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New Lag Window for Spectrum Estimation of Low Order AR Processes

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Abstract Spectral estimates depend for their specification on a truncation point and Lag window. New Lag window is suggested to estimate the spectral density function of Low order autoregressive processes, AR(1) and AR(2). This New Lag window based on a family of densities suggested by Johnson, Tietjen and Beckman (JTB). A comparison among a wide range of Lag windows is presented based upon empirical experiments. The New Lag window was the best of all other Lag windows in most of cases.

Keywords Lag Window, Spectral Density Estimation, AR Processes

1. Introduction

Let $X_t, t = \cdots, -2, -1, 0, 1, 2, \ldots$ be a real valued, weakly stationary, discrete stochastic process (time series) with zero mean and autocorrelation function

$$\rho_v = \frac{R_v}{R_0} \quad , \text{where } \{R_v = E[X_t X_{t-v}], v = 0, \pm 1, \pm 2, \ldots\} \quad (1)$$

If one has series of size n, then the consistent form to estimate the spectral density function is[10]

$$f(\omega) = \frac{1}{2\pi} \sum_{v=-T+1}^{T-1} \hat{\rho}_v \, k_T(v) \cos(v\omega) \quad , -\pi \leq \omega \leq \pi \quad (2)$$

Where T is the truncation point $0 \leq T \leq n$ and $\hat{\rho}_v = \frac{\hat{R}_v}{\hat{R}_0}$, where

$$\hat{R}_v = \frac{1}{n} \sum_{t=1}^{n-|v|} X_t X_{t+|v|} \quad , v < n \quad , \quad (3)$$

And $k_T(v)$ is the lag window.

To get a good estimate of $f(\omega)$, one must select an appropriate value of T and an appropriate function of $k_T(v)$.

The above approach is shown to be a special case of smoothing a sample spectrum estimator by giving decreasing weight to the autocovariances as the lag increases. The weighting function is known as the lag window (kernel) and leads to a smoothed spectral estimator.

NEAVE 1972[8] explained that Spectrum estimates depend for their specification on a truncation point and a "covariance averaging kernel" or "lag window," which is derived from a "lag window generator." A comparison of such generators is presented, based upon their negated derivatives, which are shown to be the approximate weightings in a particular type of representation of the spectrum estimate. It is clearly shown why the well-known generators of Parzen and Tukey are superior to most (but not all) of their rivals. Neave stated that "These still have weaknesses, however, and the article concludes with a suggestion how these may be overcome".

In 1978, Harris[4] paper includes a comprehensive catalog of data windows along with their significant performance parameters from which the different windows can be compared. Also, an example demonstrates the use and value of windows to resolve closely spaced harmonic signals characterized by large differences in amplitude.

In 1981 Nuttall[9] studied some windows with very good sidelobes behavior some windows with very good sidelobes behavior which means in terms of bias due to nearby sidelobes and bias due to distant sidelobes.

Hannover in 2002[5] gave a practical overview about the estimation of power spectral densities using the discrete Fourier transform and the fast Fourier transform. And emphasized on the relationship between estimates of power spectra and power spectral densities which is given by the effective noise bandwidth (ENBW). Included is a detailed list of common and useful window functions, among them the often neglected flat-top windows. Special highlights are a procedure to test new programs, a table of comprehensive graphs for each window and the introduction of a whole family of new flat-top windows.

In 2009 Hongwei[6] used multi-instrument to evaluation of various window functions which are analyze the window function WAV file with no window function and analyze a unit DC signal with the window function to be evaluated, and then evaluate of window function parameters

