frac{z}{q}\right)}{2(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n-1)}{2}+n\alpha} z^n (-z;q)_n}{(z^2/q;q^2)_{n+1}} = \frac{z}{2}\left(1+\frac{z}{q}\right)D_{q,t}\rho(t,\alpha,z;q),$$

which proves Theorem 2 (ii).

Proof of (iii)

Writing α^2 for α in $\phi(t,\alpha,\beta,z;q)$, we have

$$D_{q,t}\phi(t,\alpha^2,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-2n+n\beta} z^{2n}}{(-\alpha^2 z^2/q;q^2)_n} = \left(1+\frac{\alpha^2 z^2}{q^3}\right)\nu(t,\alpha,\beta,z;q),$$

which proves Theorem 2 (iii).

Proof of (iv)

Writing $\frac{-\alpha}{q}$ for α and then α^2 for α in $\psi(t,\alpha,\beta,z;q)$ in (1.8), we have

$$\psi\left(t,\frac{-\alpha^2}{q},\beta,z;q\right) = \frac{z}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-n+n\beta} z^{2n}}{(-\alpha^2 z^2/q^3;q^2)_{n+1}} = zD_{q,t}\nu(t,\alpha,\beta,z;q)$$

which proves Theorem 2 (iv).

5. q-Integral Representation for the Generalized Functions of Sixth and Third Order Mock Theta Functions

The q-integral was defined by Thomae and Jackson[7, p. 19] as

$$\int_0^1 f(t)d_q t = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n.$$

We now give the q-integral representation for the generalized sixth order mock theta functions and also for generalized third order mock theta functions.

Theorem 3(a)

(i) $\Phi(q^t,\alpha,z;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty}\displaystyle\int_0^1 w^{t-1}(wq;q)_\infty \Phi(0,aw,z;q)d_q w.$

(ii) $\Psi(q^t,\alpha,z;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty}\displaystyle\int_0^1 w^{t-1}(wq;q)_\infty \Psi(0,aw,z;q)d_q w.$

(iii) $\rho(q^t,\alpha,z;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty}\displaystyle\int_0^1 w^{t-1}(wq;q)_\infty \rho(0,aw,z;q)d_q w.$

(iv) $\gamma(q^t,\alpha,z;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty}\displaystyle\int_0^1 w^{t-1}(wq;q)_\infty \gamma(0,aw,z;q)d_q w.$

(v) $\sigma(q^t,\alpha,z;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty}\displaystyle\int_0^1 w^{t-1}(wq;q)_\infty \sigma(0,aw,z;q)d_q w.$

Proof

We shall give the detailed proof for $\Phi(q^t,\alpha,z;q)$. The proofs for the other functions are similar, so omitted.

Limiting case of q-beta integral[7, p. 19 (1.11.7)] is

$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{x-1}(tq;q)_\infty d_q t. \tag{5.1}$$

Now

$$\Phi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n q^{n(n-3)+n\alpha} z^{2n}(z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}$$

Replacing t by q^t and q^α by a, we have

$$\Phi(q^t,\alpha,z;q) = \frac{1}{(q^t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (q^t)_n q^{n(n-3)+n\alpha} z^{2n} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-3)+n\alpha} z^{2n} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n} (q^{n+t})_\infty}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-3)+n\alpha} z^{2n} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n}} \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{n+t-1} (wq;q)_\infty d_q w , \qquad \text{by (5.1)}$$

$$= \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-3)} z^{2n} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n}} (aw)^n d_q w. \tag{5.2}$$

But

$$\Phi(0,\alpha,z;q) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-3)+n\alpha} z^{2n} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}$$

and since $q^\alpha = a$,

$$\Phi(0,a,z;q) = \sum_{n=0}^{\infty}\frac{(-1)^n (a)^n q^{n(n-3)} z^{2n} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}.$$

Hence

$$\Phi(0,aw,z;q) = \sum_{n=0}^{\infty}\frac{(-1)^n (aw)^n q^{n(n-3)} z^{2n} (z^2/q;q^2)_n}{(-z^2/q;q)_{2n}}. \tag{5.3}$$

by using (5.3), (5.2) can be written as

$$\Phi(q^t,\alpha,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty \Phi(0,aw,z;q) d_q w,$$

which proves (i). The proofs for all other functions are similar.

