

Exact Elliptic Solution for Non-Linear Klein-Gordon Equation Via Auxiliary Equation Method

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Abstract By using symbolic computation, we apply Auxiliary equation method to construct exact solutions of Non-Linear Klein-Gordon equation. We show that Auxiliary equation method provides a powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

Keywords Traveling Wave, Exact Solutions And Auxiliary Equation

1. Introduction

The investigation of the travelling wave solutions for non-linear partial differential equations plays an important role in the study of non-linear physical phenomena. Non-linear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Non-linear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in non-linear wave equations. In recent years, new exact solutions may help to find new phenomena. A variety of powerful methods, such as inverse scattering method[1, 2], bilinear transformation[3], the tanh-sech method[4-6], extended tanh method[7,8], sine-cosine method[9,10], homogeneous balance method[11], Exp-function method[12,13], improved tanh-function method[14] and Auxiliary equation method[15] were used to develop non-linear dispersive and dissipative problems.

2. Auxiliary Equation Method

Consider a given nonlinear wave equation

$$N(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \quad (2.1)$$

we seek its wave solutions.

$$\begin{aligned} u &= U(\eta), \quad v = V(\eta), \\ \eta &= k_1 x + k_2 y + k_3 z - \omega t. \end{aligned} \quad (2.2)$$

Consequently, (1.1) is reduced to the ordinary differential equation (ODE):

$$U(u, u', u'', \dots) = 0 \quad (2.3)$$

Auxiliary equation method is based on the assumption that the travelling wave solutions can be expressed in the following form

$$u(\eta) = \sum_{i=0}^{\infty} a_i [\varphi(\eta)]^i \quad (2.4)$$

where $\varphi(\eta)$ satisfies Auxiliary equation method

$$\varphi'^2(\eta) = R + Q\varphi^2(\eta) + P\varphi^4(\eta). \quad (2.5)$$

3. Non-Linear Klein-Gordon Equation

We study the following well-known the nonlinear Klein-Gordon equation:

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} + cu - r|u|^2 u = 0, \quad (3.1)$$

for $c > 0, r < 0$.

To Higgs equation and for $c = 0$, it assumes the form of Yang-Milles equation. The Klein-Gordon equation has been studied in many literatures, e.g., [16-18]. However, numerical treatment for Klein-Gordon equation is rarely reported (cf. [19-21]). Particularly, [20] concern with the decomposition method and difference method used in [21] nonlinear problems. We take $u = U + iV$, then by used the transformation (2.2) we get:

$$\begin{aligned} (\omega^2 - k_1^2 - k_2^2 - k_3^2)U'' + cU - r(U^3 + V^3U) &= 0, \\ (\omega^2 - k_1^2 - k_2^2 - k_3^2)V'' + cV - r(V^3 + U^3V) &= 0. \end{aligned} \quad (3.2)$$

Now balancing U'' with U^3 and V'' with U^3V gives $M=1$, $N=1$. Therefore we may choose:

$$\begin{aligned} u(\eta) &= a_0 + a_1\varphi + a_2\varphi^2, \\ v(\eta) &= b_0 + b_1\varphi + b_2\varphi^2. \end{aligned} \quad (3.3)$$

Substituting from equations (3.3) in equation (3.1), collect the coefficients of $\varphi(\eta)^i$ ($i = 0, \dots, 3$) and equated them to zero we obtain the system

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Published online at <http://journal.sapub.org/am>

