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Equivalent Function to the Implicit Function $X^Y = Y^X$

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Abstract This short paper deals with the implicit function $X^Y = Y^X$, X,Y > 0, and shows surprinsingly how accurately it is equivalent to another very much simpler and explicit function.

Keywords Power Exponential Function, Equivalent Function, Approximation

1. Introduction

The literature devoted to the equation $X^Y = Y^X$, $X, Y > 0$, is really limited. From[1] we know that L. Euler treated it and gave a parametric representation, from which the rational solutions were drawn. He also deduced the existence of the two asymptotes ($X = 1$ and $Y = 1$) to the curve. The same paper gives notice that also Daniel Bernouilli found the rational solutions. Later E. J. Moulton[2] writes a discussion of the curve defined by $X^Y = Y^X$, $X, Y > 0$, and recently Y. S. Kupitz and H. Martini[3] demonstrate the following two propositions: (1) *There is a non-trivial solution* $X(\neq Y)$ *to the equation* $X^Y = Y^X$, $X, Y > 0$, *if and only if* $1 < Y \neq e$, *and for such a* Y *the solution is unique*, and (2) *The only non-trivial integer solutions to the equation* $X^Y = Y^X$, $X, Y > 0$, *are* (2, 4) *and* (4, 2).

Recently this function has also focussed the attention of mathematicians[5,6], although little has been added to its knowledge and development.

In brief, it is well known that the implicit power- exponential function

$$X^Y = Y^X, \quad X, Y > 0 \tag{1}$$

admits the trivial solution, which will be named as solution (A),

$$Y_A = X \tag{2}$$

and another solution (B), which may be found either by successive iterations or by using some software, like Mathematica[4], in a computer.

Obviously, solution (B) is symmetrical with respect to the straight line defined by solution (A).

2. Non-Trivial Solution (B)

To find out the solution (B) one can proceed as follows:

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From (1)

$$Y \ln X = X \ln Y \tag{3}$$

$$\ln Y = \frac{Y \ln X}{X} \tag{4}$$

$$Y = e^{Y \ln X / X} \tag{5}$$

$$-\frac{Y \ln X}{X} = -\frac{\ln X}{X} e^{Y \ln X / X} \tag{6}$$

$$-\frac{\ln X}{X} = -\frac{Y \ln X}{X} e^{-Y \ln X / X} \tag{7}$$

And,

$$-\frac{Y \ln X}{X} = \mathrm{Pr}\,oductLog\left(-\frac{\ln X}{X}\right) \tag{8}$$

being ProductLog[z] the function which gives the principal solution for w in

$$z = we^w \tag{9}$$

as defined and tabulated by Mathematica.

Then solution (B) may be tabulated from

$$Y_B = -\frac{X}{\ln X} \mathrm{Pr}\,oductLog\left(-\frac{\ln X}{X}\right) \tag{10}$$

Both, equation (10) and direct iterations, yield the results shown in Table I, by means of which figure 1 represents the solution (B) (continuous line), together with solution (A) (discontinuous line).

3. Equivalent Function

Figure 1 shows at first glance that the function Y_B looks very close to the hyperbola

$$(X-1)(Y-1) = 3 \tag{11}$$

which, by the way, also admits the integer solutions (2, 4) and (4, 2) as equation (1).

In order to analyse how close the function (11) is to the original function Y_B, a third column (Y_{HI}) is added in Table I, showing

$$Y_{H1} = \frac{3}{X-1} + 1 \tag{12}$$

as given by (11), whereas the fifth column shows the distance $Y_{HI} - Y_B$.

Then, accounting for the fact that the curve $Y_B = Y_B(X)$ also goes through the point (*e, e*), the hyperbola

$$(X-1)(Y-1) = (e-1)^2 \tag{13}$$

is considered too and

$$Y_{H2} = \frac{(e-1)^2}{X-1} + 1 \tag{14}$$

as given by (13), is shown in the fourth column of table 1, whereas the distance Y_{H2} - Y_B appears in the sixth column.

Table 1.