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Table (1). Lag windows

windows	The form of $k_T(v)$	Notes
Rectangular	1	For all $v \le T$
Hanning	$0.5 - 0.5COS\left(\frac{\pi v}{T}\right)$	$v \le T$
Hamming	$0.54 - 0.46COS\left(\frac{\pi v}{T}\right)$	$v \le T$
Bohman	$\left[1 - \frac{v}{T}\right] cos\frac{\pi v}{T} + \frac{1}{\pi} sin\frac{\pi v}{T}$	$v \le T$
Cosine	$sin\left(\frac{\pi v}{T}\right)$	$v \le T$
Cosine^3	$sin^3\left(\frac{\pi v}{T}\right)$	$v \le T$
Cosine^4	$sin^4\left(\frac{\pi v}{T}\right)$	$v \le T$
Cosine^5	$sin^5\left(\frac{\pi v}{T}\right)$	$v \le T$
Riemann	$\frac{sin\frac{\pi v}{T}}{\frac{\pi v}{T}}$	$v \le T$
Parzen	$w(n) = \begin{cases} 1 - 6\left(\frac{v}{T}\right)^2\left(1 - \left\lvert\frac{v}{T}\right\rvert\right), & v \le \frac{1}{2}T \\ 2\left(1 - \left\lvert\frac{v}{T}\right\rvert\right)^3, & v \ge \frac{1}{2}T \end{cases}$	$v \le T$
Tukey	$\begin{cases} 0.5 + 0.5\cos\left(\pi\left(\frac{v}{N\alpha} - 1\right)\right), & \alpha\frac{T}{2} \le \lvert v\rvert \le \frac{T}{2} \\ 1, & 0 \le \lvert v\rvert \le \alpha\frac{T}{2} \end{cases}$	$v \le T, \alpha = 0.5$
Triangular	$1 - \frac{v}{T}$	$v \le T$
Gaussian	$e^{-\frac{1}{2}\left(\frac{\alpha v}{T}\right)^2}$	$v \le T, \alpha = 2.5$
Riesz	$1 - \left(\frac{v}{T}\right)^2$	$v \le T$
Bartlett–Hann	$0.62 - 0.48\left\lvert\frac{v}{T} - 0.5\right\rvert - 0.38cos\left(\frac{\pi v}{T}\right)$	$v \le T$
Blackman	$0.42 - 0.5COS\left(\frac{\pi v}{T}\right) + 0.8COS\left(\frac{2\pi v}{T}\right)$	$v \le T$
Kaiser	$\frac{I_0\left(\pi\alpha\sqrt{1 - \left(\frac{v}{T} - 1\right)^2}\right)}{I_0(\pi\alpha)}$	I_0 is the zeros order Bessel function of the first kind $\alpha = 3\ , v \le T$
Nuttall	$0.355768 - 0.487396\cos\left(\frac{2\pi n}{N-1}\right) + 0.144232\cos\left(\frac{4\pi n}{N-1}\right) - 0.012604\cos\left(\frac{6\pi n}{N-1}\right)$	$v \le T$
Blackman–Harris	$0.402 - 0.498COS\left(\frac{\pi v}{T}\right) + 0.98COS\left(\frac{2\pi v}{T}\right) + 0.001COS\left(\frac{3\pi v}{T}\right)$	$v \le T$
Blackman–Nuttall	$0.3635819 - 0.4861775COS\left(\frac{\pi v}{T}\right) + 0.1365995COS\left(\frac{2\pi v}{T}\right) + 0.0106411COS\left(\frac{3\pi v}{T}\right)$	$v \le T$
Flat top	$0.21557895 - 0.41663158COS\left(\frac{\pi v}{T}\right) + 0.277263158COS\left(\frac{2\pi v}{T}\right) + 0.083578947COS\left(\frac{3\pi v}{T}\right) + 0.006947368COS\left(\frac{4\pi v}{T}\right)$	$v \le T$
Exponential	$(0.1)^{\frac{v}{T}}$	$v \le T$
Poisson	$e^{-\alpha\frac{v}{T}}$	$e = 2.71828$ $\alpha = 2\ , v \le T$
Hanning-Poisson	$0.5 * \left[1 + cos\left(\frac{\pi v}{T}\right) exp\left(-\alpha\frac{v}{T}\right)\right]$	$\alpha = 2\ , v \le T$
Cauchy	$\frac{1}{1 + \left[\alpha * \frac{v}{T}\right]^2}$	$v \le T$

In 2010 Albrecht[1] explained tailoring of minimum sidelobes cosine-sum windows for high-resolution measurements.

In 2012 Abdus Samad[3] proposed, a novel window function. Such windows, including the Hanning, Hamming, Blackman and Gaussian windows are useful in spectral analysis applications where a sampled signal is multiplied by such a window, usually followed by a discrete Fourier transform (DFT) in order to control spectral roll-off, generally at the expense of spectral resolution.

A general feature of lag windows is that they give less weight to $\hat{\rho}_v$ as the modulus of v increases. Several different criteria have been proposed in the literature for evaluating different lag window spectral estimators. One of the more useful is the mean square error which is dependent on here.

There are a lot of Lag windows, suggested by authors, also a lot of studies to compare among them numerically and analytically. Table (1) contains a wide range of Lag windows.

In this work, we will suggest a new Lag window and then compare the results, empirically, of the spectral density functions constructed by the suggested lag window and by the other lag windows stated in table(1) for AR(1) and AR(2) processes, with a wide range of cases and series sizes. The comparison will be based on the mean square error criterion as mentioned above.

2. New Lag Window

In 1980, Johnson, Tietjen and Beckman (JTB)[7], Suggested a new family of probability distributions with the following density,

$$f(x) = \frac{A}{2\sigma\Gamma(\alpha)}\int_B^{\infty} w^{\alpha-\tau-1}e^{-w}dw\ ,$$
$$-\infty < x,\mu < \infty\ ,\ \alpha,\tau,\sigma > 0, \quad (4)$$

Where $B = ([A/\sigma][x-\mu])^{1/\tau}$ and $A = \sqrt{\frac{\Gamma(\alpha+2\tau)}{3\Gamma(\alpha)}}$. The parameters μ and σ location and scale parameters, respectively; α and τ are shape parameters. This distribution is unimodal and symmetric about μ. A random variable X with density (4) has mean μ and variance σ^2. Moreover if $\mu = 0$ and $\sigma = 1$ then,

$$E(X^k) = 0\ , \qquad k \text{ is odd},$$
$$E(X^k) = \frac{\Gamma(\alpha+k\tau)}{(k+1)\Gamma(\alpha)}\ , \qquad k \text{ is even},$$

In particular, the coefficient of kurtosis β_2 is,

$$\beta_2 = \frac{9\Gamma(\alpha+4\tau)\Gamma(\alpha)}{5\Gamma^2(\alpha+2\tau)}\ , \quad (5)$$

These moment derivations can be performed by using the moment-generating function of X,

$$M_X(s) = \frac{Ae^{s\mu}}{\sigma s\Gamma(\alpha)}\int_0^{\infty}\sinh\left(\frac{\sigma\omega^{\tau}}{A}\right)\omega^{\alpha-\tau-1}e^{-\omega}d\omega \quad ,(6)$$

Several densities for which $\beta_2 = 3$ are plotted in Figure (1). Each of these distributions has zero mean, unit variance, zero skewness, and kurtosis equal to three. A considerable degree of shape differences is observed.

