Applied Mathematics. 2011; 1(2): 99-105
DOI: 10.5923/j.am.20110102.16

# Averaging of Set Integrodifferential Equations

**Andrej V. Plotnikov[1,3,*], Tatyana A. Komleva[2]**

[1]Department Applied Mathematics, Odessa State Academy Civil Engineering and Architecture, Odessa, 65029, Ukraine
[2]Department Mathematics, Odessa State Academy Civil Engineering and Architecture, Odessa, 65029, Ukraine
[3]Department Optimal Control, Odessa National University, Odessa, 65026, Ukraine

**Abstract** In this article we prove the substantiation of the method of averaging for the set integrodifferential equations with small parameter. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

**Keywords** Set Integrodifferential Equation, Method of Averaging

## 1. Introduction

Many important problems of analytical dynamics are described by the nonlinear mathematical models that as a rule are presented by the nonlinear differential or the integrodifferential equations. The absence of exact universal research methods for nonlinear systems has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

The averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. Averaging theory for ordinary differential equations has a rich history, dating to back to the work of N.M. Krylov and N.N. Bogoliubov[1], and has been used extensively in engineering applications[2-6]. Books that cover averaging theory for differential equations and inclusions include[7-10].

In recent years the development of the calculus in metric spaces has attracted some attention[8-14]. Earlier, F.S. de Blasi, F. Iervolino[15] started the investigation of set differential equations (SDEs) in semilinear metric spaces. This has now evolved into the theory of SDEs as an independent discipline: properties of solutions[8-10,12-43], the impulse equations[8,9,44], control systems[45-48] and asymptotic methods[8-10,49-53]. On the other hand, SDEs are useful in other areas of mathematics. For example, SDEs are used, as an auxiliary tool, to prove existence results for differential inclusions[8,33,38,42]. Also, one can employ SDEs in the investigation of fuzzy differential equations[9,13,28-30, 32,33]. Moreover, SDEs are a natural generalization of the usual ordinary differential equations in finite (or infinite) dimensional Banach spaces.

* Corresponding author:
a-plotnikov@ukr.net (Andrej V. Plotnikov)
Published online at http://journal.sapub.org/am


In this article we prove the substantiation of the method of averaging for the set integrodifferential equations with small parameter. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

## 2. Preliminaries

Let $comp\left(R^n\right)\left(conv\left(R^n\right)\right)$ be a set of all nonempty (convex) compact subsets from the space $R^n$,

$$h(A,B)=\min_{r\geq 0}\left\{S_r(A)\supset B, S_r(B)\supset A\right\}$$

be Hausdorff distance between sets $A$ and $B$, $S_r(A)$ is $r$-neighborhood of set $A$.

Let $A, B, C$ be in $conv\left(R^n\right)$. The set $C$ is the Hukuhara difference of $A$ and $B$, if $B+C=A$, i.e.

$$C = A \overset{H}{-} B$$

From Radstrom's Cancellation Lemma[54], it follows that if this difference exists, then it is unique.

**Definition**[55]. A mapping $X:[0,T]\to conv\left(R^n\right)$ is differentiable in the sense of Hukuhara at $t\in[0,T]$ if for some $\delta>0$ the Hukuhara differences

$$X(t+\Delta)\overset{H}{-}X(t),\quad X(t)\overset{H}{-}X(t-\Delta)$$

exists in $conv\left(R^n\right)$ for all $0<\Delta<\delta$ and there exists an $DX(t)\in conv\left(R^n\right)$ such that

$$\lim_{\Delta\to 0_+} h\left(\Delta^{-1}\left(X(t+\Delta)\overset{H}{-}X(t)\right), DX(t)\right)=0$$

and

$$\lim_{\Delta\to 0_+} h\left(\Delta^{-1}\left(X(t)\overset{H}{-}X(t-\Delta)\right), DX(t)\right)=0 \cdot$$

Here $DX(t)$ is called the Hukuhara derivative of $X(t)$ at $t$.

## 3. The First Scheme of an Average

Consider the Cauchy problem with small parameter

$$DX = \varepsilon\left[ F(t,X) + \int_0^t \Phi(t,s,X(s))ds \right],\ X(0) = X_0, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, $t,s \in R_+$, $F : R_+ \times conv(R^n) \to conv(R^n)$ is a multivalued mapping, $\Phi : R_+ \times R_+ \times conv(R^n) \to comp(R^n)$ is a multivalued mapping, Here the integral is understood in the sense of[55] (the integral exists for example if $X(\cdot)$ is measurable and the real mapping $t \to h(X(t),\{0\})$ is integrable on $I \subset R_+$), $X_0 \in conv(R^n)$.

**Definition.** A mapping $X : [0,T] \to conv(R^n)$ is a solution to the problem (1) if and only if it is continuous and satisfies the integral equation

$$X(t) = X_0 + \varepsilon\int_0^t \left[ F(\tau, X(\tau)) + \int_0^\tau \Phi(\tau,s,X(s))ds \right] d\tau$$

for all $t \in [0,T]$.

