American Journal of Systems Science 2017, 5(1): 13-27
DOI: 10.5923/j.ajss.20170501.02

# [0,1] Truncated Fréchet-Uniform and Exponential Distributions

**Salah H. Abid***, **Russul K. Abdulrazak**

Mathematics department, Education College, Al-Mustansiriya University, Baghdad, Iraq

**Abstract** In this paper, we introduce a new family of continuous distributions based on [0,1] truncated Fréchet distribution. [0,1] truncated Fréchet Uniform ([0,1] TFU ) and [0,1] truncated Fréchet Exponential ([0,1] TFE ) distributions are discussed as special cases. The cumulative distribution function, the *r*th moment, the mean, the variance, the skewness, the kurtosis, the mode, the median, the characteristic function, the reliability function and the hazard rate function are obtained for the distributions under consideration. It is well known that an item fails when a stress to which it is subjected exceeds the corresponding strength. In this sense, strength can be viewed as "resistance to failure". Good design practice is such that the strength is always greater than the expected stress. The safety factor can be defined in terms of strength and stress as strength/ stress. So, the [0,1] TFU strength-stress and the [0,1] TFE strength-stress models with different parameters will be derived here. The Shannon entropy and Relative entropy will be derived also.

**Keywords** [0,1] TFU, [0,1] TFE, Stress-strength model, Shannon entropy, Relative entropy

## 1. Introduction

Here, we proposed a distribution with the hope it will attract wider applicability in other fields. The generalization which is motivated by the work of Eugene et al. [2] will be our guide. Eugene et al. (2002) defined the beta G distribution from a quite arbitrary cumulative distribution function (cdf), G(x) by

$$F(x) = (1/\beta(a,b)) \int_0^{G(x)} w^{a-1} (1-w)^{b-1}\, dw \quad (1)$$

where a > 0 and b > 0 are two additional parameters whose role is to introduce skewness and to vary tail weight and $\beta(a,b) = \int_0^1 w^{a-1} (1-w)^{b-1}\, dw$ is the beta function. The class of distributions (1) has an increased attention after the works by Eugene et al. (2002) [2] and Jones (2004) [5]. Application of $X = G^{-1}(V)$ to the random variable V following a beta distribution with parameters a and b, V ~ B(a, b) say, yields X with cdf (1). Eugene et al. (2002) defined the beta normal (BN) distribution by taking G(x) to be the cdf of the normal distribution and derived some of its first moments. General expressions for the moments of the BN distribution were derived (Gupta and Nadarajah, 2004) [4]. An extensive review of scientific literature on this subject is available in Abid and Hassan (2015) [1]. We can write (1) as,

$$F(x) = I_{G(x)}(a,b) \quad (2)$$

Where, $I_y(a,b) = (1/B(a,b) \int_0^y w^{a-1} (1-w)^{b-1}\, dw$, denotes the incomplete beta function ratio, i.e., the cdf of the beta distribution with parameters $a$ and $b$. For general $a$ and $b$, we can express (2) in terms of the well-known hypergeometric function defined by,

$${}_2F_1(\alpha,\beta,\gamma;x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i\,(\beta)_i}{(\gamma)_i\, i!}\, x^i$$

Where $(\alpha)_i = \alpha(\alpha+1)\ldots(\alpha+i-1)$ denotes the ascending factorial. We obtain,

$$F(x) = \frac{G(x)^a}{a\,B(a,b)}\ {}_2F_1(a, 1-b, a+1; G(x))$$

The properties of the cdf, F(x) for any beta G distribution defined from a parent $G(x)$ in (1), could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of Gradshteyn and Ryzhik (2000) [3]. The probability density function (pdf) corresponding to (1) can be written in the form,

$$f(x) = \frac{1}{B(a,b)} G(x)^{a-1} \left(1 - G(x)\right)^{b-1} g(x) \quad (3)$$

where $g(x) = \partial G(x)/\partial x$ is the pdf of the parent distribution.

Now, since the pdf and cdf of [0,1] truncated Fréchet distribution are respectively,

$$h(x) = \frac{ab}{e^{-a}}\, x^{-(b+1)}\, e^{-ax^{-b}} \quad 0 < x < 1 \quad (4)$$

$$H(x) = \frac{1}{e^{-a}}\, e^{-ax^{-b}} \quad (5)$$

* Corresponding author:
abidsalah@uomustansiriyah.edu.iq (Salah H. Abid)
Published online at http://journal.sapub.org/ajss

Graphs for some arbitrary parameters values of pdf and cdf are shown in figure (1) and figure (2) respectively,

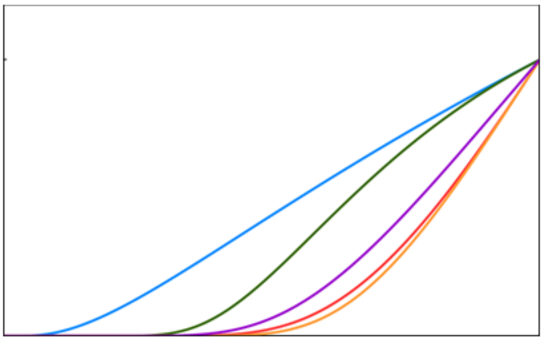

Figure (2) : cdf of (0,1)truncated Fréchet distribution with parameters a and b (a=2,b=0.5), (a=2,b=1.5),(a=0.5,b=2),(a=1.5,b=2) and (a=b=1.5)

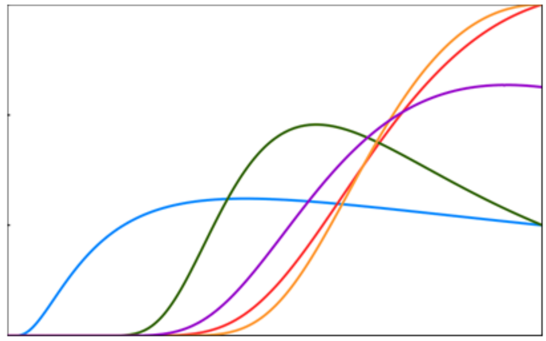

Figure (1) : pdf of (0,1)truncated Fréchet distribution with parameters a and b (a=2,b=0.5), (a=2,b=1.5),(a=0.5,b=2),(a=1.5,b=2) and (a=b=1.5)

Now, Given two absolutely continuous cdfs, H and G, so that h and g are their corresponding pdfs. We suggest a new distribution F by composing H with G, so that $F(x) = H(G(x))$ is a CDF,

$$F(x) = \int_0^{G(x)} \frac{ab}{e^{-a}}\, t^{-(b+1)}\, e^{-at^{-b}}\, dt \;=\; \frac{1}{e^{-a}}\, e^{-at^{-b}}\Big]_0^{G(x)} \;=\; \frac{1}{e^{-a}}\, e^{-aG(x)^{-b}} \tag{6}$$

With pdf,

$$f(x) = \frac{\partial}{\partial x}\,\mathrm{F(x)} \;= \frac{\partial}{\partial x}\frac{e^{-aG(x)^{-b}}}{e^{-a}} \;=\; \frac{ab}{e^{-a}}\, e^{-aG(x)^{-b}}\, \left(G(x)\right)^{-(b+1)}\, g(x) \tag{7}$$

With $G(x)$ being a baseline distribution, we define in (6) and (7) above, a generalized class of distributions. We will name it the [0,1] truncated Fréchet -G distribution.

In the following two sections, we will assume that G are Uniform and Exponential distributions respectively.