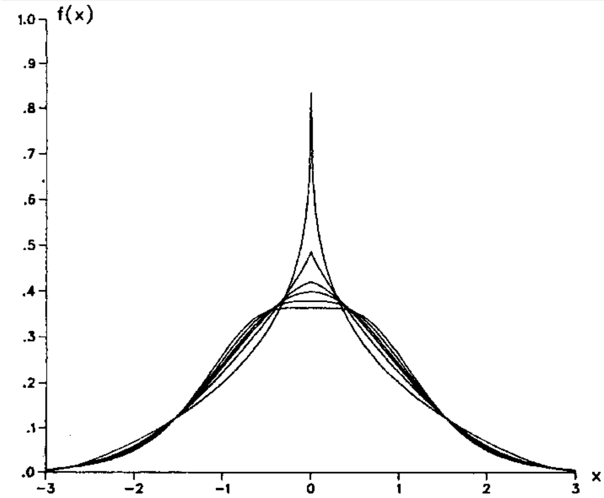

Figure (1). illustrated Six Densities Having $\mu = 0, \sigma^2 = 1, \beta_1 = 0$ and $\beta_2 = 3$ [7]

Figure (2) provides additional density plots for $\mu = 0, \sigma^2 = 1, \tau = 0.5$, and α varying as indicated. These densities look very similar to those in Figure (2). Here, however, the corresponding kurtosis values are given, as follows:

α	0.75	1	1.5	2	4	9
β_2	4.2	3.6	3	2.7	2.25	2

By comparing the two figures in the range $|x| > 2$, it is clear that kurtosis is a measure of tail behavior.

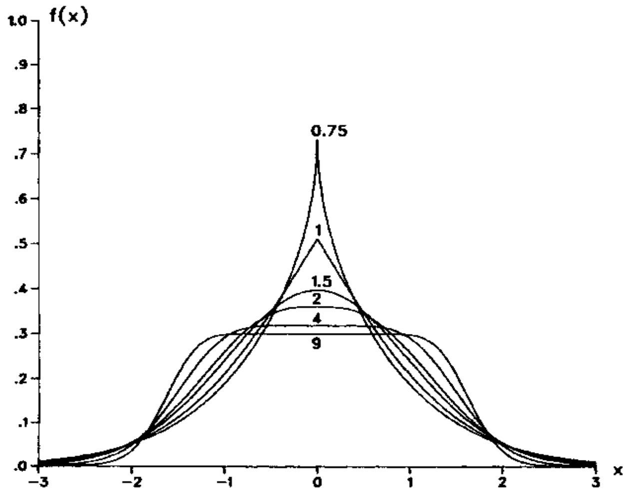

Figure (2). illustrated Six Densities Having $\mu = 0, \sigma^2 = 1, \tau = 0.5$ and α as indicated [7]

There are two reasons to suggest this family of densities as Lag window, the first one is the high flexibility since it is contains four parameters. The second reason is that in a lot of cases, the shape of JTB family of densities is close to the ideal window shape. Since the JTB Family satisfy $f(x) \geq 0$ and $f(x) = f(-x)$ and it is continuous for all x, then one can use it as Lag window function.

Table (2). values of the parameter of $AR(1)$

AR(1) process	stationary	the Bounded value	nonstationary
The value of α	$-0.5, -0.3, 0.1, 0.5, 0.9$	$-1, 1$	$-1.2, 1.1, 1.3$

Table (3). values of the parameters of $AR(2)$

AR(2) process	stationary		the Bounded value		Nonstationary			
	α_1	α_2	α_1	α_2	α_1	α_2	α_1	α_2
Couple of values of α_1 and α_2	1	0.5	0.2	0.5	0.6	1.2	0.7	-1.2
	-1	0.5	0.5	0.2	1.2	0.6	1.1	1.2
	1	1.2	-0.3	-0.1	-0.9	-1.2	1.2	1.1
	-1	1.2	-0.1	-0.3	-1.2	-0.9	-1.1	-1.2
			-0.1	-0.7	0.9	1.2	-1.2	-1.1
			0.7	-0.1	1.2	0.9	-1.1	1.2
			0.3	-0.3	-1.2	0.7	1.1	-1.2
			-0.3	0.3				

3. Empirical Aspect

A simulation Experiment was conducted to satisfy our Research goal, by using Matlab software according to the following assumptions,

1) Generate AR(1) process, $x_t = \alpha x_{t-1} + \epsilon_t$, and AR(2), $x_t = \alpha_1 x_{t-2} + \alpha_2 x_{t-1} + \epsilon_t$, With the following values α , α_1 and α_2 ,

2) The series sizes $n = 10, 20, 50, 75, 100$.

3) The Run size value was $k = 500$.

4) The truncation point value T was calculated according to the closing window algorithm.

5) The values of ω are$[-\pi(0.0251)\pi]$where the number of values is $L = 250$, and the autocorrelation function ACF,

$$\hat{\rho}_v = \frac{\sum_{t=1}^{n-|v|} X_t X_{t+|v|}}{\sum_{t=1}^{n} {x_t}^2} , \tag{7}$$

and the lag windows $k_T(v)$ which are defined in table (1) and spectral density function of autoregressive model, and the consistent estimate of the spectral density function in the formula (2).