In this section we associate with the equation (1) the following averaged integrodifferential equation

$$DY = \varepsilon\left[ \overline{F}(Y) + \int_0^t \overline{\Phi}(t,Y(s))ds \right], \qquad Y(0) = X_0, \tag{2}$$

where

$$\lim_{T\to\infty} h\left( \frac{1}{T}\int_0^T F(t,X)dt, \overline{F}(X) \right) = 0, \tag{3}$$

$$\lim_{T\to\infty} h\left( \frac{1}{T}\int_0^T\int_0^t \Phi(t,s,X)ds\,dt, \frac{1}{T}\int_0^T\int_0^t \overline{\Phi}(t,X)ds\,dt \right) = 0. \tag{4}$$

**Remark.** In this paper we will consider a case when the limits (3),(4) exist.

**Definition.** We say that the limits (3),(4) exist uniformly in $X$, if for any $\alpha > 0$ and any $X \in G \in conv(R^n)$ there exists $T(\alpha)$ such that

$$h\left( \frac{1}{T}\int_0^T F(t,X)dt, \overline{F}(X) \right) < \alpha,$$

$$h\left( \frac{1}{T}\int_0^T\int_0^t \Phi(t,s,X)ds\,dt, \frac{1}{T}\int_0^T\int_0^t \overline{\Phi}(t,X)ds\,dt \right) < \alpha.$$

for all $T > T(\alpha)$.

**Theorem.** Let in domain

$$Q = \left\{ (t,X) \mid t \ge 0,\ X \in G \in conv(R^n) \right\}$$

the following hold:

1) $F(t,X)$ is continuous in $(t,X) \in R_+ \times G$;

2) $\Phi(t,s,X)$ is continuous in $(t,s,X) \in R_+ \times R_+ \times G$;

3) there exist continuous functions $K(t), P(t,s)$, and constants $K_0, P_0$ such that

$$h(F(t,X),\{0\}) \le K(t),\quad h(\Phi(t,s,X),\{0\}) \le P(t,s),$$

$$\int_{t_1}^{t_2} K(t)dt \le K_0(t_2 - t_1),\quad \int_{t_1}^{t_2}\int_0^t P(t,s)dsdt \le P_0(t_2 - t_1)$$

for any $0 \le t_1 \le t_2 < \infty$;

4) there exist continuous functions $N(t)$, $M(t,s)$, and constants $N_0$, $M_0$, $M_1$ such that

$$h(F(t,X_1),F(t,X_2)) \le N(t)h(X_1,X_2),$$

$$h(\Phi(t,s,X_1),\Phi(t,s,X_2)) \le M(t,s)h(X_1,X_2),$$

$$\int_{t_1}^{t_2} N(t)dt \le N_0(t_2 - t_1),\quad \int_{t_1}^{t_2}\int_0^t M(t,s)dsdt \le M_0(t_2 - t_1),$$

$$\int_{t_1}^{t_2}\int_0^t M(t,s)(t-s)dsdt \le M_1(t_2 - t_1)$$

for any $0 \le t_1 \le t_2 < \infty$;

5) there exist continuous function $V(t)$, and constants $\lambda$, $V_0$, $V_1$ such that

$$h(\overline{F}(X_1),\overline{F}(X_2)) \le \lambda\, h(X_1,X_2),$$

$$h(\overline{\Phi}(t,X_1),\overline{\Phi}(t,X_2)) \le V(t)h(X_1,X_2),$$

$$\int_{t_1}^{t_2} t\, V(t)dt \le V_0(t_2 - t_1),\quad \int_{t_1}^{t_2} t^2\, V(t)dt \le V_1(t_2 - t_1)$$

for any $0 \le t_1 \le t_2 < \infty$;

6) the limits (3),(4) exist uniformly in $X \in G$;

7) for any $X_0 \in G' \subset G$ and $t \ge 0$ the solution of the equation (2) together with a $\sigma$-neighborhood belong to the domain $G$.

Then for any $\eta > 0$ and $L > 0$ there exists $\varepsilon^0(\eta,L) \in (0,\sigma]$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statement fulfill:

$$h(X(t),Y(t)) < \eta, \tag{5}$$

where $X(t)$, $Y(t)$ are the solutions of the initial and the averaged equations.

**Proof.** Since

$$X(t) = X_0 + \varepsilon\int_0^t \left[ F(\tau, X(\tau)) + \int_0^\tau \Phi(\tau,s,X(s))ds \right] d\tau,$$

$$Y(t) = X_0 + \varepsilon\int_0^t \left[ \overline{F}(Y(\tau)) + \int_0^\tau \overline{\Phi}(\tau,Y(s))ds \right] d\tau$$

we have

$$h(X(t),Y(t)) \le \varepsilon h\left( \int_0^t F(\tau,X(\tau))d\tau, \int_0^t \overline{F}(Y(\tau))d\tau \right) +$$

$$+\varepsilon h\left( \int_0^t\int_0^\tau \Phi(\tau,s,X(s))dsd\tau, \int_0^t\int_0^\tau \overline{\Phi}(\tau,Y(s))dsd\tau \right).$$