## 2. [0,1] Truncated Fréchet Uniform Distribution

Assume that $g(x) = 1/\theta$ and $G(x) = x/\theta$ ( $0 < x < \theta$) are pdf and cdf of Uniform random variable respectively, then, by applying (6) and (7) above, we get the cdf and pdf of [0,1] TFU random variable as follows,

$$F(x) = \frac{1}{e^{-a}}\, e^{-a\left(\frac{x}{\theta}\right)^{-b}}, 0 < x < \theta \tag{8}$$

$$f(x) = ab\theta^{b}\, x^{-(b+1)}\, e^{a\left(1-\left(\frac{x}{\theta}\right)^{-b}\right)}, 0 < x < \theta \tag{9}$$

So, the reliability and hazard rate functions are respectively

$$R(x) = 1 - F(x) = 1 - e^{a\left(1-\left(\frac{x}{\theta}\right)^{-b}\right)},$$

$$\lambda(x) = \frac{f(x)}{R(x)} = \frac{ab\theta^{b}\, x^{-(b+1)}\, e^{a\left(1-\left(\frac{x}{\theta}\right)^{-b}\right)}}{1 - e^{a\left(1-\left(\frac{x}{\theta}\right)^{-b}\right)}}$$

The rth raw moment can be derived as follows,

$$E(x^r) = \int_0^{\theta} x^r\, \frac{ab\,\theta^{b}}{e^{-a}}\, x^{-(b+1)}\, e^{-a\left(\frac{x}{\theta}\right)^{-b}}\, dx$$

$$= \frac{1}{e^{-a}} \int_0^{\theta} ab\theta^{b}\, x^{r-(b+1)}\, e^{-a\left(\frac{x}{\theta}\right)^{-b}}\, dx$$

$let\ y = a\left(\frac{x}{\theta}\right)^{-b} \;\Rightarrow\; x = \theta\, \left(\frac{y}{a}\right)^{\frac{-1}{b}} \;\Rightarrow dx = \frac{-\theta}{ab}\, \left(\frac{y}{a}\right)^{\frac{-1}{b}-1}\, dy$, then,

$$E(x^r) = \frac{1}{e^{-a}} \int_a^\infty ab\theta^b \left(\theta \left(\frac{y}{a}\right)^{\frac{-1}{b}}\right)^{r-(b+1)} e^{-y} \frac{\theta}{ab} \left(\frac{y}{a}\right)^{\frac{-1}{b}-1} dy$$

$$= \frac{1}{e^{-a}} \int_a^\infty \theta^b\, \theta^r\, \theta^{-b-1}\, \theta \left(\frac{y}{a}\right)^{\frac{-r}{b}} \left(\frac{y}{a}\right)^{1+\frac{1}{b}} e^{-y} \left(\frac{y}{a}\right)^{\frac{-1}{b}-1} dy$$

$$= \frac{\theta^r\, a^{\frac{r}{b}}}{e^{-a}} \int_a^\infty y^{\frac{-r}{b}} e^{-y}\, dy$$

$$= \frac{\theta^r\, a^{\frac{r}{b}}}{e^{-a}} \Gamma\left(1-\frac{r}{b}, a\right) \tag{10}$$

And then, the characteristic function is

$$\phi_x(t) = E(e^{ixt}) = E\left(\sum_{r=0}^{\infty} \frac{(ixt)^r}{r!}\right) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r)$$

$$= e^a \sum_{r=0}^{\infty} \frac{\left(it\theta a^{\frac{1}{b}}\right)^r}{r!} \Gamma\left(1-\frac{r}{b}, a\right)$$

So, the mean and variance of the of [0,1] TFU random variable are,

$$\mu = E(x) = \frac{\theta\, a^{\frac{1}{b}}}{e^{-a}} \Gamma\left(1-\frac{1}{b}, a\right) \tag{11}$$

$$\sigma^2 = \theta^2\, a^{\frac{2}{b}} \left(\frac{\Gamma\left(1-\frac{2}{b}, a\right)}{e^{-a}} - \frac{\Gamma^2\left(1-\frac{1}{b}, a\right)}{e^{-2a}}\right) \tag{12}$$

The mode $\mathrm{M}_o$ and the median $\mathrm{M}_e$ can be derived as,

$$f'(x) = \frac{-ab\theta^b}{e^{-a}} (b+1)\, x^{-(b+2)}\, e^{-a\left(\frac{x}{\theta}\right)^{-b}} + \frac{ab\theta^b}{e^{-a}} x^{-(b+1)}\, e^{-a\left(\frac{x}{\theta}\right)^{-b}} \frac{ab}{\theta} \left(\frac{x}{\theta}\right)^{-(b+1)} = 0$$

$$\Longrightarrow (b+1)\, x^{-(b+1)}\, x^{-1} = x^{-(b+1)} \frac{ab}{\theta} \frac{x^{-b-1}}{\theta^{-b-1}} \Longrightarrow (b+1) = ab\theta^b\, x^{-b}$$

$$x = \mathrm{M}_o = \theta\, a^{\frac{1}{b}} \left(\frac{b}{b+1}\right)^{\frac{1}{b}} \tag{13}$$

$$F(x) = \frac{1}{e^{-a}} e^{-a\left(\frac{x}{\theta}\right)^{-b}} = \frac{1}{2}$$

$$\Longrightarrow x = \mathrm{M}_e = \theta\left[1 + \frac{\ln(2)}{a}\right]^{\frac{-1}{b}} \tag{14}$$

The skewness of [0,1] TFU random variable will be,

$$sk = \frac{(\mu - \mathrm{M}_o)}{\sigma} = \frac{\frac{\theta\, a^{\frac{1}{b}}}{e^{-a}} \Gamma\left(1-\frac{1}{b}, a\right) - \theta\, a^{\frac{1}{b}} \left(\frac{b}{b+1}\right)^{\frac{1}{b}}}{\left\{\theta^2\, a^{\frac{2}{b}} \left[\frac{\Gamma\left(1-\frac{2}{b}, a\right)}{e^{-a}} - \frac{\Gamma^2\left(1-\frac{1}{b}, a\right)}{e^{-2a}}\right]\right\}^{\frac{1}{2}}}$$

$$= \frac{\frac{1}{e^{-a}} \Gamma\left(1-\frac{1}{b}, a\right) - \left(\frac{b}{b+1}\right)^{\frac{1}{b}}}{\left\{\frac{\Gamma\left(1-\frac{2}{b}, a\right)}{e^{-a}} - \frac{\Gamma^2\left(1-\frac{1}{b}, a\right)}{e^{-2a}}\right\}^{\frac{1}{2}}} \tag{15}$$

Also, the kurtosis is,

$$kr = \frac{\mu_4}{\sigma^4} - 3 = \frac{E(x^4) - 4\mu E(x^3) + 6\,\mu^2 E(x^2) - 3\mu^4}{\mu_2^2} - 3$$

$$= \frac{\left\{\begin{array}{c} \frac{\theta^4 a^{\frac{4}{b}}}{e^{-a}} \Gamma\left(1-\frac{4}{b}, a\right) - 4\left(\frac{\theta a^{\frac{1}{b}}}{e^{-a}}\right)\left\{\Gamma\left(1-\frac{1}{b}, a\right)\right\}\left(\frac{\theta^3 a^{\frac{3}{b}}}{e^{-a}}\right) \\ \left\{\Gamma\left(1-\frac{3}{b}, a\right)\right\} + 6\left[\left(\frac{\theta a^{\frac{1}{b}}}{e^{-a}}\right)\left\{\Gamma\left(1-\frac{1}{b}, a\right)\right\}\right]^2 \left(\frac{\theta^2 a^{\frac{2}{b}}}{e^{-a}}\right) \\ \left\{\Gamma\left(1-\frac{2}{b}, a\right)\right\} - 3\left[\left(\frac{\theta a^{\frac{1}{b}}}{e^{-a}}\right)\left\{\Gamma\left(1-\frac{1}{b}, a\right)\right\}\right]^4 \end{array}\right\}}{\left[\left(\frac{\theta^2 a^{\frac{2}{b}}}{e^{-a}}\right)\Gamma\left(1-\frac{2}{b}, a\right) - \left(\frac{\theta^2 a^{\frac{2}{b}}}{e^{-2a}}\right)\Gamma^2\left(1-\frac{1}{b}, a\right)\right]^2} - 3$$

$$= \frac{\frac{\theta^4 a^{\frac{4}{b}}}{e^{-4a}}\left\{\begin{array}{c} e^{-3a}\,\Gamma\left(1-\frac{4}{b}, a\right) - 4\,e^{-2a}\left\{\Gamma\left(1-\frac{1}{b}, a\right)\right\} \\ \left\{\Gamma\left(1-\frac{3}{b}, a\right)\right\} + 6\,e^{-a}\left\{\Gamma^2\left(1-\frac{1}{b}, a\right)\right\} \\ \left\{\Gamma\left(1-\frac{2}{b}, a\right)\right\} - 3\left\{\Gamma^4\left(1-\frac{1}{b}, a\right)\right\} \end{array}\right\}}{\frac{\theta^4 a^{\frac{4}{b}}}{e^{-4a}}\left[e^{-a}\,\Gamma\left(1-\frac{2}{b}, a\right) - \Gamma^2\left(1-\frac{1}{b}, a\right)\right]^2} - 3$$

$$= \frac{\left\{\begin{array}{c} e^{-3a}\,\Gamma\left(1-\frac{4}{b}, a\right) - 4\,e^{-2a}\left\{\Gamma\left(1-\frac{1}{b}, a\right)\right\} \\ \left\{\Gamma\left(1-\frac{3}{b}, a\right)\right\} + 6\,e^{-a}\left\{\Gamma^2\left(1-\frac{1}{b}, a\right)\right\} \\ \left\{\Gamma\left(1-\frac{2}{b}, a\right)\right\} - 3\left\{\Gamma^4\left(1-\frac{1}{b}, a\right)\right\} \end{array}\right\}}{\left[e^{-a}\,\Gamma\left(1-\frac{2}{b}, a\right) - \Gamma^2\left(1-\frac{1}{b}, a\right)\right]^2} - 3 \tag{16}$$