X	Y_B	Y_{H1}	Y_{H2}	Y_{H1}-Y_B	Y_{H2}-Y_B
e	e	2.7459	2.7183	0.0276	0.0000
2.8	2.6405	2.6667	2.6403	0.0262	-0.0002
2.9	2.5548	2.5790	2.5539	0.0242	-0.0009
3.0	2.4781	2.5000	2.4763	0.0219	-0.0018
3.5	2.1897	2.2000	2.1810	0.0103	-0.0087
4.0	2.0000	2.0000	1.9842	0.0000	-0.0158
4.5	1.8655	1.8571	1.8436	-0.0084	-0.0219
5.0	1.7649	1.7500	1.7381	-0.0149	-0.0268
6.0	1.6242	1.6000	1.5905	-0.0242	-0.0337
7.0	1.5301	1.5000	1.4921	-0.0301	-0.0380
8.0	1.4625	1.4286	1.4218	-0.0339	-0.0407
9.0	1.4114	1.3750	1.3691	-0.0364	-0.0423
10.0	1.3713	1.3333	1.3281	-0.0380	-0.0432
12.0	1.3122	1.2727	1.2684	-0.0395	-0.0438
14.0	1.2707	1.2308	1.2271	-0.0399	-0.0436
16.0	1.2396	1.2000	1.1968	-0.0396	-0.0428
18.0	1.2155	1.1765	1.1737	-0.0390	-0.0418
20.0	1.1962	1.1579	1.1554	-0.0383	-0.0408
25.0	1.1613	1.1250	1.1230	-0.0363	-0.0383
30.0	1.1377	1.1034	1.1018	-0.0343	-0-0359
35.0	1.1206	1.0882	1.0868	-0.0324	-0.0338
40.0	1.1075	1.0769	1.0757	-0.0306	-0.0318
45.0	1.0973	1.0682	1.0671	-0.0291	-0.0302
50.0	1.0889	1.0612	1.0603	-0.0277	-0.0286
60.0	1.0762	1.0508	1.0500	-0.0254	-0.0262
70.0	1.0669	1.0435	1.0428	-0.0234	-0.0241
80.0	1.0598	1.0380	1.0374	-0.0218	-0.0224
90.0	1.0541	1.0337	1.0332	-0.0204	-0.0209
100.0	1.0495	1.0303	1.0298	-0.0192	-0.0197
125.0	1.0410	1.0242	1.0238	-0.0168	-0.0172
150.0	1.0352	1.0201	1.0198	-0.0151	-0.0154
175.0	1.0309	1.0172	1.0170	-0.0137	-0.0139
200.0	1.0276	1.0151	1.0148	-0.0125	-0.0128
250.0	1.0228	1.0120	1.0119	-0.0108	-0.0109
300.0	1.0196	1.0100	1.0099	-0.0096	-0.0097
400.0	1.0153	1.0075	1.0074	-0.0078	-0.0079
500.0	1.0127	1.0060	1.0059	-0.0067	-0.0068

Thus, direct reading of table I shows that the hyperbola (11) is closer to Y_B than the hyperbola (13), and that

$$\left|Y_{H1} - Y_B\right| < 0.04 \tag{15}$$

for two reasons: 1) this value is not reached before $X = 150$, and 2) for $X = 150$ and onwards the distance between Y_B and the asymptote $Y = 1$, as well as between Y_{HI} and the same asymptote, is less than 0.04, which implies (15).

In fact, in figure 1 the points representing Y_{HI} are plotted over the curve Y_B and the closeness is very evident.

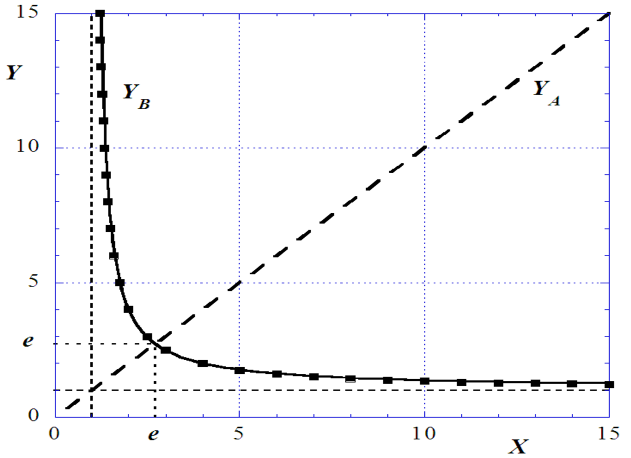

Figure 1. Trivial solution (A) (discontinuous straight line) and solution (B) (full line curve). Overlapping the curve the dots representing the equivalent hyperbolic function

4. Conclusions

The little difference between the two functions Y_{HI} and Y_B, which remains always under 0.04, means that the much simpler hyperbola given by equation (11) is a very good approximation to the implicit power-exponential function defined by equation (1).

REFERENCES

[1] R. C. Archivald, “Problem notes, No. 9”, Amer. Math. Monthly, vol. 28, pp. 141-143, 1921

[2] E. J. Moulton, “The real function defined by $x^y = y^x$ “, Amer. Math. Monthly, vol. 23, pp. 233-237, 1916

[3] Y. S. Kupitz, and H. Martini, C. “On the equation $x^y = y^x$”, Elemente der Mathematik, vol. 55, pp. 95–101, 2000

[4] Mathematica, Trade Mark. Wolfram Research, Inc., 100 Trade Center Drive, Champaign, IL 61820-7237

[5] Mitteldorf, “Solutions to $x^y = y^x$ “ [on line]. Available from: http://mathforum.org/library/drmath/view/53229.html (accessed November 2011)

[6] Vogler, “Solving the equation $x^y = y^x$ “[on line]. Available from: http://mathforum.org/library/drmath/view/66166.html (accessed November 2011)