Hence

$$h(X(t),Y(t)) \le \varepsilon h\left( \int_0^t F(\tau,X(\tau))d\tau, \int_0^t \overline{F}(X(\tau))d\tau \right) +$$

$$+\varepsilon h\left( \int_0^t \overline{F}(X(\tau))d\tau, \int_0^t \overline{F}(Y(\tau))d\tau \right) +$$

$$+\varepsilon h\left( \int_0^t\int_0^\tau \Phi(\tau,s,X(s))dsd\tau, \int_0^t\int_0^\tau \overline{\Phi}(\tau,X(s))dsd\tau \right) +$$

$$+\varepsilon h\left( \int_0^t\int_0^\tau \overline{\Phi}(\tau,X(s))dsd\tau, \int_0^t\int_0^\tau \overline{\Phi}(\tau,Y(s))dsd\tau \right) \le$$

$$\le \varepsilon h\left( \int_0^t F(\tau,X(\tau))d\tau, \int_0^t \overline{F}(X(\tau))d\tau \right) +$$

$$+\varepsilon\int_0^t h\left(\bar F\left(X(\tau)\right),\bar F\left(Y(\tau)\right)\right)d\tau+$$
$$+\varepsilon h\left(\int_0^t\int_0^\tau \Phi\left(\tau,s,X(s)\right)dsd\tau,\int_0^t\int_0^\tau \bar\Phi\left(\tau,X(s)\right)dsd\tau\right)+$$
$$+\varepsilon\int_0^t\int_0^\tau h\left(\bar\Phi\left(\tau,X(s)\right),\bar\Phi\left(\tau,Y(s)\right)\right)dsd\tau\le$$
$$\le\varepsilon\lambda\int_0^t h\left(X(\tau),Y(\tau)\right)d\tau+\varepsilon\int_0^t\int_0^\tau V(\tau)h\left(X(s),Y(s)\right)dsd\tau+$$
$$+\varepsilon h\left(\int_0^t F\left(\tau,X(\tau)\right)d\tau,\int_0^t \bar F\left(X(\tau)\right)d\tau\right)+$$
$$+\varepsilon h\left(\int_0^t\int_0^\tau \Phi\left(\tau,s,X(s)\right)dsd\tau,\int_0^t\int_0^\tau \bar\Phi\left(\tau,X(s)\right)dsd\tau\right). \qquad (6)$$

Now we will estimate last summands in (6). Divide the interval $[0,L\varepsilon^{-1}]$ into partial intervals by the points

$$t_i=\frac{iL}{m\varepsilon},\quad i=0,...,m,\quad m\in\mathrm{N}.$$

Then

$$\varepsilon h\left(\int_0^t F\left(\tau,X(\tau)\right)d\tau,\int_0^t \bar F\left(X(\tau)\right)d\tau\right)+$$
$$+\varepsilon h\left(\int_0^t\int_0^\tau \Phi\left(\tau,s,X(s)\right)dsd\tau,\int_0^t\int_0^\tau \bar\Phi\left(\tau,X(s)\right)dsd\tau\right)\le$$
$$\le\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5,$$

where

$$\Sigma_1=\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} h\left(F\left(\tau,X(\tau)\right),F\left(\tau,X(\tau_i)\right)\right)d\tau,$$
$$\Sigma_2=\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau h\left(\Phi\left(\tau,s,X(s)\right),\Phi\left(\tau,s,X(t_i)\right)\right)dsd\tau,$$
$$\Sigma_3=\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau h\left(\bar\Phi\left(\tau,X(s)\right),\bar\Phi\left(\tau,X(t_i)\right)\right)dsd\tau$$
$$\Sigma_4=\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} h\left(\bar F\left(X(\tau)\right),\bar F\left(X(\tau_i)\right)\right)d\tau,$$
$$\Sigma_5=\varepsilon\sum_{i=0}^{m-1}\left[h\left(\int_{t_i}^{t_{i+1}} F\left(\tau,X(t_i)\right)d\tau,\int_{t_i}^{t_{i+1}} \bar F\left(X(t_i)\right)d\tau\right)+\right.$$
$$\left.+h\left(\int_{t_i}^{t_{i+1}}\int_0^\tau \Phi\left(\tau,s,X(t_i)\right)dsd\tau,\int_{t_i}^{t_{i+1}}\int_0^\tau \bar\Phi\left(\tau,X(t_i)\right)dsd\tau\right)\right].$$

By condition 3) of the theorem, we get

$$h\left(X(\tau),X(t_i)\right)\le\frac{L}{m}(K_0+P_0),$$
$$h(X(\tau),X(s))\le\varepsilon(K_0+P_0)(\tau-s), \qquad (7)$$

for $\tau\in[t_i,t_{i+1}]$.

From (7), and conditions of the theorem, we have

$$\Sigma_1\le\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} N(\tau)h\left(X(\tau),X(t_i)\right)d\tau\le$$
$$\le\varepsilon\frac{L}{m}(K_0+P_0)\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} N(\tau)d\tau\le$$
$$\le LN_0\frac{L}{m}(K_0+P_0),$$
$$\Sigma_2\le\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau M(\tau,s)h(X(s),X(t_i))dsd\tau\le$$
$$\le\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau M(\tau,s)[h(X(s),X(\tau))+h(X(\tau),X(t_i))]dsd\tau\le$$
$$\le\varepsilon^2(K_0+P_0)\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau M(\tau,s)(\tau-s)dsd\tau+$$
$$+\varepsilon\frac{L}{m}(K_0+P_0)\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau M(\tau,s)dsd\tau\le$$
$$\le(K_0+P_0)L\left(\varepsilon M_1+\frac{LM_0}{m}\right),$$
$$\Sigma_3\le\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau V(\tau)h\left(X(s),X(t_i)\right)dsd\tau\le$$
$$\le\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau V(\tau)\left[h\left(X(s),X(\tau)\right)+h\left(X(\tau),X(t_i)\right)\right]dsd\tau\le$$
$$\le\varepsilon^2(K_0+P_0)\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau V(\tau)\tau^2dsd\tau+$$
$$+\varepsilon\frac{L}{m}(K_0+P_0)\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau V(\tau)dsd\tau\le$$
$$\le L(K_0+P_0)\left(\varepsilon\frac{V_1}{2}+L\frac{V_0}{2}\right),$$
$$\Sigma_4\le\varepsilon\lambda\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} h\left(X(\tau),X(t_i)\right)d\tau\le\frac{\lambda}{2}(K_0+P_0)\frac{L^2}{m}.$$