The quantile function $x_q$ of [0,1] TFU random variable can be derived as,

$$q = \mathrm{p}(x \le x_q) = F(x_q)\ 0 < q < 1, x_q > 0$$

$$q = \frac{1}{e^{-a}}\, e^{-a\left(\frac{x_q}{\theta}\right)^{-b}} \Rightarrow q\, e^{-a} = e^{-a\left(\frac{x_q}{\theta}\right)^{-b}} \Rightarrow -a\left(\frac{x_q}{\theta}\right)^{-b} = \ln(q\, e^{-a})$$

$$x_q = F^{-1}(q) = \theta\left[1 - \frac{\ln(q)}{a}\right]^{\frac{-1}{b}} \tag{17}$$

So by using the inverse transform method, we can generate [0,1] TFU random variable as follows,

$$x = \theta\left[1 - \frac{\ln(u)}{a}\right]^{\frac{-1}{b}}$$

Where $u$ is a random number distributed uniformly in the unit interval [0,1].

### 2.1. Shannon and Relative Entropies

An entropy of a random variable $X$ is a measure of variation of the uncertainty. The Shannon entropy of [0,1] TFU$(a, b, \theta)$ random variable $X$ can be found as follows,

$$H = E(-\ln(f(x)))$$

$$= \int_0^\theta \frac{ab\theta^b}{e^{-a}}\, x^{-(b+1)}\, e^{-a\left(\frac{x}{\theta}\right)^{-b}}\left[-\ln\left(\frac{ab\theta^b}{e^{-a}}\right) + (b+1)\ln(x) + a\left(\frac{x}{\theta}\right)^{-b}\right] dx$$

$$= -\ln\left(\frac{ab\theta^b}{e^{-a}}\right) + (b+1)\,E(\ln(x)) + a\,\theta^b\,E\left(\frac{1}{x^b}\right)$$

Let $I_1 = (b+1)\,E(\ln(x))$

$$= \frac{ab\theta^b}{e^{-a}}\,(b+1)\int_0^\theta \ln(x)\; x^{-(b+1)}\, e^{-a\theta^b x^{-b}}\, dx$$

$let\ y = x^{-b} \Rightarrow x = y^{\frac{-1}{b}} \Rightarrow dx = \frac{-1}{b}\, y^{\frac{-1}{b}-1}\, dy,$ then,

$$I_1 = \frac{ab\theta^b}{e^{-a}}\,(b+1)\int_{\theta^{-b}}^{\infty} \ln\left(y^{\frac{-1}{b}}\right)\left(y^{\frac{-1}{b}}\right)^{-(b+1)} e^{-a\theta^b y}\,\frac{1}{b}\,y^{\frac{-1}{b}-1}\,dy$$

$$= \frac{ab\theta^b(b+1)}{e^{-a}}\,\frac{-1}{b^2}\int_{\theta^{-b}}^{\infty} \ln(y)\;e^{-a\theta^b y}\;dy$$

$$= \frac{-a(b+1)\theta^b e^a}{b}\left[\int_0^{\infty} \ln(y)\;e^{-a\theta^b y}\,dy - \int_0^{\theta^{-b}} \ln(y)\;e^{-a\theta^b y}\,dy\right]$$

Since $\int_0^{\infty} x^{s-1}\;e^{-mx}\;\ln(x)\;\mathrm{dx} = m^{-s}\,\Gamma(s)\,\{\Psi(s) - \ln(m)\}$, where $\Psi(1) = -\ln(\gamma) \sim 0.5772$ and $\gamma = 0.5772$ is an Euler constant, then $\int_0^{\infty} \ln(y)\;e^{-a\theta^b y}\;dy = \frac{1}{a\theta^b}\,\{-\gamma - \ln(a\theta^b)\}$.

For, $I_{11} = \int_0^{\theta^{-b}} \ln(y)\;e^{-a\theta^b y}\;dy$, let $z = \theta^b\,y \longrightarrow y = \theta^{-b}\,z \longrightarrow dy = \theta^{-b}\,dz$, then,

$$I_{11} = \int_0^1 \ln(\theta^{-b}\,z)\;e^{-az}\;\theta^{-b}\;dz$$

$$= \theta^{-b}\;\ln(\theta^{-b})\int_0^1 e^{-az}\;dz + \theta^{-b}\int_0^1 \ln(z)\;e^{-az}\;dz\,,$$

$$= \frac{b}{a}\,\theta^{-b}\;\ln(\theta)\;e^{-az}\Big]_0^1 + \theta^{-b}\int_0^1 \ln(z)\sum_{m=0}^{\infty}\frac{(-a\,z)^m}{m!}\;dz\,,\;\; since\; e^{-az} = \sum_{m=0}^{\infty}\frac{(-a\,z)^m}{m!}$$

$$= \frac{b\theta^{-b}}{a}\;\ln(\theta)\;[e^{-a} - 1] + \theta^{-b}\sum_{m=0}^{\infty}\frac{(-a)^m}{m!}\int_0^1 \ln(z)\;z^m\;dz$$

$since\;\int x^m\;\ln(x)\;dx = x^{m+1}\left\{\frac{\ln(x)}{m+1} - \frac{1}{(m+1)^2}\right\}$, then,

$$I_{11} = \frac{b\ln(\theta)}{a\theta^b}\;(e^{-a} - 1) - \frac{1}{\theta^b}\sum_{m=0}^{\infty}\frac{(-a)^m}{m!\,(m+1)^2},\;\text{so,}$$

$$I_1 = \frac{b+1}{b}\;e^a\,\{\gamma + \ln(a\,\theta^b)\} + (b+1)\,e^a\;\ln(\theta)\;(e^{-a} - 1) - \frac{a(b+1)e^a}{b}\sum_{m=0}^{\infty}\frac{(-a)^m}{m!\;(m+1)^2}$$

$$I_2 = a\,\theta^b\,E\left(\frac{1}{x^b}\right)$$

$$= \frac{a^2 b\theta^{2b}}{e^{-a}}\int_0^{\theta} x^{-(2b+1)}\;e^{-a\left(\frac{x}{\theta}\right)^{-b}}\;dx$$

let $y = a\left(\frac{x}{\theta}\right)^{-b} \Longrightarrow x = \theta\left(\frac{y}{a}\right)^{\frac{-1}{b}} \Longrightarrow dx = \frac{-\theta}{ab}\left(\frac{y}{a}\right)^{\frac{-1}{b}-1}\;dy$, then,

$$I_2 = \frac{a^2 b\theta^{2b}}{e^{-a}}\int_a^{\infty}\left(\theta\left(\frac{y}{a}\right)^{\frac{-1}{b}}\right)^{-2b-1} e^{-y}\;\frac{\theta}{ab}\left(\frac{y}{a}\right)^{\frac{-1}{b}-1}\;dy$$

$= \frac{a}{e^{-a}}\int_a^{\infty}\frac{y}{a}\;e^{-y}\;dy \;=\; e^a\;\Gamma(2,a)$, then the Shannon entropy is

$$H = -\ln(a\,b\,\theta^b e^a) + \frac{b+1}{b}\;e^a\,\{\gamma + \ln(a\,\theta^b)\} + (b+1)\,e^a\;\ln(\theta)\;(e^{-a} - 1)$$
$$-\frac{a(b+1)}{b}\,e^a\sum_{m=0}^{\infty}\frac{(-a)^m}{m!\,(m+1)^2} + e^a\;\Gamma(2,a) \quad (18)$$

The relative entropy (or the Kullback–Leibler divergence) is a measure of the difference between two probability distributions $F_1$ and $F_2$. It is not symmetric in $F_1$ and $F_2$. In applications, $F_1$ typically represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution, while $F_2$ typically represents a theory, model, description, or approximation of $F_1$. Specifically, the Kullback–Leibler divergence of $F_2$ from $F_1$, denoted $D_{KL}(F_1 \parallel F_2)$, is a measure of the information gained when one revises ones beliefs from the prior probability distribution $F_2$ to the posterior probability distribution $F_1$. More exactly, it is the amount of information that is *lost* when $F_2$ is used to approximate $F_1$, defined operationally as the expected extra number of bits required to code samples from $F_1$ using a code optimized for $F_2$ rather than the code optimized for $F_1$.