6) The criterion used to evaluate the kernels performance was the square error (MSE) with the following formula,

$$MSE = \frac{\sum_{j=1}^{k} \sum_{i=1}^{L} \left(f_j(\omega_i) - \hat{f}_j(\omega_i)\right)^2}{k\, L} , \tag{8}$$

Where k and L were defined in (3) and (5) respectively, and $\hat{f}_j(\omega_i)$ is the spectrum estimation according to the consistent spectral density function formula in (2).

and $f_j(\omega_i)$ is the spectrum of AR(p), for AR(1) process is,

$$f(\omega) = \frac{1-\alpha^2}{2\pi(1 - 2\alpha\cos\omega + \alpha^2)} , \tag{9}$$

and the spectrum for AR(2) process is,

$$f(\omega) = \frac{(1-\alpha_2)\left[(1-\alpha_2)^2 - \alpha_1^2\right]}{2\pi(1+\alpha_2)\left[(1-\alpha_2)^2 + \alpha_1^2 + 2\alpha_1(1+\alpha_2)\cos\omega + 4\alpha_2 \cos^2\omega\right]}, \tag{10}$$

4. The Results Discussion

The simulation program was carried out. Tables from (4) to (23) contain the results of MSE Criterion. To discuss the results in efficient way, let us take every process separately,

4.1. The First Order Autoregressive Model AR(1)

If one noticed carefully, the tables from (4) to (8), resulted from the simulation experiment, he can divide the kernels used according to their behavior to two groups,

A - Rectangle, Triangle, Hamming, Exponential, Gaussian, Riesz, Riemann, Tukey, Parzen, Bohman, Poisson, Hann-Poisson, Cauchy, NLW.

B - Hanning, Blackman, Exact Blackman, Blackman-Harris, Blackman-Nuttall, Flat-Top, Cosine, Bartlett-Hanning.

In the following, some important results, which we reached,

1) If one used group (A) kernels, the values of MSE in the nonstationary case ($\alpha = -1.2, 1.1, 1.3$) and in the boundary values case ($\alpha = -1, 1$) are clearly, greater than matching values if one used group (B) kernels.

2) If one used group (B) kernels, the values of MSE in the nonstationary case ($\alpha = -1.2, 1.1, 1.3$) and in the boundary values case ($\alpha = -1, 1$) are very small, and the differences among are very small also.

3) If one used group (A) kernels, the values of MSE in the stationary case ($\alpha = -0.5, -0.3, 0.1, 0.5, 0.9$) are smaller than matching values if one used group (B) kernels.

4) In general, the values of MSE for positive values of α in the stationary case, are smaller than MSE for negative values except for $\alpha = 0.9$ since it is near to the boundary values.

5) The Blackman kernel has greatest MSE for all n and $\alpha \geq 1$.

6) The values of MSE are closed for Triangle and Rectangle kernels for all n.

7) The Flat-Top kernel was the best for the nonstationary cases.

8) The cosine[5] kernel was the best for the boundary

values cases.

9) The Riemann kernel was the best only in the case $n = 10$ and $\alpha = -0.5, -0.3$.

10) The suggested kernel (NLW) was the best for all other stationary values and all series sizes n.

11) If one used NLW for the stationary cases, There is an extrusive relationship between the values of MSE and α.

Table (4). MSE values of AR(1) spectrum estimation with (n=10)

Windows	α=-1.2	α =-1	α=-0.5	α =-0.3	α=0.1	α=0.5	α =0.9	α =1	α =1.1	α =1.3
Rectangle	24.999	6.332	5.7E-05	2.4E-05	2.1E-05	2.4E-05	3.19E-05	6.329	25.35	25.35
Triangle	24.997	6.331	4.1E-05	1.1E-05	5.6E-06	1.1E-05	1.77E-05	6.330	25.35	25.35
Hanning	6.169	7.8E-06	6.299	6.317	6.336	6.34	6.348	9.1E-06	6.347	6.346
Hamming	24.997	6.331	4.3E-05	1.2E-05	6.9E-06	1.2E-05	1.89E-05	6.329	25.35	25.35
Blackman	46.394	18.729	3.304	3.290	3.277	3.273	3.269	18.727	46.88	46.88
Exact Blackman	6.255	0.0003	6.213	6.231	6.249	6.257	6.261	0.001	6.435	6.434
Blackman-Harris	6.170	5.3E-06	6.298	6.317	6.335	6.343	6.347	6.1E-06	6.349	6.347
Blackman-Nuttall	6.211	7.7E-05	6.257	6.275	6.293	6.301	6.305	7.6E-05	6.391	6.389
Flat-Top	6.163	4.6E-06	6.304	6.323	6.341	6.349	6.353	5.3E-06	6.343	6.341
Exponential	25.002	6.332	9.2E-05	5.2E-05	4.7E-05	4.5E-05	5.68E-05	6.328	25.35	25.35
Gaussian	24.997	6.331	4.1E-05	1.1E-05	5.5E-06	1.1E-05	1.76E-05	6.329	25.35	25.35
Riesz	24.995	6.331	4.6E-05	1.6E-05	1.2E-05	1.8E-05	2.47E-05	6.329	25.35	25.35
cosine	6.168	2.3E-35	6.299	6.318	6.335	6.343	6.348	2.4E-35	6.348	6.346
cosine	6.168	1.5E-35	6.299	6.318	6.335	6.343	6.348	1.2E-35	6.348	6.346
cosine	6.169	5.6E-06	6.299	6.317	6.336	6.344	6.348	6.5E-06	6.348	6.346
cosine	6.168	8.6E-36	6.299	6.318	6.335	6.343	6.348	8.3E-36	6.348	6.346
Riemann	24.997	6.331	3.9E-05	6.5E-06	1.7E-06	7.2E-06	1.33E-05	6.330	25.35	25.35
Parzen	25.006	6.332	0.001	0.001	0.001	0.001	0.001	6.328	25.35	25.35
Tukey	24.998	6.331	5.3E-05	2.2E-05	1.8E-05	2.3E-05	3.04E-05	6.329	25.35	25.35
Bohman	24.996	6.331	4.0E-05	9.8E-06	4.6E-06	1.1E-05	1.65E-05	6.329	25.35	25.35
Poisson	24.998	6.331	4.1E-05	9.2E-06	3.E-06	9.1E-06	1.50E-05	6.330	25.35	25.35
Hanning-Poisson	24.997	6.332	4.1E-05	8.1E-06	2.5E-06	7.7E-06	1.4E-05	6.331	25.358	25.356
Cauchy	24.998	6.332	5.2E-05	2.1E-05	1.6E-05	2.1E-05	2.8E-05	6.329	25.355	25.355
Bartlett-Hann	6.169	4.3E-06	6.299	6.317	6.335	6.344	6.348	4.9E-06	6.3480	6.346
NLW	25.002	6.332	4.3E-05	7.9E-06	3.5E-07	4.8E-06	9.7E-06	6.333	25.361	25.359