Hence, for any $\eta>0$, there exist $m$, and $\varepsilon_1>0$ such that the following estimate is true for $0<\varepsilon<\varepsilon_1$:

$$\sum_{i=1}^{4}\Sigma_i\le LN_0\frac{L}{m}(K_0+P_0)+$$
$$+L(K_0+P_0)\left[\varepsilon M_1+\frac{LM_0}{m}+\frac{\lambda L}{2m}+\varepsilon\frac{V_1}{2}+\frac{LV_0}{m}\right]\le$$
$$\le\frac{\eta}{2}e^{-L(\lambda+V_0)}. \qquad (8)$$

From condition 6) of the theorem it follows that there exist the increasing functions $\theta(t)$ and $\varphi(t)$ such that

1) $\lim\limits_{t\to\infty}\theta(t)=0$, $\lim\limits_{t\to\infty}\phi(t)=0$;

2) $h\left(\int_0^t F(\tau,X)d\tau,\int_0^t \bar F(X)d\tau\right)\le t\theta(t)$,

$$h\left(\int_0^t\int_0^\tau \Phi(\tau,s,X)dsd\tau,\int_0^t\int_0^\tau \bar\Phi(\tau,X)dsd\tau\right)\le t\phi(t).$$

Then

$$\Sigma_5\le\varepsilon\sum_{i=0}^{m-1}h\left(\int_{t_i}^{t_{i+1}} F\left(\tau,X(t_i)\right)d\tau,\int_{t_i}^{t_{i+1}} \bar F\left(X(t_i)\right)d\tau\right)+$$
$$+\varepsilon\sum_{i=0}^{m-1}h\left(\int_{t_i}^{t_{i+1}}\int_0^\tau \Phi\left(\tau,s,X(t_i)\right)dsd\tau,\int_{t_i}^{t_{i+1}}\int_0^\tau \bar\Phi\left(\tau,X(t_i)\right)dsd\tau\right)\le$$

$$\le \varepsilon \sum_{i=0}^{m-1} h\left( \int_0^{t_i} F(\tau, X(t_i))d\tau, \int_0^{t_i} \overline{F}(X(t_i))d\tau \right) +$$

$$+ \varepsilon \sum_{i=0}^{m-1} h\left( \int_0^{t_{i+1}} F(\tau, X(t_i))d\tau, \int_0^{t_{i+1}} \overline{F}(X(t_i))d\tau \right) +$$

$$+ \varepsilon \sum_{i=0}^{m-1} h\left( \int_0^{t_i}\int_0^{\tau} \Phi(\tau, s, X(t_i))dsd\tau, \int_0^{t_i}\int_0^{\tau} \overline{\Phi}(\tau, X(t_i))dsd\tau \right) +$$

$$+ \varepsilon \sum_{i=0}^{m-1} h\left( \int_0^{t_{i+1}}\int_0^{\tau} \Phi(\tau, s, X(t_i))dsd\tau, \int_0^{t_{i+1}}\int_0^{\tau} \overline{\Phi}(\tau, X(t_i))dsd\tau \right) \le$$

$$\le 2m(\psi_1(\varepsilon) + \psi_2(\varepsilon)),$$

where

$$\psi_1(\varepsilon) = \sup_{\tau\in[0,L]} \left( \tau\theta\left(\frac{\tau}{\varepsilon}\right)\right), \quad \psi_2(\varepsilon) = \sup_{\tau\in[0,L]} \left( \tau\phi\left(\frac{\tau}{\varepsilon}\right)\right), \quad \tau = \varepsilon\, t .$$

Fix $m$. Then for any $\eta > 0$, there exists $\varepsilon_2 > 0$ such that the following estimate is true for $0 < \varepsilon < \varepsilon_2$:

$$\Sigma_5 \le 2m(\psi_1(\varepsilon) + \psi_2(\varepsilon)) \le \frac{\eta}{2} e^{-L(\lambda+V_0)}. \tag{9}$$

Now, we get $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Combining (6), (8), and (9), we obtain

$$h(X(t),Y(t)) \le$$

$$\le \varepsilon \int_0^t \left[ \lambda h(X(\tau),Y(\tau)) + V(\tau)\int_0^{\tau} h(X(s),Y(s))ds \right] d\tau + \eta e^{-L(\lambda+V_0)}$$

Using Gronwall-Bellman's inequality, we obtain

$$h(X(t),Y(t)) < \eta .$$

This concludes the proof.