Theorem 3(b)

The q-integral representation for the generalized third order mock theta functions:

(i) $f(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty f(0,\alpha,aw,z;q) d_q w.$

(ii) $\phi(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty \phi(0,\alpha,aw,z;q) d_q w.$

(iii) $\psi(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty \psi(0,\alpha,aw,z;q) d_q w.$

(iv) $\nu(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty \nu(0,\alpha,aw,z;q) d_q w.$

(v) $\chi(q^t,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty \chi(0,aw,z;q) d_q w.$

(vi) $\rho(q^t,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty \rho(0,aw,z;q) d_q w.$

(vii) $\omega(q^t,\alpha,\beta,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1} (wq;q)_\infty \omega(0,\alpha,aw,z;q) d_q w.$

Proof

The proofs are similar to given above for $\Phi(q^t,\alpha,z;q)$, so the Theorem 3(b) follows.

6. Multibasic Expansions of Generalized Functions of Sixth and Third Order Mock Theta Functions

Using the summation formula[7, (3.6.7), p. 71] and[8, Lemma 10, p. 57], we have the multibasic expansion

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_kq^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k}\sum_{m=0}^{\infty}\alpha_{m+k} = \sum_{m=0}^{\infty}\frac{(ap,bp;p)_m(cq,aq/bc;q)_m}{(q,aq/b;q)_m(ap/c,bcp;p)_m}\alpha_m. \quad (6.1)$$

Corollary 1

Letting $q \to q^2$ and $c \to \infty$ in (6.1) we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{2k})(1-bp^kq^{-2k})(a,b;p)_kq^{k^2+k}}{(1-a)(1-b)(q^2,aq^2/b;q^2)_kb^kp^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k} = \sum_{m=0}^{\infty}\frac{(ap,bp;p)_mq^{m^2+m}}{(q^2,aq^2/b;q^2)_mb^mp^{\frac{m^2+m}{2}}}\alpha_m. \quad (6.2)$$

Corollary 2

Letting $q \to q^3$ and $c \to \infty$ in (6.1) we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{3k})(1-bp^kq^{-3k})(a,b;p)_kq^{\frac{3k^2+3k}{2}}}{(1-a)(1-b)(q^3,aq^3/b;q^3)_kb^kp^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k} = \sum_{m=0}^{\infty}\frac{(ap,bp;p)_mq^{\frac{3m^2+3m}{2}}}{(q^3,aq^3/b;q^3)_mb^mp^{\frac{m^2+m}{2}}}\alpha_m. \quad (6.3)$$

Corollary 3

Letting $q \to q^5$ and $c \to \infty$ in (6.1) we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{5k})(1-bp^kq^{-5k})(a,b;p)_kq^{\frac{5k^2+5k}{2}}}{(1-a)(1-b)(q^5,aq^5/b;q^5)_kb^kp^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k} = \sum_{m=0}^{\infty}\frac{(ap,bp;p)_mq^{\frac{5m^2+5m}{2}}}{(q^5,aq^5/b;q^5)_mb^mp^{\frac{m^2+m}{2}}}\alpha_m. \quad (6.4)$$

In the following Theorem we now give multibasic expansions for generalized functions of sixth order mock theta functions and third order mock theta functions. We give detailed proof for $\Phi(t,\alpha,z;q)$, and for other functions we write only the specialized parameters.