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$$\begin{aligned}
rb_2^3 + rb_2a_2^2 &= 0, \\
a_2^3 + ra_2b_2^2 &= 0, \\
3rb_1b_2^2 + rb_1a_2^2 + 2rb_2a_1a_2 &= 0, \\
ra_0b_2^2 + 2ra_1b_1b_2 + ra_2(2b_0b_2 + b_1^2) &= 0, \\
2ra_2b_0b_1 + r(4a_0a_1a_2 + a_1(2a_0a_2 + a_1^2)) &= 0, \\
2(a_1^2a_0 + a_2a_0^2) + 4(w^2 - k_1^2 - k_2^2 - k_3^2)a_1Q &= 0, \\
2(w^2 - k_1^2 - k_2^2 - k_3^2)a_2R + ca - ra_0b_2 - ra_0^3 &= 0, \\
2rb_2a_0a_1 + r(4b_0b_1b_2 + b_1(2b_0b_2 + b_1^2)) &= 0, \\
6(w^2 - k_1^2 - k_2^2 - k_3^2)b_2P - r(b_0b_2^2 + 2b_1^2b_2) &= 0, \\
2(w^2 - k_1^2 - k_2^2 - k_3^2)b_2R + cb_0 - rb_0a_0^2 - rb_0^3 &= 0, \\
b_2(2b_0b_2 + b_1^2) - rb_0a_2^2 - 2rb_1a_1a_2 - rb_2 & \\
(2a_0a_2 + a_1^2) &= 0, \\
2ra_0b_0b_1 - ra_1b_0^2 + (w^2 - k_1^2 - k_2^2 - k_3^2)a_1Q & \\
+ ca_1 - 3ra_0^2a_1 &= 0, \\
2ra_0b_0b_1 + ra_1b_0^2 - (w^2 - k_1^2 - k_2^2 - k_3^2)a_1Q & \\
- ca_1 + 3ra_0^2a_1 &= 0, \\
ca_2 - ra_0(2b_0b_2 + b_1^2) - 2ra_1b_0b_1 - ra_2b_0^2 & \\
- ra_0(2a_0a_2 + a_1^2) &= 0, \\
6(w^2 - k_1^2 - k_2^2 - k_3^2)a_2P - r(a_0a_2^2 + 2a_1^2a_2 & \\
+ a_2(2a_0a_2 + a_1^2)) &= 0.
\end{aligned} \tag{3.4}$$

Solving the system of algebraic equations with the aid of Maple, in Eq.(3.3), we obtain the following results:

$$\begin{aligned}
a_0 &= a_2 = b_0 = b_2 = 0, \\
a_1^2 + b_1^2 &= \frac{-2cP}{rQ}, \\
c &= -(w^2 - k_1^2 - k_2^2 - k_3^2)Q,
\end{aligned} \tag{3.5}$$

Substituting these results into (3.3) and with the aid of Appendix A, we obtain the following multiple soliton-like and triangular periodic solutions for (DSW) system:

$$U_1 = \sqrt{\frac{2cm^2}{r(1+m^2)}} - b_1^2 \operatorname{sn}(\eta), \quad V_1 = b_1 \operatorname{sn}(\eta). \tag{3.6}$$

$$U_2 = \sqrt{\frac{2cm^2}{r(m^2-1)}} - b_1^2 \operatorname{sn}(\eta), \quad V_2 = b_1 \operatorname{sn}(\eta). \tag{3.7}$$

$$U_3 = \sqrt{\frac{2c}{r(2-m^2)}} - b_1^2 \operatorname{dn}(\eta), \quad V_3 = b_1 \operatorname{dn}(\eta). \tag{3.8}$$

$$U_4 = \sqrt{\frac{2c}{r(1-2m^2)}} - b_1^2 \operatorname{ds}(\eta), \quad V_4 = b_1 \operatorname{ds}(\eta). \tag{3.9}$$

$$U_5 = \sqrt{\frac{2c}{r(m^2-2)}} - b_1^2 \operatorname{cs}(\eta), \quad V_5 = b_1 \operatorname{cs}(\eta). \tag{3.10}$$

$$U_6 = \sqrt{\frac{cm^2}{r(2-m^2)}} - b_1^2 \left(\frac{\operatorname{sn}(\eta)}{1 \pm \operatorname{dn}(\eta)} \right), \quad V_6 = b_1 \left(\frac{\operatorname{sn}(\eta)}{1 \pm \operatorname{dn}(\eta)} \right). \tag{3.11}$$

$$\begin{aligned}
U_7 &= \sqrt{\frac{cm^2}{r(2-m^2)}} - b_1^2 (\operatorname{sn}(\eta) \pm i \operatorname{cn}(\eta)), \\
V_7 &= b_1 (\operatorname{sn}(\eta) \pm i \operatorname{cn}(\eta)).
\end{aligned} \tag{3.12}$$