# 4. The Second Scheme of an Average

In this section we associate with the equation (1) the averaged integrodifferential equation (2), where

$$\lim_{T\to\infty} h\left( \frac{1}{T}\int_0^T F(t,X)dt, \overline{F}(X)\right) = 0 , \tag{10}$$

$$\overline{\Phi}(t,X) = \begin{cases} \frac{1}{t}\Phi_1(t,X), & t>0, \\ \lim\limits_{t\to 0+} \frac{1}{t}\Phi_1(t,X), & t=0, \end{cases} \quad \Phi_1(t,X) = \int_0^t \Phi(t,s,X)ds . \tag{11}$$

**Remark**. In this paper we will consider a case when the limits (10),(11) exist.

**Theorem.** Let in domain

$$Q = \{(t,X) \mid t \ge 0,\ X \in G \in conv(R^n)\}$$

the following hold:

1) $F(t,X)$ is continuous in $(t,X) \in R_+ \times G$ ;

2) $\Phi(t,s,X)$ is continuous in $(t,s,X) \in R_+ \times R_+ \times G$ ;

3) there exist continuous functions $K(t)$, $P(t,s)$, and constants $K_0, P_0$ such that

$$h(F(t,X),\{0\}) \le K(t), \quad h(\Phi(t,s,X),\{0\}) \le P(t,s),$$

$$\int_{t_1}^{t_2} K(t)dt \le K_0(t_2-t_1), \quad \int_{t_1}^{t_2}\int_0^t P(t,s)dsdt \le P_0(t_2-t_1)$$

for any $0 \le t_1 \le t_2 < \infty$ ;

4) there exist continuous functions $N(t)$, $M(t,s)$, and constants $N_0$, $M_0$, $M_1$ such that

$$h(F(t,X_1),F(t,X_2)) \le N(t)h(X_1,X_2),$$

$$h(\Phi(t,s,X_1),\Phi(t,s,X_2)) \le M(t,s)h(X_1,X_2),$$

$$\int_{t_1}^{t_2} N(t)dt \le N_0(t_2-t_1), \quad \int_{t_1}^{t_2}\int_0^t M(t,s)dsdt \le M_0(t_2-t_1),$$

$$\int_{t_1}^{t_2}\int_0^t M(t,s)(t-s)dsdt \le M_1(t_2-t_1)$$

for any $0 \le t_1 \le t_2 < \infty$;

5) there exist continuous function $V(t)$, and constants $\lambda$, $V_0$, $V_1$ such that

$$h(\overline{F}(X_1),\overline{F}(X_2)) \le \lambda h(X_1,X_2),$$

$$h(\overline{\Phi}(t,X_1),\overline{\Phi}(t,X_2)) \le V(t)h(X_1,X_2),$$

$$\int_{t_1}^{t_2} t\, V(t)dt \le V_0(t_2-t_1), \quad \int_{t_1}^{t_2} t^2\, V(t)dt \le V_1(t_2-t_1)$$

for any $0 \le t_1 \le t_2 < \infty$;

6) the limits (3),(4) exist uniformly in $X \in G$;

7) for any $X_0 \in G' \subset G$ and $t \ge 0$ the solution of the equation (2) together with a $\sigma$-neighborhood belong to the domain $G$.

Then for any $\eta > 0$ and $L > 0$ there exists $\varepsilon^0(\eta,L) \in (0,\sigma]$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statement fulfill:

$$h(X(t),Y(t)) < \eta ,$$

where $X(t)$, $Y(t)$ are the solutions of the initial and the averaged equations.

**Proof.** Since

$$X(t) = X_0 + \varepsilon\int_0^t \left[ F(\tau,X(\tau)) + \int_0^{\tau} \Phi(\tau,s,X(s))ds \right] d\tau ,$$

$$Y(t) = X_0 + \varepsilon\int_0^t \left[ \overline{F}(Y(\tau)) + \int_0^{\tau} \overline{\Phi}(\tau,Y(s))ds \right] d\tau ,$$

we have $h(X(t),Y(t)) \le$

$$\begin{aligned} &\le \varepsilon\lambda\int_0^t h(X(\tau),Y(\tau))d\tau + \varepsilon\int_0^t\int_0^{\tau} V(\tau)h(X(s),Y(s))dsd\tau + \\ &+\varepsilon h\left( \int_0^t F(\tau,X(\tau))d\tau, \int_0^t \overline{F}(X(\tau))d\tau \right) + \\ &+\varepsilon h\left( \int_0^t\int_0^{\tau} \Phi(\tau,s,X(s))dsd\tau, \int_0^t\int_0^{\tau} \overline{\Phi}(\tau,X(s))dsd\tau \right) \end{aligned} \tag{12}$$