Theorem 4(a)

The expansions for generalized functions of sixth order mock theta functions.

$$\text{(i)}\Phi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}(z^2/q;q^2)_kq^{k^2-3k+k\alpha}z^{2k}}{(1-q^{k+2})(-z^2/q;q)_{2k}}$$
$$\times\phi\left[\begin{matrix}q,0:z^2q^{2k-1},0:tq^{3k},q^{3k+3}:\\q^{k+3}:-z^2q^{2k-1},-z^2q^{2k}:0,0:\end{matrix};q,q^2,q^3;-z^2q^{\alpha-2}\right].$$

$$\text{(ii)}\Psi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}(z^2/q;q^2)_kq^{k^2-k+k\alpha}z^{2k+1}}{(1-q^{k+2})(-z^2/q;q)_{2k+1}}$$
$$\times\phi\left[\begin{matrix}q,0:z^2q^{2k-1},0:tq^{3k},q^{3k+3}:\\q^{k+3}:-z^2q^{2k},-z^2q^{2k+1}:0,0:\end{matrix};q,q^2,q^3;-z^2q^{\alpha}\right].$$

$$\text{(iii)}\rho(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{3k-1})(1-q^{-k+2})(t;q)_{k-1}(-z;q)_kq^{\frac{k^2-3k}{2}+k\alpha}z^k}{(1-q^{k+2})(z^2/q;q^2)_{k+1}}$$
$$\times\phi\left[\begin{matrix}q,-zq^k:tq^{2k-1},q^{2k+2}:\\q^{k+3}:z^2q^{2k+1},0:\end{matrix};q,q^2;zq^{\alpha}\right].$$

$$\text{(iv)}\gamma(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}q^{k^2-3k+k\alpha}z^{2k}}{(1-q^{k+2})(v^2z;q)_k(v^4z;q)_k}$$
$$\times\phi\left[\begin{matrix}q,0,0,0:tq^{3k},q^{3k+3}:\\q^{k+3},v^2zq^k,v^4zq^k:0,0:\end{matrix};q,q^3;z^2q^{\alpha-2}\right].$$

$$\text{(v)}\sigma(t,\alpha,z;q) = \frac{1}{2(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{3k-1})(1-q^{-k+1})(t;q)_{k-1}(-z/q;q)_{k+1}q^{\frac{k^2-k}{2}+k\alpha}z^{k+1}}{(1-q^{k+1})(z^2/q;q^2)_{k+1}}$$
$$\times\phi\left[\begin{matrix}q,-zq^k:tq^{2k},q^{2k+2}:\\q^{k+2}:z^2q^{2k+1},0:\end{matrix};q,q^2;zq^{\alpha}\right].$$

Proof of (i)

Take $a = \frac{t}{q}$, $b = q^2, p = q$ and $\alpha_m = \frac{(-1)^mq^{m\alpha-2m}(q^3;q^3)_m(t;q^3)_m(z^2/q;q^2)_mz^{2m}}{(q^3;q)_m(-z^2/q;q)_{2m}}$

in (6.3) to get

$$\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t/q,q^2;q)_kq^{\frac{3k^2+3k}{2}}}{(1-t/q)(1-q^2)(t,q^3;q^3)_kp^{\frac{k^2+5k}{2}}}$$
$$\times\sum_{m=0}^{\infty}\frac{(-1)^{m+k}(q^3;q^3)_{m+k}(t;q^3)_{m+k}z^{2(m+k)}(z^2/q;q^2)_{m+k}q^{(m+k)\alpha-2(m+k)}}{(q^3;q)_{m+k}(-z^2/q;q)_{2m+2k}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2-3m+m\alpha}(t;q)_m (z^2/q\,;q^2)_m z^{2m}}{(-z^2/q\,;q)_{2m}}.$$

The right hand side of (6.5) is equal to

$$(t;q)_\infty \Phi(t,\alpha,z;q)$$

The left hand side of (6.5) is equal to

$$\sum_{k=0}^{\infty} \frac{(-1)^k(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}(z^2/q\,;q^2)_k q^{k^2-3k+k\alpha} z^{2k}}{(1-q^{k+2})(-z^2/q;q)_{2k}}$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m (q^{3k+3};q^3)_m (tq^{3k};q^3)_m (z^2q^{2k-1};q^2)_m q^{m\alpha-2m} z^{2m}}{(q^{k+3};q)_m (-z^2q^{2k-1};q^2)_m (-z^2q^{2k};q^2)_m}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}(z^2/q;q^2)_k q^{k^2-3k+k\alpha} z^{2k}}{(1-q^{k+2})(-z^2/q;q)_{2k}}$$

$$\times \phi\left[\begin{matrix} q,0:z^2q^{2k-1},0:tq^{3k},q^{3k+3}: \\ q^{k+3}:-z^2q^{2k-1},-z^2q^{2k}:0,0: \end{matrix}; q,q^2,q^3;-z^2q^{\alpha-2}\right],$$

which proves Theorem 4(a)(i).