So, the relative entropy $Dkl(F_1\|F_2)$ for a random variable $[0,1]$ TFU$(a,b,\theta)$ can be found as follows,

since, $\frac{f_1(x)}{f_2(x)} = \frac{a\;b\;\theta^b\;e^{-a}\;x^{-(b+1)}\;e^{-a\left(\frac{x}{\theta}\right)^{-b}}}{\alpha\;\beta\;\theta_1^{\beta}\;e^{-\alpha}\;x^{-(\beta+1)}\;e^{-\alpha\left(\frac{x}{\theta_1}\right)^{-\beta}}}$, then

$$Dkl(F_1\|F_2) = \int_0^\theta f_1(x)\ \ln\left(\frac{f_1(x)}{f_2(x)}\right)\ dx$$

$$= \int_0^\theta \frac{ab\theta^b}{e^{-a}}\ x^{-(b+1)}\ e^{-a\left(\frac{x}{\theta}\right)^{-b}}\ \ln\left(\frac{ab\theta^b e^{-\alpha}\ x^{-(b+1)}\ e^{-a\left(\frac{x}{\theta}\right)^{-b}}}{\alpha\beta\theta_1^\beta e^{-a}\ x^{-(\beta+1)}\ e^{-\alpha\left(\frac{x}{\theta_1}\right)^{-\beta}}}\right) dx$$

$$= \int_0^\theta \frac{ab\theta^b}{e^{-a}}\ x^{-(b+1)}\ e^{-a\left(\frac{x}{\theta}\right)^{-b}}\ \left[\ln\left(\frac{ab\theta^b e^{-\alpha}}{\alpha\beta\theta_1^\beta e^{-a}}\right) + (\beta-b)\ \ln(x) - a\left(\frac{x}{\theta}\right)^{-b}\ \alpha\left(\frac{x}{\theta_1}\right)^{-\beta}\right] dx$$

$$= \ln\left(\frac{ab\theta^b e^{-\alpha}}{\alpha\beta\theta_1^\beta e^{-a}}\right) + (\beta-b)\ E(\ln(x)) - a\ \theta^b\ E\left(\frac{1}{x^b}\right) + \alpha\ \theta_1^\beta\ E\left(\frac{1}{x^\beta}\right)$$

Since, $I_1 = (\beta-b)\ E(\ln(x))$

$$= (\beta-b)\int_0^\theta \ln(x)\ \frac{ab\theta^b}{e^{-a}}\ x^{-(b+1)}\ e^{-a\left(\frac{x}{\theta}\right)^{-b}}\ dx$$

$$= \frac{(\beta-b)\ e^a}{b}\ \{\gamma + \ln(a\ \theta^b)\} + (\beta-b)\ e^a\ \ln(\theta)\ (e^{-a}-1) - \frac{a\ (\beta-b)\ e^a}{b}\sum_{m=0}^{\infty}\frac{(-a^m)}{m!\ (m+1)^2}$$

And, $I_2 = -a\ \theta^b\ E\left(\frac{1}{x^b}\right)$

$$= -a\ \theta^b \int_0^\theta \frac{1}{x^b}\ \frac{ab\theta^b}{e^{-a}}\ x^{-(b+1)}\ e^{-a\left(\frac{x}{\theta}\right)^{-b}}\ dx = -e^a\ \Gamma(2,a)$$

And, $I_3 = \alpha\ \theta_1^\beta\ E\left(\frac{1}{x^\beta}\right)$

$$= \alpha\ \theta_1^\beta\ \int_0^\theta \frac{1}{x^\beta}\ \frac{ab\,\theta^b}{e^{-a}}\ x^{-(b+1)}\ e^{-a\left(\frac{x}{\theta}\right)^{-b}}\ dx$$

let $y = a\left(\frac{x}{\theta}\right)^{-b} \Rightarrow x = \theta\left(\frac{y}{a}\right)^{\frac{-1}{b}} \Rightarrow dx = \frac{-\theta}{ab}\left(\frac{y}{a}\right)^{\frac{-1}{b}-1} dy,$ then,

$I_3 = \frac{\alpha\theta_1^\beta}{e^{-a}\theta^\beta a^{\frac{\beta}{b}}}\int_a^\infty y^{\frac{\beta}{b}}\ e^{-y}\ dy = \frac{\alpha\ e^a\ (\theta_1/\theta)^\beta\ \Gamma\left(\frac{\beta}{b}+1,a\right)}{a^{\frac{\beta}{b}}}$, so the relative entropy is,

$$Dkl(F_1\|F_2) = \ln\left(\frac{ab\theta^b e^{-\alpha}}{\alpha\beta\theta_1^\beta e^{-a}}\right) + \frac{(\beta-b)\ e^a}{b}\ \{\gamma + \ln(a\ \theta^b)\} + (\beta-b)\ e^a \ln(\theta)\ (e^{-a}-1)$$

$$- \frac{a\ (\beta-b)\ e^a}{b}\sum_{m=0}^{\infty}\frac{(-a)^m}{m!\ (m+1)^2} - e^a\ \Gamma(2,a) + \frac{\alpha\ e^a\ (\theta_1/\theta)^\beta\ \Gamma\left(\frac{\beta}{b}+1,a\right)}{a^{\frac{\beta}{b}}} \tag{19}$$

**2.2. Stress-Strength Reliability**

Inferences about R = P[Y < X], where X and Y are two independent random variables, is very common in the reliability literature. For example, if X is the strength of a component which is subject to a stress Y, then R is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength. Let Y and X be the stress and the strength random variables, independent of each other, follow respectively $[0,1]\ \mathrm{TFU}(a,b,\theta)$ and $[0,1]\ \mathrm{TFU}(\alpha,\beta,\theta_1)$, then,

$$R = P(y < x) = \int_0^\theta f_x(x)\ F_y(x)\ dx$$

$$= \int_0^\theta \frac{ab\theta^b}{e^{-a}}\ x^{-(b+1)}\ e^{-a\left(\frac{x}{\theta}\right)^{-b}}\ \frac{1}{e^{-\alpha}}\ e^{-\alpha\left(\frac{x}{\theta_1}\right)^{-\beta}}\ dx$$

since $e^{-\alpha\left(\frac{x}{\theta_1}\right)^{-\beta}} = \sum_{m=0}^{\infty} \frac{\left(-\alpha\, \theta_1^{\beta}\right)^m}{m!}\, x^{-\beta m}$, then,

$$R = \frac{1}{e^{-a} e^{-\alpha}} \sum_{m=0}^{\infty} \frac{\left(-\alpha\, \theta_1^{\beta}\right)^m}{m!} \int_0^{\theta} ab\theta^b\, x^{-(b+1)}\, e^{-a\left(\frac{x}{\theta}\right)^{-b}}\, x^{-\beta m}\, dx$$

let $y = a\left(\frac{x}{\theta}\right)^{-b} \Rightarrow x = \theta\left(\frac{y}{a}\right)^{\frac{-1}{b}} \Rightarrow dx = \frac{-\theta}{ab}\left(\frac{y}{a}\right)^{\frac{-1}{b}-1} dy$, so,

$$R = \frac{1}{e^{-a} e^{-\alpha}} \sum_{m=0}^{\infty} \frac{\left(-\alpha\, \theta_1^{\beta}\right)^m}{m!} \int_a^{\infty} ab\theta^b \left(\theta\left(\frac{y}{a}\right)^{\frac{-1}{b}}\right)^{-(b+1)} e^{-y} \left(\theta\left(\frac{y}{a}\right)^{\frac{-1}{b}}\right)^{-\beta m} \frac{\theta}{ab}\left(\frac{y}{a}\right)^{\frac{-1}{b}-1} dy$$

$$= \frac{1}{e^{-a} e^{-\alpha}} \sum_{m=0}^{\infty} \frac{\left(-\alpha\theta_1^{\beta}\right)^m}{m!} \int_a^{\infty} \frac{\theta^{-\beta m}}{a^{\frac{\beta m}{b}}}\, y^{\frac{\beta m}{b}}\, e^{-y}\, dy$$

$$= \frac{1}{e^{-a} e^{-\alpha}} \sum_{m=0}^{\infty} \frac{\left(-\alpha\theta_1^{\beta}\right)^m}{m!\; \theta^{\beta m}\; a^{\frac{\beta m}{b}}}\, \Gamma\left(\frac{\beta m}{b} + 1, a\right)$$

$$= e^{a+\alpha} \sum_{m=0}^{\infty} \frac{\left(-\alpha\left(\theta_1/\theta a^{\frac{1}{b}}\right)^{\beta}\right)^m}{m!}\, \Gamma\left(\frac{\beta m}{b} + 1, a\right) \tag{20}$$