Now we will estimate last summands in (12). Divide the interval $[0, L\varepsilon^{-1}]$ into partial intervals by the points

$$t_i = \frac{iL}{m\varepsilon}, \quad i = 0,...,m, \quad m \in \mathrm{N} .$$

Then

$$\varepsilon h\left(\int_0^t F(\tau,X(\tau))d\tau,\int_0^t \overline{F}(X(\tau))d\tau\right)+$$

$$+\varepsilon h\left(\int_0^t\int_0^\tau \Phi(\tau,s,X(s))dsd\tau,\int_0^t\int_0^\tau \overline{\Phi}(\tau,X(s))dsd\tau\right)\le$$

$$\le \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6,$$

where

$$\Sigma_1=\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} h\left(F(\tau,X(\tau)),F(\tau,X(\tau_i))\right)d\tau,$$

$$\Sigma_2=\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau h\left(\Phi(\tau,s,X(s)),\Phi(\tau,s,X(t_i))\right)dsd\tau,$$

$$\Sigma_3=\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_0^\tau h\left(\overline{\Phi}(\tau,X(s)),\overline{\Phi}(\tau,X(t_i))\right)dsd\tau,$$

$$\Sigma_4=\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} h\left(\overline{F}(X(\tau)),\overline{F}(X(\tau_i))\right)d\tau,$$

$$\Sigma_5=\varepsilon\sum_{i=0}^{m-1} h\left(\int_{t_i}^{t_{i+1}}F(\tau,X(t_i))d\tau,\int_{t_i}^{t_{i+1}}\overline{F}(X(t_i))d\tau\right),$$

$$\Sigma_6=\varepsilon\sum_{i=0}^{m-1} h\left(\int_{t_i}^{t_{i+1}}\int_0^\tau \Phi(\tau,s,X(t_i))dsd\tau,\int_{t_i}^{t_{i+1}}\int_0^\tau \overline{\Phi}(\tau,X(t_i))dsd\tau\right).$$

By condition 3) of the theorem, we get

$$h(X(\tau),X(t_i))\le \frac{L}{m}(K_0+P_0),$$

$$h(X(\tau),X(s))\le \varepsilon(K_0+P_0)(\tau-s), \tag{13}$$

for $\tau\in[t_i,t_{i+1}]$.

From (13), and conditions of the theorem, we have

$$\Sigma_1\le LN_0\frac{L}{m}(K_0+P_0),\quad \Sigma_2\le(K_0+P_0)L\left(\varepsilon M_1+\frac{LM_0}{m}\right),$$

$$\Sigma_3\le L(K_0+P_0)\left(\varepsilon\frac{V_1}{2}+L\frac{V_0}{2}\right),$$

$$\Sigma_4\le\frac{\lambda}{2}(K_0+P_0)\frac{L^2}{m}.$$

Hence, for any $\eta>0$, there exist $m$, and $\varepsilon_1>0$ such that the following estimate is true for $0<\varepsilon<\varepsilon_1$:

$$\sum_{i=1}^4\Sigma_i\le LN_0\frac{L}{m}(K_0+P_0)+$$

$$+L(K_0+P_0)\left[\varepsilon M_1+\frac{LM_0}{m}+\frac{\lambda L}{2m}+\varepsilon\frac{V_1}{2}+\frac{LV_0}{m}\right]\le$$

$$\le\frac{\eta}{2}e^{-L(\lambda+V_0)}. \tag{14}$$

From condition 6) of the theorem, it follows that there exists the increasing function $\theta(t)$ such that

1) $\lim\limits_{t\to\infty}\theta(t)=0$;

2) $h\left(\int_0^t F(\tau,X)d\tau,\int_0^t\overline{F}(X)d\tau\right)\le t\theta(t)$.

Then

$$\Sigma_5\le\varepsilon\sum_{i=0}^{m-1}h\left(\int_{t_i}^{t_{i+1}}F(\tau,X(t_i))d\tau,\int_{t_i}^{t_{i+1}}\overline{F}(X(t_i))d\tau\right)\le$$

$$\le\varepsilon\sum_{i=0}^{m-1}h\left(\int_0^{t_i}F(\tau,X(t_i))d\tau,\int_0^{t_i}\overline{F}(X(t_i))d\tau\right)+$$

$$+\varepsilon\sum_{i=0}^{m-1}h\left(\int_0^{t_{i+1}}F(\tau,X(t_i))d\tau,\int_0^{t_{i+1}}\overline{F}(X(t_i))d\tau\right)\le$$

$$\le 2m\psi(\varepsilon),$$

where

$$\psi(\varepsilon)=\sup_{\tau\in[0,L]}\left(\tau\theta\left(\frac{\tau}{\varepsilon}\right)\right),\quad \tau=\varepsilon t.$$

Fix $m$. Then for any $\eta>0$, there exists $\varepsilon_2>0$ such that the following estimate is true for $0<\varepsilon<\varepsilon_2$:

$$\Sigma_5\le 2m\psi(\varepsilon)\le\frac{\eta}{2}e^{-L(\lambda+V_0)}. \tag{15}$$

By (11), we have

$$\Sigma_6=\varepsilon\sum_{i=0}^{m-1}h\left(\int_{t_i}^{t_{i+1}}\int_0^\tau\Phi(\tau,s,X(t_i))dsd\tau,\int_{t_i}^{t_{i+1}}\int_0^\tau\overline{\Phi}(\tau,X(t_i))dsd\tau\right)=$$

$$=\varepsilon\sum_{i=0}^{m-1}h\left(\int_{t_i}^{t_{i+1}}\int_0^\tau\Phi(\tau,s,X(t_i))dsd\tau,\int_{t_i}^{t_{i+1}}\tau\,\overline{\Phi}(\tau,X(t_i))d\tau\right)\le$$

$$\le\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}h\left(\int_0^\tau\Phi(\tau,s,X(t_i))ds,\tau\,\overline{\Phi}(\tau,X(t_i))\right)d\tau=0.$$

Now, we get $\varepsilon_0=\min\{\varepsilon_1,\varepsilon_2\}$. Combining (12), (14), and (15), we obtain

$$h(X(t),Y(t))\le$$

$$\le\varepsilon\int_0^t\left[\lambda h(X(\tau),Y(\tau))+V(\tau)\int_0^\tau h(X(s),Y(s))ds\right]d\tau+\eta e^{-L(\lambda+V_0)}.$$

Using Gronwall-Bellman's inequality, we obtain

$$h(X(t),Y(t))<\eta.$$

This concludes the proof.