Proof of (ii)

Take $a = \frac{t}{q}, b = q^2, p = q$ and $\alpha_m = \frac{(-1)^m q^{m\alpha}(q^3;q^3)_m (t;q^3)_m (z^2/q\,;q^2)_m z^{2m+1}}{(q^3;q)_m (-z^2/q\,;q)_{2m+1}}$ in (6.3)

Proof of (iii)

Take $a = \frac{t}{q}, b = q^2, p = q$ and $\alpha_m = \frac{q^{m\alpha}(q^2;q^2)_m (t/q\,;q^2)_m (-z;q)_m z^m}{(q^3;q)_m (z^2/q\,;q^2)_{m+1}}$ in (6.2)

Proof of (iv)

Take $a = \frac{t}{q}, b = q^2, p = q$ and $\alpha_m = \frac{q^{m(\alpha-2)}(q^3;q^3)_m (t;q^3)_m z^{2m}}{(q^3;q)_m (\nu^2 z;q)_m (\nu^4 z;q)_m}$ in (6.3)

Proof of (v)

Take $a = \frac{t}{q}, b = q, p = q$ and $\alpha_m = \frac{q^{m\alpha}(q^2;q^2)_m (t;q^2)_m (-z/q\,;q)_{m+1} z^{m+1}}{(q^2;q)_m (z^2/q\,;q^2)_{m+1}}$ in (6.2)

Theorem 4(b)

The expansions for generalized functions of mock theta functions of third order.

(i) $$f(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty} \sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1} q^{k^2-4k+k\beta} \alpha^k z^{2k}}{(1-q^{k+2})(-z;q)_k (-z\alpha/q;q)_k}$$

$$\times \phi\left[\begin{matrix} q,0,0,0:tq^{3k},q^{3k+3}: \\ q^{k+3},-zq^k,-z\alpha q^{k-1}:0,0: \end{matrix}; q,q^3;\alpha z^2 q^{\beta-3}\right].$$

(ii) $$\phi(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty} \sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1} q^{k^2-3k+k\beta} z^{2k}}{(1-q^{k+2})(-\alpha z^2/q\,;q^2)_k}$$

$$\times \phi\left[\begin{matrix} q,0:0:tq^{3k},q^{3k+3}: \\ q^{k+3}:-\alpha z^2 q^{2k-1}:0,0: \end{matrix}; q,q^2,q^3;z^2q^{\beta-2}\right].$$

(iii) $$\psi(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty} \sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1} q^{k^2-k+k\beta} z^{2k+1}}{(1-q^{k+2})(\alpha z^2/q^2;q^2)_{k+1}}$$

$$\times \phi\left[\begin{matrix} q,0:0:tq^{3k},q^{3k+3}: \\ q^{k+3}:\alpha z^2 q^{2k}:0,0: \end{matrix}; q,q^2,q^3;z^2q^{\beta}\right].$$

(iv) $$\nu(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty} \sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1} q^{k^2-2k+k\beta} z^{2k}}{(1-q^{k+2})(-\alpha^2 z^2/q^3;q^2)_{k+1}}$$

$$\times \phi\left[\begin{matrix} q,0:0:tq^{3k},q^{3k+3}: \\ q^{k+3}:-\alpha^2 z^2 q^{2k-1}:0,0: \end{matrix}; q,q^2,q^3;z^2q^{\beta-1}\right].$$