## 3. [0,1] Truncated Fréchet Exponential Distribution

Assume that $g(x) = \theta\,\mathrm{Exp}\{-\theta x\}$ and $G(x) = 1 - \mathrm{Exp}\{-\theta x\}$ $(0 < x)$ are pdf and cdf of Exponential random variable respectively, then, by applying (6) and (7) above, we get the pdf and cdf of [0,1] TFE random variable as follows,

$$F(x) = \frac{1}{e^{-a}}\, e^{-a\left(1-e^{-\theta x}\right)^{-b}} \tag{21}$$

$$f(x) = \frac{\theta ab}{e^{-a}}\, e^{-\theta x}\left(1 - e^{-\theta x}\right)^{-(b+1)} e^{-a\left(1-e^{-\theta x}\right)^{-b}} \quad x \geq 0 \tag{22}$$

So, the reliability $R(x)$ and hazard rate $\lambda(x)$ functions are respectively

$$R(x) = 1 - \frac{1}{e^{-a}}\, e^{-a\left(1-e^{-\theta x}\right)^{-b}}$$

$$= 1 - e^{-a\left[\left(1-e^{-\theta x}\right)^{-b} - 1\right]}$$

$$\lambda(x) = \frac{\frac{\theta ab}{e^{-a}}\, e^{-\theta x}\left(1 - e^{-\theta x}\right)^{-(b+1)} e^{-a\left(1-e^{-\theta x}\right)^{-b}}}{1 - e^{-a\left[\left(1-e^{-\theta x}\right)^{-b} - 1\right]}}$$

The r*th* raw moment can be derived as follows,

$$E(x^r) = \frac{\theta ab}{e^{-a}} \int_0^{\infty} x^r\, e^{-\theta x}\left(1 - e^{-\theta x}\right)^{-(b+1)} e^{-a\left(1-e^{-\theta x}\right)^{-b}}\, dx$$

by using poisson series, $e^{-a\left(1-e^{-\theta x}\right)^{-b}} = \sum_{i=0}^{\infty} \frac{(-a)^i}{i!}\left(1 - e^{-\theta x}\right)^{-bi}$, we get,

$$E(x^r) = \frac{\theta ab}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\, a^i \int_0^{\infty} x^r\, e^{-\theta x}\left(1 - e^{-\theta x}\right)^{-(b+1)}\left(1 - e^{-\theta x}\right)^{-bi}\, dx$$

$$= \frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\, a^{i+1} \int_0^{\infty} x^r\, e^{-\theta x}\left(1 - e^{-\theta x}\right)^{-[b(i+1)+1]}\, dx$$

By using the series expansion $(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{j!\,\Gamma(k)}\, z^j \;\; |z| < 1, k > 0,$ we get,
$\left(1 - e^{-\theta x}\right)^{-[b(i+1)+1]} = \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\, e^{-\theta jx}$, and then,

$$E(x^r) = \frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \int_0^{\infty} x^r e^{-\theta x} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} e^{-\theta j x}\, dx$$

$$= \frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \int_0^{\infty} x^r e^{-(1+j)\theta x}\, dx$$

$$= \frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])} \frac{\Gamma(r+1)}{[(1+j)\theta]^{r+1}} \quad (23)$$

And then, the characteristic function is

$$\phi_x(t) = E(e^{ixt})$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r), \ \ since\ e^{ixt} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} x^r$$

$$= \frac{\theta b}{e^{-a}} \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{\Gamma(r+1)}{[(1+j)\theta]^{r+1}}$$

So, the mean $\mu$ and variance $\sigma^2$ of the of [0,1] TFE random variable are,

$$\mu = E(x) = \frac{b}{\theta e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^2} \quad (24)$$

$$= E(x^2) - (Ex)^2$$

$$\sigma^2 = \frac{2b}{\theta^2 e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^3} -$$

$$\left\{\frac{b}{\theta e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^2}\right\}^2 \quad (25)$$

The mode $M_o$ has not closed form, whilst the median $M_e$ can be derived as,

Since, $F(x) = \frac{e^{-a\left(1-e^{-\theta x}\right)^{-b}}}{e^{-a}} = \frac{1}{2}$, then,

$$x = M_e = \frac{-1}{\theta} \ln\left[1 - \left(1 + \frac{\ln(2)}{a}\right)^{\frac{-1}{b}}\right] \quad (26)$$

The skewness of [0,1] TFE random variable will be,

$$sk = \frac{\mu_3}{\mu_2^{3/2}} = \frac{E(x^3) - 3\mu E(x^2) + \mu^3}{(\sigma^2)^{3/2}} = \frac{\left(\begin{array}{c} \frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{\Gamma(4)}{[(1+j)\theta]^4} - 3 \\ \left\{\frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{\Gamma(2)}{[(1+j)\theta]^2}\right\} \\ \left\{\frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{\Gamma(3)}{[(1+j)\theta]^3}\right\} + 2 \\ \left\{\frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{\Gamma(2)}{[(1+j)\theta]^2}\right\}^3 \end{array}\right)}{\left(\begin{array}{c} \frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{\Gamma(3)}{[(1+j)\theta]^3} - \\ \left\{\frac{\theta b}{e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{\Gamma(2)}{[(1+j)\theta]^2}\right\}^2 \end{array}\right)^{3/2}}$$

$$= \frac{\left(\begin{array}{c} \frac{6b}{\theta^3 e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^4} - 3 \\ \left\{\frac{b}{\theta e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^2}\right\} \\ \left\{\frac{2b}{\theta^2 e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^3}\right\} + 2 \\ \left\{\frac{b}{\theta e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^2}\right\}^3 \end{array}\right)}{\left(\begin{array}{c} \frac{2b}{\theta^2 e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^3} - \\ \left\{\frac{b}{\theta e^{-a}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])} \frac{1}{(1+j)^2}\right\}^2 \end{array}\right)^{3/2}} \quad (27)$$

Also, the kurtosis is,

$$kr=\frac{\mu_4}{\mu_2^2}-3=\frac{E(x^4)-4\mu E(x^3)+6\mu^2E(x^2)-3\mu^4}{(\sigma^2)^2}-3$$

$$=\frac{\left\{\begin{array}{l}\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{\Gamma(5)}{[(1+j)\theta]^5}-4\\ \left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{\Gamma(2)}{[(1+j)\theta]^2}\right\}\\ \left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{\Gamma(4)}{[(1+j)\theta]^4}\right\}+\\ 6\left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{\Gamma(2)}{[(1+j)\theta]^2}\right\}^2\\ \left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{\Gamma(3)}{[(1+j)\theta]^3}\right\}-\\ 3\left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{\Gamma(2)}{[(1+j)\theta]^2}\right\}^4\end{array}\right\}}{\left\{\begin{array}{l}\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{\Gamma(3)}{[(1+j)\theta]^3}-\\ \left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{\Gamma([b(i+1)+1])}\frac{\Gamma(2)}{[(1+j)\theta]^2}\right\}^2\end{array}\right\}^2}-3$$

$$=\frac{\left\{\begin{array}{l}\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{24}{[(1+j)\theta]^5}-4\\ \left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{[(1+j)\theta]^2}\right\}\\ \left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{6}{[(1+j)\theta]^4}\right\}+\\ 6\left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{[(1+j)\theta]^2}\right\}^2\\ \left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{2}{[(1+j)\theta]^3}\right\}-\\ 3\left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{[(1+j)\theta]^2}\right\}^4\end{array}\right\}}{\left\{\begin{array}{l}\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{2}{[(1+j)\theta]^3}-\\ \left\{\frac{\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{[(1+j)\theta]^2}\right\}^2\end{array}\right\}^2}-3$$