# 5. Conclusions

Here we used the approach of Hukuhara at definition of the derivative which has essential shortages. However the given approach is well investigated by many authors. Also in the literature exist other approaches to definition of the derivative[9,10,16,28,36,43], but also they have the shortages. It is easily possible to show that this outcome will be true for some other cases with little changes.

# REFERENCES

[1] N. M. Krylov and N. N. Bogoliubov, Introduction to nonlinear mechanics. Princeton, Princeton University Press, 1947

[2] N. N. Bogoliubov and Yu. A. Mitropolsky, Asymptotic methods in the theory of non-linear oscillations. New York, Gordon and Breach, 1961

[3] Th. J. Coakley, A Generalized Averaging Method for Linear Differential Equations with Almost Periodic Coefficients. National Aeronautics and Space Administration, Washington, D. C., 1969

[4] Ju. A. Mitropolsky, Method of an average in the nonlinear mechanics. Kiev, Naukova dumka, 1971

[5] J. A. Sanders and F. Verhulst, Averaging methods in nonlinear dynamical systems. Appl. Math. Sci., Springer-Verlag, New York, 1985, vol. 59

[6] V. M. Volosov and B. I. Morgunov, Metod of average in the theory of nonlinear oscillations. Moscow, Moscow State University Publishing house, 1971

[7] P. Lochak, C. Meunier, Multiphase averaging for classical systems. Appl. Math. Sci., Springer-Verlag, New York, 1988, vol. 72

[8] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko and N. V. Skripnik, Differential equations with impulse effects: multivalued right-hand sides with discontinuities. De Gruyter Studies in Mathematics. Berlin/Boston: Walter De Gruyter GmbH\&Co., 2011, vol. 40

[9] A. V. Plotnikov and N. V. Skripnik, Differential equations with "clear" and fuzzy multivalued right-hand sides. Asymptotics Methods. AstroPrint, Odessa, 2009

[10] V. A. Plotnikov, A. V. Plotnikov and A. N. Vityuk, Differential equations with a multivalued right-hand side: Asymptotic methods. AstroPrint, Odessa, 1999

[11] L. Ambrosio and L. Tilli, Topics on analysis in metric spaces. Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2004, vol. 25

[12] V. Lakshmikantham, T. Granna Bhaskar and J. Vasundhara Devi, Theory of set differential equations in metric spaces. Cambridge Scientific Publishers. 2006

[13] V. Lakshmikantham and R. N. Mohapatra, Theory of Fuzzy Differential Equations and Inclusions. London, Taylor & Francis, 2003

[14] Plotnikov, A.V. and Skripnik, N.V., 2011, Set-valued differential equations with generalized derivative., J. Advanced Research in Pure Mathematics, 13(1), 144-160

[15] de Blasi, F. S. and Iervolino, F., 1969, Equazioni differentiali con soluzioni a valore compatto convesso., Boll. Unione Mat. Ital., 2(4-5), 491-501

[16] Bede, B. and Stefanini, L., 2008, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations., Univ. Urbino "Carlo Bo", Working Paper Series in Economics, Math. and Statistics. WP-EMS#2008/03

[17] de Blasi, F. S. and Iervolino, F., 1971, Euler method for differential equations with set-valued solutions., Boll. Unione Mat. Ital., 4(4), 941-949

[18] Brandao Lopes Pinto, A. J., de Blasi, F. S. and Iervolino, F., 1970, Uniqueness and existence theorems for differential equations with compact convex valued solutions., Boll. Unione Mat. Ital., (4), 534-538

[19] Dabrowska, R.. and Janiak, T., 1993, Stability of functional-differential equations with compact convex valued solutions., Discuss. Math., (13), 87-92

[20] Drici, Z., Mcrae, F. A. and Vasundhara Devi, J., 2005, Set differential equations with causal operators., Mathematical Problems in Engineering, 2005:2, 185–194

[21] Galanis, G. N., Gnana Bhaskar, T., Lakshmikantham, V., and Palamides, P. K., 2005, Set value functions in Frechet spaces: Continuity, Hukuhara differentiability and applications to set differential equations., J. Nonlinear Analysis, 61, 559-575

[22] Galanis, G. N., Tenali, G. B. and Lakshmikantham, V., 2008, Set differential equations in Frechet spaces., J. Appl. Anal., 14, 103-113

[23] Gnana Bhaskar, T. and Lakshmikantham, V., 2003, Set differential equations and flow invariance., Appl. Anal. (82), 357-368

[24] Gnana Bhaskar, T. and Lakshmikantham, V., 2004, Lyapunov stability for set differential equations., Dynam. Systems Appl., (13), 1-10