(v) $$\chi(t,\beta,z;q) = \frac{1}{(t)_\infty} \sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+1})(t;q)_{k-1} q^{k^2-3k+k\beta} z^{2k}}{(1-q^{k+1})(\nu z;q)_k (-\nu^2 z;q)_k}$$

$$\times \phi\left[\begin{matrix} q,0,0,0:tq^{3k+1},q^{3k+3}: \\ q^{k+2},\nu z q^k,-\nu^2 z q^k:0,0: \end{matrix}; q,q^3;z^2q^{\beta-3}\right].$$

$$(\text{vi})\rho(t,\beta,z;q)=\frac{z^4}{q^4(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+4})(t;q)_{k-1}q^{2k^2-3k+k\beta}z^{4k}}{(1-q^{k+4})(v^2z^2/q;q^2)_{k+1}(v^{-2}z^2/q;q^2)_{k+1}}$$
$$\times\phi\left[\begin{matrix}q,0:0,0:tq^{5k},q^{5k+5}:\\ q^{k+5}:v^2z^2q^{2k+1},v^{-2}z^2q^{2k+1}:0,0:\end{matrix};q,q^2,q^5;z^4q^{\beta-1}\right].$$

$$(\text{vii})\ \omega(t,\alpha,\beta,z;q)=\frac{z^4}{q^4(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+4})(t;q)_{k-1}q^{2k^2-5k+k\beta}\alpha^{2k}z^{4k}}{(1-q^{k+4})(z^2/q;q^2)_{k+1}(\alpha^2z^2/q^3;q^2)_{k+1}}$$
$$\times\phi\left[\begin{matrix}q,0:0,0:tq^{5k},q^{5k+5}:\\ q^{k+5}:z^2q^{2k+1},\alpha^2z^2q^{2k-1}:0,0:\end{matrix};q,q^2,q^5;\alpha^2z^4q^{\beta-3}\right].$$

Proof of (i)

Take $a=\frac{t}{q}, b=q^2, p=q$ and $\alpha_m=\frac{q^{m\beta-3m}(q^3;q^3)_m(t;q^3)_m\alpha^m z^{2m}}{(q^3;q)_m(-z;q)_m(-\alpha z/q;q)_m}$ in (6.3)

Proof of (ii)

Take $a=\frac{t}{q}, b=q^2, p=q$ and $\alpha_m=\frac{q^{m\beta-2m}(q^3;q^3)_m(t;q^3)_m z^{2m}}{(q^3;q)_m(-\alpha z^2/q;q^2)_m}$ in (6.3)

Proof of (iii)

Take $a=\frac{t}{q}, b=q^2, p=q$ and $\alpha_m=\frac{q^{m\beta}(q^3;q^3)_m(t;q^3)_m z^{2m}}{(q^3;q)_m(vz;q)_m(\alpha z^2/q^2;q^2)_{m+1}}$ in (6.3)

Proof of (iv)

Take $a=\frac{t}{q}, b=q^2, p=q$ and $\alpha_m=\frac{q^{m\beta-m}(q^3;q^3)_m(t;q^3)_m z^{2m}}{(q^3;q)_m(-\alpha^2z^2/q^3;q^2)_{m+1}}$ in (6.3)

Proof of (v)

Take $a=\frac{t}{q}, b=q, p=q$ and $\alpha_m=\frac{q^{m\beta-3m}(q^3;q^3)_m(tq;q^3)_m z^{2m}}{(q^2;q)_m(vz;q)_m(-v^2z;q)_m}$ in (6.3)

Proof of (vi)

Take $a=\frac{t}{q}, b=q^4, p=q$ and $\alpha_m=\frac{q^{m\beta-m}(q^5;q^5)_m(t;q^5)_m z^{4m}}{(q^5;q)_m(v^2z^2/q;q^2)_{m+1}(v^{-2}z^2/q;q^2)_{m+1}}$ in (6.4)

Proof of (vii)

Take $a=\frac{t}{q}, b=q^4, p=q$ and $\alpha_m=\frac{q^{m\beta-3m}(q^5;q^5)_m(t;q^5)_m\alpha^{2m}z^{4m}}{(q^5;q)_m(z^2/q;q^2)_{m+1}(\alpha^2z^2/q^3;q^2)_{m+1}}$ in (6.4)

7. Conclusions

Mock theta functions are mysterious functions. These investigations will be helpful in understanding more about these functions. Being shown that they belong to the class of F_q-functions more properties can be established and relations between these mock theta functions can also be derived.

ACKNOWLEDGEMENTS

I am thankful to Dr. Bhaskar Srivastava for his help and guidance.

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