Table (5). MSE values of AR(1) spectrum estimation with (n=20)

Windows	α=-1.2	α =-1	α=-0.5	α =-0.3	α=0.1	α=0.5	α =0.9	α =1	α =1.1	α =1.3
rectangle	25	6.331	6.35E-05	2.84E-05	1.9E-05	2.3E-05	3.15E-05	6.331	25.362	25.357
triangle	24.999	6.330	4.70E-05	1.30E-05	6.8E-06	9.4E-06	1.76E-05	6.331	25.36	25.359
Hanning	6.1686	8.5E-06	6.29865	6.31852	6.335	6.343	6.3482	7.9E-06	6.3489	6.3464
Hamming	25	6.331	4.85E-05	1.41E-05	7.9E-06	1.1E-05	1.85E-05	6.332	25.361	25.358
Blackman	46.398	18.725	3.3061	3.29215	3.279	3.273	3.27118	18.735	46.892	46.886
Exact Blackman	6.2548	0.001	6.21208	6.23179	6.248	6.256	6.26118	0.001	6.4364	6.4339
Blackman-Harris	6.1692	5.7E-06	6.29793	6.31773	6.334	6.343	6.34726	5.5E-06	6.3496	6.3472
Blackman-Nuttall	6.2106	7.8E-05	6.25629	6.27602	6.292	6.301	6.30546	7.8E-05	6.3915	6.3891
Flat-Top	6.1631	4.9E-06	6.30412	6.32381	6.341	6.349	6.35315	4.5E-06	6.3433	6.3411
Exponential	25.002	6.333	9.83E-05	6.24E-05	5.1E-05	5.4E-05	6.23E-05	6.332	25.364	25.355
Gaussian	24.999	6.331	4.69E-05	1.28E-05	6.6E-06	9.3E-06	1.72E-05	6.331	25.361	25.358
Riesz	24.998	6.331	5.35E-05	1.97E-05	1.4E-05	1.4E-05	2.56E-05	6.331	25.359	25.359
cosine	6.1686	4.5E-35	6.29958	6.31842	6.335	6.343	6.34823	4.1E-35	6.3483	6.347
cosine	6.1686	2.3E-35	6.29958	6.31842	6.335	6.343	6.34823	2.4E-35	6.3483	6.347
cosine	6.1685	6.1E-06	6.29868	6.3185	6.335	6.343	6.34805	5.8E-06	6.3488	6.3464
cosine	6.1686	1.5E-35	6.29958	6.31842	6.335	6.343	6.34823	1.5E-35	6.3483	6.347
Riemann	24.999	6.331	4.37E-05	9.24E-06	2.7E-06	7.4E-06	1.26E-05	6.332	25.361	25.358
Parzen	25.01	6.338	0.00022	0.00019	0.001	0.001	0.00019	6.332	25.369	25.352
Tukey	24.999	6.331	6.11E-05	2.61E-05	1.7E-05	2.1E-05	2.97E-05	6.332	25.362	25.357
Bohman	24.999	6.331	4.61E-05	1.22E-05	5.8E-06	8.6E-06	1.64E-05	6.331	25.36	25.358
Poisson	24.999	6.331	4.53E-05	1.09E-05	4.3E-06	7.9E-06	1.48E-05	6.332	25.361	25.358
Hanning-Poisson	24.999	6.331	4.39E-05	9.74E-06	3.3E-06	7.1E-06	1.34E-05	6.332	25.361	25.358
Cauchy	25	6.331	5.86E-05	2.37E-05	1.5E-05	1.9E-05	2.75E-05	6.332	25.362	25.357
Bartlett-Hann	6.1686	4.7E-06	6.2989	6.31846	6.335	6.343	6.34822	4.4E-06	6.3487	6.3466
NLW	25.001	6.332	4.31E-05	7.92E-06	3.6E-07	4.7E-06	9.67E-06	6.332	25.362	25.359