$$U_8 = \sqrt{\frac{c}{r(2m^2-1)}} - b_1^2 \left(\frac{\operatorname{sn}(\eta)}{1 \pm \operatorname{dn}(\eta)} \right), \quad V_8 = b_1 \left(\frac{\operatorname{sn}(\eta)}{1 \pm \operatorname{dn}(\eta)} \right). \tag{3.13}$$

$$U_9 = \sqrt{\frac{c(1-m^2)}{r(1+m^2)}} - b_1^2 \left(\frac{\operatorname{dn}(\eta)}{1 \pm m \operatorname{sn}(\eta)} \right), \quad V_9 = b_1 \left(\frac{\operatorname{dn}(\eta)}{1 \pm m \operatorname{sn}(\eta)} \right). \tag{3.14}$$

$$\begin{aligned}
U_{10} &= \sqrt{\frac{c(1-m^2)}{r(1+m^2)}} - b_1^2 \left(\frac{\operatorname{sn}(\eta)}{1 \pm \operatorname{sn}(\eta)} \right), \\
V_{10} &= b_1 \left(\frac{\operatorname{sn}(\eta)}{1 \pm \operatorname{sn}(\eta)} \right).
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
U_{11} &= \sqrt{\frac{c}{r(m^2+1)}} - b_1^2 (m \operatorname{cn}(\eta) \pm \operatorname{dn}(\eta)), \\
V_{11} &= b_1 (m \operatorname{cn}(\eta) \pm \operatorname{dn}(\eta)).
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
U_{12} &= \sqrt{\frac{-c(1-m^2)}{r(1+m^2)}} - b_1^2 \left(\frac{\operatorname{sn}(\eta)}{\operatorname{dn}(\eta) \pm \operatorname{cn}(\eta)} \right), \\
V_{12} &= b_1 \left(\frac{\operatorname{sn}(\eta)}{\operatorname{dn}(\eta) \pm \operatorname{cn}(\eta)} \right).
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
U_{13} &= \sqrt{\frac{cm^2}{r(2-m^2)}} - b_1^2 \left(\frac{\operatorname{cn}(\eta)}{\sqrt{1-m^2 \pm \operatorname{dn}(\eta)}} \right), \\
V_{13} &= b_1 \left(\frac{\operatorname{cn}(\eta)}{\sqrt{1-m^2 \pm \operatorname{dn}(\eta)}} \right).
\end{aligned} \tag{3.18}$$

Some soliton solutions of Eq. (3.1) can be obtained in the limited case when the modulus $m \rightarrow 1$ (see Appendix B)[22], as follows.

$$U_1^1 = \sqrt{\frac{c}{r}} - b_1^2 \tanh(\eta), \quad V_1^1 = b_1 \tanh(\eta). \tag{3.19}$$

$$U_3^1 = \sqrt{\frac{2c}{r}} - b_1^2 \operatorname{sech}(\eta), \quad V_3^1 = b_1 \operatorname{sech}(\eta). \tag{3.20}$$

$$U_4^1 = \sqrt{b_1^2 - \frac{2c}{r}} \operatorname{csch}(\eta), \quad V_4^1 = b_1 \operatorname{csch}(\eta). \tag{3.21}$$

$$U_5^1 = \sqrt{b_1^2 - \frac{2c}{r}} \operatorname{csch}(\eta), \quad V_5^1 = b_1 \operatorname{csch}(\eta). \tag{3.22}$$

Some trigonometric-function solutions of Eq. (3.1) can be obtained in the limited case when the modulus $m \rightarrow 0$ (see Appendix B). For example,

$$U_1^2 = ib_1 \sin(\eta), \quad V_1^2 = b_1 \sin(\eta). \tag{3.23}$$

$$U_4^2 = \sqrt{\frac{2c}{r}} - b_1^2 \operatorname{csch}(\eta), \quad V_4^2 = b_1 \operatorname{csch}(\eta). \tag{3.24}$$

$$U_5^2 = \sqrt{b_1^2 - \frac{c}{r}} \cot(\eta), \quad V_5^2 = b_1 \cot(\eta). \tag{3.25}$$

4. Conclusions

In this study, we have applied Auxiliary equation method to obtain the generalized solitary wave solutions of Non-Linear Klein-Gordon equation. As we can see in the example of (K. G.) equation, the main advantage of this method over the other methods is that it can be applied to a wide class of nonlinear evolution equations including those in which the odd and even-order derivative terms are coexist. It may be concluded that, Auxiliary equation method can be easily extended to all kinds of nonlinear equations.