[25] Gnana Blaskar, T. and Vasundhara Devi, J., 2005, Set differential systems and vector Lyapunov functions., Appl. Math. Comput., 165(3), 539-548

[26] Kisielewicz, M., 1975, Description of a class of differential equations with set-valued solutions., Lincei-Rend. Sc. fis. mat. e nat., 58, 158-162

[27] Kisielewicz, M., Serafin, B. and Sosulski, W., 1975, Existence theorem for functional-differential equation with compact convex valued solutions., Demonstratio math., 13(2), 229-237

[28] Komleva, T. A., Plotnikov, A. V. and Skripnik, N. V., 2008, Differential equations with set-valued solutions., Ukr. Math. J., 60(10), 1540-1556

[29] Lakshmikantham, V., 2004, The connection between set and fuzzy differential equations., Facta Univ. Ser. Mech. Automat. Control. Robot., 4, 1-10

[30] Lakshmikantham, V., 2005, Set differential equations versus fuzzy differential equations., J. Applied Mathematics and Computation, 164, 277-294

[31] Laksmikantham, V., Leela, S. and Vatsala, A. S., 2002, Set-valued hybrid differential equations and stability in terms of two measures., J. Hybrid Systems, (2), 169-187

[32] Laksmikantham, V., Leela, S. and Vatsala, A. S., 2003, Interconnection between set and fuzzy differential equations., Nonlinear Anal., 54, 351-360

[33] Lakshmikantham, V. and Tolstonogov, A. A., 2003, Existence and interrelation between set and fuzzy differential equations., Nonlinear Anal., 55, 255-268

[34] Piszczek, M., 2006, Second Hukuhara derivative and cosine family of linear set-valued functions., An. Acad. Paedagogicae Cracoviensis. Studia Math., 33, 87-98

[35] Piszczek, M., 2008, On a multivalued second order differential problem with Hukuhara derivative., Opuscula Math., 28(2), 151-161

[36] Plotnikov, A. V., 2000, Differentiation of multivalued mappings. T-derivative., Ukr. Math. J., 52(8), 1282-1291

[37] Plotnikov, A. V. and Tumbrukaki, A. V., 2000, Integro-differential equations with multivalued solutions., Ukr. Math. J., 52(3), 413-423

[38] Plotnikova, N. V., 2006, Approximation of a set of solutions of linear differential inclusions., Nonlinear Oscil., 9(3), 375-390

[39] Skripnik, N. V., 2008, The Krasnoselskii-Krein theorem for differential equations with multivalued solutions., Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh., 826(58), 87-99

[40] Smajdor, A., 2003, On a multivalued differential problem., Internat. J. Bifur. Chaos Appl. Sci. Engrg., 13, 1877-1882

[41] Tise, I., 2007, Data dependence of the solutions for set differential equations., Carpathian J. Math., 23(1-2), 192-195

[42] A. Tolstonogov, Differential inclusions in a Banach space, Dordrecht, Kluwer Academic Publishers, 2000

[43] Vityuk, A. N., 2003, Differential equations of fractional order with set-valued solutions., Visn. Odes. Derzh. Univ., Ser. Fiz.-Mat. Nauky., 8(2), 108-112

[44] Ahmad, B. and Sivasundaram, S., 2008, $\varphi_0$-stability of impulsive hybrid setvalued differential equations with delay by perturbing lyapunov functions., Communications in Applied Analysis, 12(2), 137-146

[45] Arsirii, A. V. and Plotnikov, A. V., 2009, Systems of control over set-valued trajectories with terminal quality criterion., Ukr. Math. J., 61(8), 1349-1356

[46] N. D. Phu and T. T. Tung, Multivalued Differential Equations, VNU - HCM City: Publishing House, 2005

[47] Phu, N. D. and Tung, T. T., 2007, Existence of solutions of set control differential equations., J. Sci. Tech. Devel., 10(6), 5-14

[48] A. V. Plotnikov, "Differential inclusions with Hukuhara derivative and some control problems," Rep. VINITI 26.04.82, 2036-82, 1982

[49] Janiak, T. and Łuczak-Kumorek, E., 2003, Method on partial averaging for functional-differential equations with Hukuhara's derivative. Studia Univ. "Babeş-Bolyai", Math., XLVIII(2), 65-72

[50] Kichmarenko, O. D., 2009, Averaging of differential equations with Hukuhara derivative with maxima., Int. J. Pure Appl. Math., 57(3), 447-457

[51] Kisielewicz, M., 1976, Method of Averaging for Differential Equations with Compact Convex Valued Solutions., Rend. Math., 9(3), 397-408

[52] Plotnikov, V. A. and Kichmarenko, O. D., 2006, Averaging of controlled equations with the Hukuhara derivative., Nonlinear Oscil., 9(3), 365-374

[53] Plotnikov, V. A. and Rashkov, P. I., Averaging in differential equations with Hukuhara derivative and delay., Funct. Differ. Equ., (8), 371-381

[54] Radstrom, H, 1952, An embeldding theorem for spaces of convex sets., Proc. Amer. Math. Soc., (3), 165-169

[55] Hukuhara, M., 1967, Integration des applications mesurables dont la valeur est un compact convexe., Funkcial. Ekvac., (10), 205-223