Table (6). MSE values of AR(1) spectrum estimation with (n=50)

Windows	α=-1.2	α =-1	α=-0.5	α =-0.3	α=0.1	α=0.5	α =0.9	α =1	α =1.1	α =1.3
rectangle	25.0009	6.33155	6.42E-05	3.05E-05	2.03E-05	2.79E-05	3.33E-05	6.33271	25.3607	25.3573
triangle	24.9998	6.33307	4.80E-05	1.49E-05	6.23E-06	1.32E-05	1.80E-05	6.33165	25.3607	25.3581
Hanning	6.16893	1.02E-05	6.29882	6.31832	6.33619	6.34405	6.34903	9.28E-06	6.34812	6.34634
Hamming	25.0002	6.33211	4.95E-05	1.59E-05	8.49E-06	1.44E-05	1.94E-05	6.33187	25.3611	25.3584
Blackman	46.4016	18.7345	3.30597	3.29296	3.2813	3.27349	3.27108	18.7317	46.892	46.8844
Exact Blackman	6.25523	0.0003	6.21226	6.23159	6.24917	6.25705	6.262	0.00031	6.43565	6.43374
Blackman-Harris	6.16971	6.69E-06	6.29811	6.31755	6.3351	6.34309	6.34808	6.40E-06	6.34889	6.34689
Blackman-Nuttall	6.21106	7.70E-05	6.25647	6.27584	6.29334	6.3013	6.30628	8.05E-05	6.39085	6.38884
Flat-Top	6.1637	5.19E-06	6.30431	6.32364	6.34089	6.34898	6.35396	5.00E-06	6.34285	6.34058
Exponential	25.0022	6.32969	9.88E-05	7.09E-05	5.36E-05	6.53E-05	7.02E-05	6.33364	25.3613	25.357
Gaussian	25	6.33259	4.79E-05	1.44E-05	6.47E-06	1.28E-05	1.76E-05	6.33172	25.3609	25.3582
Riesz	24.9992	6.33384	5.51E-05	2.36E-05	1.27E-05	2.24E-05	2.71E-05	6.33153	25.3602	25.3579
cosine	6.16858	1.04E-34	6.29958	6.31842	6.33558	6.34359	6.34823	9.17E-35	6.34832	6.34695
cosine	6.16858	4.90E-35	6.29958	6.31842	6.33558	6.34359	6.34823	4.92E-35	6.34832	6.34695
cosine	6.16896	7.13E-06	6.29886	6.31831	6.3359	6.34388	6.34887	6.73E-06	6.34813	6.34615
cosine	6.16858	3.29E-35	6.29958	6.31842	6.33558	6.34359	6.34823	3.42E-35	6.34832	6.34695
Riemann	25.0005	6.33247	4.41E-05	1.06E-05	2.82E-06	8.57E-06	1.30E-05	6.33152	25.361	25.3575
Parzen	25.0046	6.3256	0.00024	0.00027	0.00023	0.00027	0.00027	6.33368	25.3646	25.3603
Tukey	25.0006	6.33203	6.21E-05	2.80E-05	1.80E-05	2.60E-05	3.12E-05	6.33262	25.3605	25.3571
Bohman	24.9999	6.33283	4.71E-05	1.40E-05	5.75E-06	1.25E-05	1.70E-05	6.33165	25.3607	25.3581
Poisson	25.0002	6.33248	4.61E-05	1.24E-05	4.51E-06	1.05E-05	1.53E-05	6.33188	25.361	25.358
Hanning-Poisson	25.0002	6.3325	4.47E-05	1.11E-05	3.51E-06	9.27E-06	1.39E-05	6.33176	25.3611	25.3581
Cauchy	25.0006	6.33192	5.94E-05	2.55E-05	1.60E-05	2.33E-05	2.88E-05	6.33248	25.3607	25.3575
Bartlett-Hann	6.16881	5.38E-06	6.299	6.31835	6.336	6.34389	6.34884	5.00E-06	6.34814	6.34646
NLW	25.0012	6.33257	4.31E-05	7.92E-06	3.56E-07	4.79E-06	9.67E-06	6.33257	25.3618	25.359

Table (7). MSE values of AR(1) spectrum estimation with (n=75)