Appendix A

Cases	R	Q	P	solution of Auxiliary equation
1	1	$-(1+m^2)$	m^2	$\text{sn}(\eta), \text{cd}(\eta)$
2	$1-m^2$	$2m^2-1$	$-m^2$	$\text{cn}(\eta)$
3	m^2-1	$2-m^2$	-1	$\text{dn}(\eta)$
4	$m^2(m^2-1)$	$2m^2-1$	1	$\text{ds}(\eta)$
5	$1-m^2$	$2-m^2$	1	$\text{cs}(\eta)$
6	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\frac{\text{sn}(\eta)}{1 \pm \text{dn}(\eta)}$
7	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\text{sn}(\eta) \pm \text{I cn}(\eta)$
$\frac{\text{dn}(\eta)}{\text{I}\sqrt{1-m^2\text{sn}(\eta) \pm \text{cn}(\eta)}}$,				
8	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\frac{\text{dn}(\eta)}{\text{cn}(\eta) \pm \text{I}\sqrt{1-m^2}}$
$\text{sn}(\eta) \pm \text{I dn}(\eta)$				
9	$\frac{m^2-1}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4}$	$\frac{\text{dn}(\eta)}{1 \pm m.\text{cn}(\eta)}$
10	$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$\frac{\text{cn}(\eta)}{1 \pm \text{sn}(\eta)}$
11	$\frac{-(1-m^2)^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1}{4}$	$m \text{cn}(\eta) \pm \text{dn}(\eta)$
12	$\frac{1}{4}$	$\frac{m^2+1}{2}$	$\frac{(1-m^2)^2}{4}$	$\frac{\text{sn}(\eta)}{\text{dn}(\eta) \pm \text{cn}(\eta)}$
13	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\frac{\text{cn}(\eta)}{\text{I}\sqrt{1-m^2 \pm \text{dn}(\eta)}}$

where m is the modulus of the Jacobi elliptic functions which satisfies $(0 \leq m \leq 1)$.

Appendix B

The Jacobi elliptic functions degenerate as hyperbolic functions when $m \rightarrow 1$.

$\text{sn}(\eta)$	$\text{cn}(\eta)$	$\text{dn}(\eta)$	$\text{sc}(\eta)$	$\text{sd}(\eta)$	$\text{cd}(\eta)$
$\tanh(\eta)$	$\text{sech}(\eta)$	$\text{sech}(\eta)$	$\sinh(\eta)$	$\sinh(\eta)$	1
$\text{ns}(\eta)$	$\text{nc}(\eta)$	$\text{nd}(\eta)$	$\text{cs}(\eta)$	$\text{ds}(\eta)$	$\text{dc}(\eta)$
$\coth(\eta)$	$\cosh(\eta)$	$\cosh(\eta)$	$\text{csch}(\eta)$	$\text{csch}(\eta)$	1

The Jacobi elliptic functions degenerate as trigonometric functions when $m \rightarrow 0$,

$\text{sn}(\eta)$	$\text{cn}(\eta)$	$\text{dn}(\eta)$	$\text{sc}(\eta)$	$\text{sd}(\eta)$	$\text{cd}(\eta)$
$\sin(\eta)$	$\cos(\eta)$	1	$\tan(\eta)$	$\sin(\eta)$	$\cos(\eta)$
$\text{ns}(\eta)$	$\text{nc}(\eta)$	$\text{nd}(\eta)$	$\text{cs}(\eta)$	$\text{ds}(\eta)$	$\text{dc}(\eta)$
$\text{csc}(\eta)$	$\text{sec}(\eta)$	1	$\cot(\eta)$	$\text{csch}(\eta)$	$\text{sec}(\eta)$

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