$$=\frac{\left\{\begin{array}{l}\frac{24b}{\theta^4e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=o}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{(1+j)^5}-4\\ \left\{\frac{b}{\theta e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{(1+j)^2}\right\}\\ \left\{\frac{6b}{\theta^3e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{(1+j)^4}\right\}+6\\ \left\{\frac{b}{\theta e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{(1+j)^2}\right\}^2\\ \left\{\frac{2b}{\theta^2e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{(1+j)^3}\right\}-\\ 3\left\{\frac{b}{\theta e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{(1+j)^2}\right\}^4\end{array}\right\}}{\left\{\begin{array}{l}\frac{2b}{\theta^2e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+1)+1])}\frac{1}{(1+j)^3}-\\ \left\{\frac{b}{\theta e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\;\Gamma([b(i+)+1])}\frac{1}{(1+j)^2}\right\}^2\end{array}\right\}^2}-3$$

$$= \frac{\left\{\begin{array}{c} \frac{24e^{-3a}}{b^3}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{1}{(1+j)^5}- \\ 4\frac{e^{-2a}}{b^2}\left\{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{1}{(1+j)^2}\right\} \\ \left\{6\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{1}{(1+j)^4}\right\}+ \\ 6\frac{e^{-a}}{b}\left\{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{1}{(1+j)^2}\right\}^2 \\ \left\{2\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{1}{(1+j)^3}\right\}- \\ 3\left\{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{\Gamma([b(i+1)+1])}\frac{1}{(1+j)^2}\right\}^4 \end{array}\right\}}{\left\{\begin{array}{c} \frac{2e^{-a}}{b}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{1}{(1+j)^3} \\ -\left\{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{1}{(1+j)^2}\right\}^2 \end{array}\right\}^2} - 3 \quad (28)$$

The quantile function $x_q$ of [0,1] TFE random variable can be derived as,

$$q = P(x \le x_q) = F_1x(x_q) = \frac{e^{-a(1-e^{-\theta x})^{-b}}}{e^{-a}} \quad 0 < q < 1 \; x_q > 0$$

$$\Rightarrow x_q = F_1^{-1}(q) = \frac{-1}{\theta}\ln\left[1-\left(1-\frac{\ln(q)}{a}\right)^{\frac{-1}{b}}\right] \quad (29)$$

So by using the inverse transform method, we can generate [0,1] TFE random variable as follows,

$$x = \frac{-1}{\theta}\ln\left[1-\left(1-\frac{\ln(u)}{a}\right)^{\frac{-1}{b}}\right]$$

Where u is a random number distributed uniformly in the unit interval [0,1].

### 3.1. Shannon and Relative Entropies

The Shannon entropy of [0,1] TFE$(a,b,\theta)$ random variable $X$ can be found as follows,

$$H = -\int_{-\infty}^{\infty} f(x)\,\ln(f_1(x))\,dx$$

$$= -\int_0^{\infty} f(x)\,\ln\left(\frac{\theta ab}{e^{-a}}\,e^{-\theta x}\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-a(1-e^{-\theta x})^{-b}}\right)dx$$

$$= \ln\left(\frac{e^{-a}}{\theta ab}\right) + \theta E(x) + (b+1)E\left(\ln\left(1-e^{-\theta x}\right)\right) + aE\left(\left(1-e^{-\theta x}\right)^{-b}\right)$$

Let, $I_1 = \theta E(x)$

$$= \theta\,\frac{b}{\theta e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\,a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\,\frac{1}{(1+j)^2}$$

$$= be^{a}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\,a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\,\frac{1}{(1+j)^2}$$

And, $I_2 = (b+1)E\left(\ln\left(1-e^{-\theta x}\right)\right)$

$$= (b+1)\int_0^{\infty}\ln\left(1-e^{-\theta x}\right)\frac{\theta ab}{e^{-a}}\,e^{-\theta x}\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-a(1-e^{-\theta x})^{-b}}\,dx$$

$$\Rightarrow let\; y = \left(1-e^{-\theta x}\right)^{-b} \Rightarrow x = \frac{-1}{\theta}\ln\left(1-y^{\frac{-1}{b}}\right) \Rightarrow dx = \frac{-1}{\theta b\left(1-y^{\frac{-1}{b}}\right)}\,y^{\frac{-1}{b}-1}\,dy$$

$$I_2 = \frac{(b+1)\theta ab}{e^{-a}} \int_1^\infty \ln\left(y^{\frac{-1}{b}}\right) \left(1 - y^{\frac{-1}{b}}\right) \left(y^{\frac{-1}{b}}\right)^{-b-1} e^{-ay} \frac{1}{\theta b\left(1-y^{\frac{-1}{b}}\right)} y^{\frac{-1}{b}-1} dy$$

$$= \frac{-(b+1)a}{e^{-a}b} \int_1^\infty \ln(y)\ e^{-ay}\ dy$$

$$= \frac{-(b+1)a}{e^{-a}b} \left[\int_0^\infty \ln(y)\, e^{-ay}\, dy - \int_0^1 \ln(y)\ e^{-ay}\ dy\right]$$

$$I_{21} = \int_0^\infty \ln(y)\ e^{-ay}\ dy = \frac{1}{a}\{-\gamma - \ln(a)\} = \frac{-1}{a}\{\gamma + \ln(a)\}$$

since $\int_0^\infty x^{s-1}\ e^{-mx}\ \ln(x)\ dx = m^{-s}\,\Gamma(s)\,\{\Psi(s) - \ln(m)\}$, where $\Psi(1) = -\gamma$ $= 0.57721$ is an Eualer constant.

$$I_{22} = \int_0^1 \ln(y)\ e^{-ay}\, dy = \sum_{m=0}^\infty \frac{(-a)^m}{m!} \int_0^1 y^m\ \ln(y)\ dy = -\sum_{m=0}^\infty \frac{(-a)^m}{m!\,(m+1)^2},$$

since $e^{-ay} = \sum_{m=0}^\infty \frac{(-ay)^m}{m!}$ And $\int x^m\ \ln(x)\ dx = x^{m+1}\left\{\frac{\ln(x)}{m+1} - \frac{1}{(m+1)^2}\right\}$

$$I_2 = \frac{(b+1)}{e^{-a}b}\left\{\{\gamma + \ln(a)\} - \sum_{m=0}^\infty \frac{(-1)^m a^{m+1}}{m!\,(m+1)^2}\right\}$$

And, $I_3 = aE\left(\left(1 - e^{-\theta x}\right)^{-b}\right)$

$$= a\int_0^\infty \left(1 - e^{-\theta x}\right)^{-b} f(x)\, dx$$

$$= a\int_0^\infty \left(1 - e^{-\theta x}\right)^{-b} \frac{\theta ab}{e^{-a}} e^{-\theta x} \left(1 - e^{-\theta x}\right)^{-(b+1)} e^{-a\left(1-e^{-\theta x}\right)^{-b}}\ dx$$

$$= b\theta e^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \int_0^\infty e^{-\theta x} \left(1 - e^{-\theta x}\right)^{-(2b+1)} \left(1 - e^{-\theta x}\right)^{-bi} dx$$

since $e^{-a\left(1-e^{-\theta x}\right)^{-b}} = \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^i \left(1 - e^{-\theta x}\right)^{-bi}$, then,

$$I_3 = b\theta e^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \int_0^\infty e^{-\theta x} \left(1 - e^{-\theta x}\right)^{-(bi+2b+1)} dx$$

By using the series expansion $(1 - z)^{-k} = \sum_{j=0}^\infty \frac{\Gamma(k+j)}{j!\,\Gamma(k)} z^j$, we get,
$\left(1 - e^{-\theta x}\right)^{-(bi+2b+1)} \sum_{j=0}^\infty \frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])} e^{-j\theta x}$, then,

$$I_3 = b\theta e^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \int_0^\infty e^{-\theta x} \sum_{j=0}^\infty \frac{\Gamma([b(i+2)+1]+j)}{j!\ \Gamma([b(i+2)+1])} e^{-j\theta x}\ dx$$

$$= b\theta e^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \sum_{j=0}^\infty \frac{\Gamma([b(i+2)+1]+j)}{j!\ \Gamma([b(i+2)+1])} \int_0^\infty e^{-(j+1)\theta x}\ dx$$

$$= be^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+2} \sum_{j=0}^\infty \frac{\Gamma([b(i+2)+1]+j)}{j!\ \Gamma([b(i+2)+1])} \frac{1}{(j+1)}$$

$$H = \ln\left(\frac{e^{-a}}{\theta ab}\right) + be^a \sum_{i=0}^\infty \frac{(-1)^i}{i!} a^{i+1} \sum_{j=0}^\infty \frac{\Gamma([b(i+1)+1]+j)}{j!\ \Gamma([b(i+1)+1])} \frac{1}{(1+j)^2}$$

$$+\frac{(b+1)}{e^{-a}b}\left\{\{\gamma+\ln(a)\}-\sum_{m=0}^{\infty}\frac{(-1)^m a^{m+1}}{m!\,(m+1)^2}\right\}+be^{a}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}$$

$$a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])}\frac{1}{(j+1)} \quad (30)$$