Windows	α=-1.2	α =-1	α=-0.5	α =-0.3	α=0.1	α=0.5	α =0.9	α =1	α =1.1	α =1.3
rectangle	24.998	6.33037	6.17E-05	2.49E-05	1.99E-05	2.37E-05	2.70E-05	6.33263	25.359	25.359
triangle	24.998	6.33141	4.88E-05	1.50E-05	6.99E-06	1.22E-05	1.71E-05	6.33239	25.36	25.359
Hanning	6.1683	1.79E-05	6.3001	6.31726	6.33523	6.34333	6.34817	8.82E-06	6.3477	6.347
Hamming	24.998	6.33081	4.92E-05	1.65E-05	8.29E-06	1.24E-05	1.63E-05	6.33262	25.361	25.359
Blackman	46.399	18.7316	3.30593	3.29321	3.28046	3.27517	3.27122	18.7322	46.892	46.891
Exact Blackman	6.2547	0.00031	6.21342	6.23077	6.24838	6.2565	6.26133	0.00031	6.4353	6.4346
Blackman-Harris	6.1693	9.72E-06	6.29917	6.31694	6.33444	6.34268	6.34758	5.87E-06	6.3486	6.3479
Blackman-Nuttall	6.2107	8.07E-05	6.25753	6.27522	6.29268	6.30089	6.30576	7.68E-05	6.3905	6.3898
Flat-Top	6.1636	5.75E-06	6.30505	6.32349	6.34048	6.34886	6.35388	5.21E-06	6.3427	6.3418
Exponential	24.997	6.32945	9.61E-05	4.71E-05	5.57E-05	5.93E-05	5.36E-05	6.33294	25.36	25.358
Gaussian	24.998	6.3311	4.81E-05	1.51E-05	6.73E-06	1.13E-05	1.59E-05	6.33244	25.361	25.359
Riesz	24.998	6.33151	5.73E-05	2.15E-05	1.46E-05	2.08E-05	2.59E-05	6.33235	25.359	25.359
cosine	6.1686	1.33E-34	6.29958	6.31842	6.33558	6.34359	6.34823	1.42E-34	6.3483	6.347
cosine	6.1686	5.74E-35	6.29958	6.31842	6.33558	6.34359	6.34823	6.65E-35	6.3483	6.347
cosine	6.1686	1.05E-05	6.29995	6.31765	6.33521	6.34344	6.34831	6.22E-06	6.3478	6.3471
cosine	6.1686	4.49E-35	6.29958	6.31842	6.33558	6.34359	6.34823	4.94E-35	6.3483	6.347
Riemann	24.999	6.33197	4.50E-05	1.15E-05	3.28E-06	7.54E-06	1.23E-05	6.33192	25.362	25.359
Parzen	25	6.32773	0.00029	0.00019	0.00025	0.00024	0.0002	6.33434	25.364	25.356
Tukey	24.998	6.33056	6.00E-05	2.40E-05	1.76E-05	2.14E-05	2.64E-05	6.3325	25.359	25.359
Bohman	24.998	6.33133	4.80E-05	1.46E-05	6.28E-06	1.09E-05	1.60E-05	6.33224	25.361	25.359
Poisson	24.999	6.33133	4.61E-05	1.29E-05	4.71E-06	9.09E-06	1.39E-05	6.33239	25.361	25.359
Hanning-Poisson	24.999	6.33148	4.52E-05	1.19E-05	3.76E-06	7.98E-06	1.28E-05	6.33231	25.361	25.359
Cauchy	24.998	6.33055	5.75E-05	2.21E-05	1.56E-05	1.97E-05	2.39E-05	6.33259	25.36	25.359
Bartlett-Hann	6.1684	9.88E-06	6.29994	6.31755	6.33533	6.34343	6.34828	4.79E-06	6.3478	6.347
NLW	25.001	6.33257	4.31E-05	7.92E-06	3.56E-07	4.79E-06	9.67E-06	6.33257	25.362	25.359

Table (8). MSE values of AR(1) spectrum estimation with (n=100)

Windows	α=-1.2	α =-1	α=-0.5	α =-0.3	α=0.1	α=0.5	α =0.9	α =1	α =1.1	α =1.3
rectangle	24.996	6.33082	6.81E-05	2.95E-05	1.99E-05	2.38E-05	3.46E-05	6.33345	25.361	25.36
triangle	24.998	6.33159	4.99E-05	1.46E-05	6.57E-06	1.25E-05	1.91E-05	6.33323	25.361	25.36
Hanning	6.1677	3.31E-05	6.29988	6.31849	6.33596	6.34336	6.3481	7.05E-06	6.3487	6.3475
Hamming	24.997	6.33143	5.09E-05	1.65E-05	8.47E-06	1.34E-05	2.03E-05	6.33358	25.36	25.36
Blackman	46.398	18.7326	3.30595	3.293	3.2806	3.27498	3.27131	18.7369	46.892	46.89
Exact Blackman	6.2541	0.00032	6.21322	6.23174	6.24905	6.25642	6.26118	0.0003	6.4363	6.4349
Blackman-Harris	6.1688	1.78E-05	6.29899	6.31766	6.33506	6.34252	6.34737	5.09E-06	6.3496	6.3481
Blackman-Nuttall	6.2101	8.81E-05	6.25735	6.27596	6.29329	6.30073	6.30556	7.61E-05	6.3915	6.39
Flat-Top	6.1629	9.05E-06	6.30495	6.32362	6.34092	6.34859	6.35358	4.90E-06	6.3435	6.3417
Exponential	24.995	6.33038	0.00011	6.00E-05	5.33E-05	5.35E-05	6.64E-05	6.33384	25.361	25.361
Gaussian	24.997	6.33158	4.93E-05	1.46E-05	6.68E-06	1.20E-05	1.89E-05	6.33346	25.361	25.359
Riesz	24.996	6.33141	5.87E-05	2.13E-05	1.31E-05	2.11E-05	2.77E-05	6.33304	25.361	25.36
cosine	6.1686	1.70E-34	6.29958	6.31842	6.33558	6.34359	6.34823	1.53E-34	6.3483	6.347
cosine	6.1686	8.72E-35	6.29958	6.31842	6.33558	6.34359	6.34823	8.49E-35	6.3483	6.347
cosine	6.168	1.93E-05	6.29976	6.31844	6.33585	6.34328	6.34812	5.34E-06	6.3488	6.3474
cosine	6.1686	6.68E-35	6.29958	6.31842	6.33558	6.34359	6.34823	5.54E-35	6.3483	6.347
Riemann	25	6.33214	4.54E-05	1.04E-05	3.39E-06	7.78E-06	1.30E-05	6.33365	25.361	25.36
Parzen	24.998	6.33026	0.00029	0.0002	0.00022	0.00024	0.0002	6.33567	25.358	25.36
Tukey	24.995	6.33089	6.61E-05	2.76E-05	1.76E-05	2.24E-05	3.31E-05	6.33331	25.361	25.36
Bohman	24.998	6.33185	4.88E-05	1.36E-05	5.99E-06	1.16E-05	1.80E-05	6.33342	25.361	25.359
Poisson	24.998	6.33169	4.73E-05	1.25E-05	4.67E-06	9.63E-06	1.59E-05	6.33332	25.361	25.36
Hanning-Poisson	24.999	6.33193	4.58E-05	1.11E-05	3.69E-06	8.62E-06	1.43E-05	6.33338	25.361	25.359
Cauchy	24.996	6.33092	6.27E-05	2.54E-05	1.57E-05	2.03E-05	3.03E-05	6.33339	25.361	25.36
Bartlett-Hann	6.1679	1.81E-05	6.29978	6.31842	6.33577	6.34344	6.34816	3.88E-06	6.3487	6.3473
NLW	25.001	6.33257	4.31E-05	7.92E-06	3.56E-07	4.79E-06	9.67E-06	6.33257	25.362	25.359