The relative entropy $Dkl(F_1\|F_2)$ for a random variable $[0,1]$ TFU$(a,b,\theta)$ can be found as follows,

$$\frac{f_1(x)}{f_2(x)}=\frac{e^{-\alpha}\theta ab\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-\theta x}\,e^{-a\left(1-e^{-\theta x}\right)^{-b}}}{e^{-a}\theta_1\alpha\beta(1-e^{-\theta_1 x})^{-(\beta+1)}\,e^{-\theta_1 x}\,e^{-\alpha\left(1-e^{-\theta_1 x}\right)^{-\beta}}}$$

$$Dkl(F_1\|F_2)=\int_0^{\infty}f_1(x)\,\ln\left(\frac{e^{-\alpha}\theta ab\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-\theta x}\,e^{-a\left(1-e^{-\theta x}\right)^{-b}}}{e^{-a}\theta_1\alpha\beta\,(1-e^{-\theta_1 x})^{-(\beta+1)}\,e^{-\theta_1 x}\,e^{-\alpha\left(1-e^{-\theta_1 x}\right)^{-\beta}}}\right)dx$$

$$=\int_0^{\infty}f_1(x)\,[\ln\left(\frac{e^{-\alpha}\theta ab}{e^{-a}\theta_1\alpha\beta}\right)+(\theta_1-\theta)x-(b+1)\ln\left(1-e^{-\theta x}\right)-$$

$$a\left(1-e^{-\theta x}\right)^{-b}+(\beta+1)\ln\left(1-e^{-\theta_1 x}\right)+\alpha\left(1-e^{-\theta_1 x}\right)^{-\beta}]dx$$

$$=\ln\left(\frac{e^{-\alpha}\theta ab}{e^{-a}\theta_1\alpha\beta}\right)+(\theta_1-\theta)E(x)-(b+1)E\left(\ln\left(1-e^{-\theta x}\right)\right)-$$

$$aE\left(\left(1-e^{-\theta x}\right)^{-b}\right)+(\beta+1)E\left(\ln\left(1-e^{-\theta_1 x}\right)\right)+\alpha E\left(\left(1-e^{-\theta_1 x}\right)^{-\beta}\right)$$

Let, $I_1=(\theta_1-\theta)E(x)$

$$=\frac{(\theta_1-\theta)b}{\theta e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\,a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\frac{1}{(1+j)^2}$$

And, $I_2=-(b+1)E\left(\ln\left(1-e^{-\theta x}\right)\right)$

$$=-(b+1)\int_0^{\infty}\ln\left(1-e^{-\theta x}\right)\frac{\theta ab}{e^{-a}}e^{-\theta x}\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-a\left(1-e^{-\theta x}\right)^{-b}}dx$$

$$=\frac{(b+1)}{be^{-a}}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m a^{m+1}}{m!\,(m+1)^2}-\{\gamma+\ln(a)\}\right\}$$

And, $I_3=-aE\left(\left(1-e^{-\theta x}\right)^{-b}\right)$

$$=-a\int_0^{\infty}\left(1-e^{-\theta x}\right)^{-b}\frac{\theta ab}{e^{-a}}e^{-\theta x}\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-a\left(1-e^{-\theta x}\right)^{-b}}dx$$

$$=-be^{a}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])}\frac{1}{(j+1)}$$

And, $I_4=(\beta+1)E\left(\ln\left(1-e^{-\theta_1 x}\right)\right)$

$$=(\beta+1)\frac{\theta ab}{e^{-a}}\int_0^{\infty}\ln\left(1-e^{-\theta_1 x}\right)e^{-\theta x}\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-a\left(1-e^{-\theta x}\right)^{-b}}dx$$

$\Rightarrow let\ y=1-e^{-\theta_1 x}\ \Rightarrow\ x=\frac{-1}{\theta_1}\ln(1-y)\ \Rightarrow dx=\frac{1}{\theta_1(1-y)}dy$, then,

$$I_4=(\beta+1)\frac{\theta ab}{e^{-a}}\int_0^1\ln(y)\,e^{\frac{\theta}{\theta_1}\ln(1-y)}\left(1-e^{\frac{\theta}{\theta_1}\ln(1-y)}\right)^{-(b+1)}e^{-a\left(1-e^{\frac{\theta}{\theta_1}\ln(1-y)}\right)^{-b}}\frac{1}{\theta_1(1-y)}dy$$

$$=(\beta+1)\frac{\theta ab}{e^{-a}\theta_1}\int_0^1\ln(y)\,(1-y)^{\frac{\theta}{\theta_1}}\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-(b+1)}(1-y)^{-1}\,e^{-a\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-b}}dy$$

$$I_4=(\beta+1)\frac{\theta ab}{e^{-a}\theta_1}\int_0^1\ln(y)\,(1-y)^{\frac{\theta}{\theta_1}-1}\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-(b+1)}e^{-a\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-b}}dy$$

Since, $e^{-a\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-b}}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-bi}$, then,

$$I_4=(\beta+1)\frac{\theta b}{e^{-a}\theta_1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^1\ln(y)\,(1-y)^{\frac{\theta}{\theta_1}-1}\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-(b+1)}\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-bi}dy$$

$$= (\beta+1)\ \frac{\theta b}{e^{-a}\theta_1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ a^{i+1}\int_0^1 \ln(y)\ (1-y)^{\frac{\theta}{\theta_1}-1}\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-(bi+b+1)} dy$$

*by using expansion series* $(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{j!\,\Gamma(k)}\ z^j\ |z|<1, k>0$, we get,

$\left(1-(1-y)^{\frac{\theta}{\theta_1}}\right)^{-(bi+b+1)} = \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\ (1-y)^{\frac{j\theta}{\theta_1}}$, and then,

$$I_4 = (\beta+1)\frac{\theta b}{e^{-a}\theta_1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\int_0^1 \ln(y)\,(1-y)^{\frac{\theta}{\theta_1}-1}\ (1-y)^{\frac{j\theta}{\theta_1}}\,dy$$

$$= (\beta+1)\frac{\theta b}{e^{-a}\theta_1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\,a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\int_0^1 \ln(y)\,(1-y)^{\frac{\theta}{\theta_1}(j+1)-1}\ dy$$

*by using the for formala* $(1-z)^{b-1} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(b)}{k!\,\Gamma(b-k)}\ z^k$, we get,

$(1-y)^{\frac{\theta}{\theta_1}(j+1)-1} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(\frac{\theta}{\theta_1}(j+1)\right)}{k!\,\Gamma\left(\frac{\theta}{\theta_1}(j+1)-k\right)}\ y^k$, and then,

$$I_4 = (\beta+1)\frac{\theta b}{e^{-a}\theta_1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{i!k!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\ \frac{\Gamma\left(\frac{\theta}{\theta_1}(j+1)\right)}{\Gamma\left(\frac{\theta}{\theta_1}(j+1)-k\right)}\int_0^1 \ln(y)\ y^k\ dy$$

*since* $\int x^m\ \ln(x)\ dx = x^{m+1}\left\{\frac{\ln(x)}{m+1}-\frac{1}{(m+1)^2}\right\}$, then,

$$I_4 = -(\beta+1)\frac{\theta b}{e^{-a}\theta_1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{i!k!}\ a^{i+1}\ \frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\ \frac{\Gamma\left(\frac{\theta}{\theta_1}(j+1)\right)}{\Gamma\left(\frac{\theta}{\theta_1}(j+1)-k\right)}\ \frac{1}{(k+1)^2}$$

And, $I_5 = \alpha E\left(\left(1-e^{-\theta_1 x}\right)^{-\beta}\right)$

$$= \frac{\alpha ab\theta}{e^{-a}}\int_0^{\infty}\left(1-e^{-\theta_1 x}\right)^{-\beta}\ \left(1-e^{-\theta x}\right)^{-(b+1)}e^{-\theta x}\,e^{-a\left(1-e^{-\theta x}\right)^{-b}}dx$$

$$= \frac{\alpha\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^{\infty}\left(1-e^{-\theta_1 x}\right)^{-\beta}\ \left(1-e^{-\theta x}\right)^{-(b+1)}\ e^{-\theta x}\ \left(1-e^{-\theta x}\right)^{-bi}dx$$

$$= \frac{\alpha\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^{\infty}\left(1-e^{-\theta_1 x}\right)^{-\beta}\ e^{-\theta x}\left(1-e^{-\theta x}\right)^{-[b(i+1)+1]}dx$$

$$= \frac{\alpha\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\ \Gamma([b(i+1)+1])}\int_0^{\infty}\left(1-e^{-\theta_1 x}\right)^{-\beta}\ e^{-\theta x}\ e^{-j\theta x}\ dx$$

$$= \frac{\alpha\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\ \Gamma([b(i+1)+1])}\int_0^{\infty}\left(1-e^{-\theta_1 x}\right)^{-\beta}\ e^{-\theta(1+j)x}\ dx$$

By using, $\left(1-e^{-\theta_1 x}\right)^{-\beta} = \sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{k!\,\Gamma(\beta)}\ e^{-k\theta_1 x}$, we get,

$$I_5 = \frac{\alpha\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\ \Gamma([b(i+1)+1])}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{k!\ \Gamma(\beta)}\int_0^{\infty}e^{-[k\theta_1+\theta(1+j)]x}\,dx$$

$$= \frac{\alpha\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ a^{i+1}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\ \Gamma([b(i+1)+1])}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{k!\ \Gamma(\beta)}\ \frac{1}{[k\theta_1+\theta(1+j)]}$$

Then, the relative entropy is,

$$Dkl(F_1\|F_2) = \ \ln\left(\frac{e^{-\alpha}\theta ab}{e^{-a}\theta_1\alpha\beta}\right)+\frac{(\theta_1-\theta)b}{\theta e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\ \sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\ \Gamma([b(i+1)+1])}\ \frac{1}{(1+j)^2}$$

$$+\frac{(b+1)}{be^{-a}}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m a^{m+1}}{m!\,(m+1)^2}-\{\gamma+\ln(a)\}\right\}-be^{a}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i}{i!}a^{i+2}\frac{\Gamma([b(i+2)+1]+j)}{j!\,\Gamma([b(i+2)+1])}$$
$$\frac{1}{(j+1)}-(\beta+1)\frac{\theta b}{e^{-a}\theta_1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}}{i!k!}a^{i+1}\frac{\Gamma([b(i+1)+1]+j)}{j!\Gamma([b(i+1)+1])}\frac{\Gamma\left(\frac{\theta}{\theta_1}(j+1)\right)}{\Gamma\left(\frac{\theta}{\theta_1}(j+1)-k\right)}\frac{1}{(k+1)^2}$$
$$+\frac{\alpha\theta b}{e^{-a}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{k!\,\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{\Gamma([b(i+1)+1]+j)}{j!\,\Gamma([b(i+1)+1])}\frac{1}{[k\theta_1+\theta(1+j)]} \quad (31)$$

### 3.2. Stress-Strength Reliability

Let Y and X be the stress and the strength random variables, independent of each other, follow respectively $[0,1]$ TFE$(a,b,\theta)$ and $[0,1]$ TFE$(\alpha,\beta,\theta_1)$, then,

$$R=P(y<x)=\int_0^{\infty}f_x(x)\,F_y(x)\,dx$$
$$=\int_0^{\infty}\frac{\theta ab}{e^{-a}}\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-\theta x}e^{-a\left(1-e^{-\theta x}\right)^{-b}}\frac{1}{e^{-\alpha}}e^{-\alpha\left(1-e^{-\theta_1 x}\right)^{-\beta}}\,dx$$

*since* $e^{-a\left(1-e^{-\theta x}\right)^{-b}}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^i\left(1-e^{-\theta x}\right)^{-bi}$, then,

$$R=\frac{\theta b}{e^{-a}e^{-\alpha}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^{\infty}\left(1-e^{-\theta x}\right)^{-(b+1)}e^{-\theta x}\left(1-e^{-\theta x}\right)^{-bi}e^{-\alpha\left(1-e^{-\theta_1 x}\right)^{-\beta}}\,dx$$
$$=\frac{\theta b}{e^{-a}e^{-\alpha}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}a^{i+1}\int_0^{\infty}\left(1-e^{-\theta x}\right)^{-(bi+b+1)}e^{-\theta x}e^{-\alpha\left(1-e^{-\theta_1 x}\right)^{-\beta}}\,dx$$

So, by using, $e^{-\alpha\left(1-e^{-\theta_1 x}\right)^{-\beta}}=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\alpha^j\left(1-e^{-\theta_1 x}\right)^{-j\beta}$, we get,

$$R=\frac{\theta b}{e^{-a}e^{-\alpha}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}a^{i+1}\alpha^j\int_0^{\infty}e^{-\theta x}\left(1-e^{-\theta x}\right)^{-(bi+b+1)}\left(1-e^{-\theta_1 x}\right)^{-j\beta}\,dx$$

And using, $\left(1-e^{-\theta x}\right)^{-[b(i+1)+1]}=\sum_{k=0}^{\infty}\frac{\Gamma([b(i+1)+1]+k)}{k!\,\Gamma([b(i+1)+1])}e^{-k\theta x}$, then,

$$R=\frac{\theta b}{e^{-a}e^{-\alpha}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}a^{i+1}\alpha^j\frac{\Gamma([b(i+1)+1]+k)}{k!\;\Gamma([b(i+1)+1])}\int_0^{\infty}e^{-(1+k)\theta x}\left(1-e^{-\theta_1 x}\right)^{-j\beta}\,dx$$
$$=\frac{\theta b}{e^{-a}e^{-\alpha}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}a^{i+1}\alpha^j\frac{\Gamma([b(i+1)+1]+k)}{k!\;\Gamma([b(i+1)+1])}\frac{\Gamma(j\beta+m)}{m!\;\Gamma(j\beta)}\int_0^{\infty}e^{-(1+k)\theta x}e^{-m\theta_1 x}\,dx$$
$$=\frac{\theta b}{e^{-a}e^{-\alpha}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}}{i!\,j!}a^{i+1}\alpha^j\frac{\Gamma([b(i+1)+1]+k)}{k!\;\Gamma([b(i+1)+1])}\frac{\Gamma(j\beta+m)}{m!\;\Gamma(j\beta)}\int_0^{\infty}e^{-[(1+k)\theta+m\theta_1]x}\,dx$$
$$=\theta be^{a+\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}}{i!j!}a^{i+1}\alpha^j\frac{\Gamma([b(i+1)+1]+k)}{k!\,\Gamma([b(i+1)+1])}\frac{\Gamma(j\beta+m)}{m!\Gamma(j\beta)}\frac{1}{[(1+k)\theta+m\theta_1]} \quad (32)$$

Where, $\left(1-e^{-\theta_1 x}\right)^{-j\beta}=\sum_{m=0}^{\infty}\frac{\Gamma(j\beta+m)}{m!\,\Gamma(j\beta)}e^{-m\theta_1 x}$.

# 4. Summary and Conclusions

In statistical analysis, a lot of distributions are used to represent set(s) data. Recently .New distributions are derived to extend some of well-known families of distributions, such that the new distributions are more flexible than the others to model real data. The composing of some distributions with each other's in some way has been in the foreword of data modeling.

In this paper,, we presented a new family of continuous distributions based on [0,1] truncated Fréchet distribution. [0,1] truncated Fréchet Uniform ([0,1] TFU ) and [0,1] truncated Fréchet Exponential ([0,1] TFE ) distributions are discussed as special cases. Properties of [0,1] TFU and [0,1] TFE is derived. We provide forms for characteristic function, *r*th raw moment, mean, variance, skewness, kurtosis, mode, median, reliability function, hazard rate function, Shannon entropy function and Relative entropy function. This paper deals also with the determination of stress-strength reliability R = P[Y < X] when X (strength) and Y (stress) are two independent [0,1] TFU ([0,1] TFE) distributions with different parameters.

---

# REFERENCES

[1] Abid, S., Hassan, H. (2015) "The Beta Marshall-Olkin Extended Uniform Distribution"; Journal of Safety Engineering 2015, 4(1): 1-7.

[2] Eugene, N., Lee, C., Famoye, F. (2002) "Beta-normal distribution and its applications" Commun.Statist. - Theory and Methods 31:497-512.

[3] Gradshteyn, I.S., Ryzhik, I.M. (2000) "Table of integrals, series, and products" Academic Press, San Diego.

[4] Gupta, A.K., Nadarajah, S. (2004) "On the moments of the beta normal distribution" Commun. Statist. - Theory and Methods 33:1-13.

[5] Jones, M.C. (2004) "Families of distributions arising from distributions of order statistics" Test 13:1-43.