4.2. The Second Order Autoregressive Model AR(2)

If one noticed carefully, the tables from (9) to (23), resulted from the simulation experiment, he can divide the kernels used according to their behavior to three groups,

A- Rectangle, Triangle, Hamming, Exponential, Gaussian, Riesz, Riemann, Parzen, Tukey, Bohman, Poisson, Hann-Poisson, Cauchy, NLW.

B- Hanning, Exact Blackman, Blackman- Harris, Blackman-Nuttall, Flat-Top, Cosine, Bartlett-Hann

C- Blackman

In the following, some important results, which we reached,

1 - If AR(2) is Boundary value process (tables from (9) to (13)) , there are different cases,

(a) If one used group (A) kernels with the following pairs of parameter values,

α_1	1	-1
α_2	0.5	0.5

Then

(i) The MSE values were very smaller than matching values if one used group (B) or group (C) kernels.

(ii) The performance of group (B) kernels was the worst from all kernels according to the MSE criterion.

(iii) The performance of the NLW was the best from all other kernels for all series sizes n.

(b) If one used group (B) kernels with the following pairs of parameter values,

α_1	1	-1
α_2	1.2	1.2

Then

(i) The MSE values were very smaller than matching values if one used group (A) or group (C) kernels.

(ii) The performance of Blackman kernel was the worst from all kernels according to the MSE criterion.
(iii) The performance of The Flat-Top kernel was the best from all other kernels for all series sizes n.

Table (9). MSE values of Boundary AR(2) spectrum estimation with (n=10)

Windows	α_1=1 α_2=0.5	α_1=-1 α_2=0.5	α_1=1 α_2=1.2	α_1=-1 α_2=1.2
rectangle	1.92E-05	3.33E-05	25.379	25.336
triangle	7.38E-06	1.99E-05	25.371	25.35
Hanning	6.32317	6.34909	6.3478	6.3443
Hamming	6.80E-06	2.29E-05	25.374	25.344
Blackman	3.28852	3.26816	46.908	46.874
Exact Blackman	6.23588	6.26165	6.4344	6.4328
Blackman-Harris	6.32137	6.34737	6.347	6.3471
Blackman-Nuttall	6.27969	6.3056	6.3891	6.389
Flat-Top	6.3265	6.35269	6.3404	6.3435
Exponential	3.47E-05	5.25E-05	25.383	25.325
Gaussian	6.72E-06	2.12E-05	25.373	25.347
Riesz	1.22E-05	2.48E-05	25.375	25.349
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32224	6.3482	6.3463	6.3461
cosine	6.32156	6.348	6.3451	6.3485
Riemann	3.97E-06	1.34E-05	25.359	25.35
Parzen	1.96E-05	6.00E-05	25.378	25.318
Tukey	1.95E-05	3.12E-05	25.379	25.339
Bohman	5.93E-06	1.95E-05	25.371	25.349
Poisson	5.45E-06	1.74E-05	25.369	25.349
Hanning-Poisson	4.44E-06	1.58E-05	25.366	25.351
Cauchy	1.62E-05	3.02E-05	25.379	25.338
Bartlett-Hann	6.3227	6.34873	6.3471	6.3456
NLW	4.53E-06	9.93E-06	25.356	25.361

Table (10). MSE values of Boundary AR(2) spectrum estimation with (n=20)

Windows	α_1=1 α_2=0.5	α_1=-1 α_2=0.5	α_1=1 α_2=1.2	α_1=-1 α_2=1.2
rectangle	2.82E-05	6.53E-05	25.373	25.362
triangle	8.89E-06	2.57E-05	25.369	25.351
Hanning	6.32625	6.34858	6.3467	6.3523
Hamming	1.12E-05	3.06E-05	25.37	25.356
Blackman	3.29002	3.26956	46.925	46.885
Exact Blackman	6.23907	6.26103	6.4344	6.4398
Blackman-Harris	6.32469	6.34663	6.3478	6.353
Blackman-Nuttall	6.28297	6.30488	6.3897	6.395
Flat-Top	6.32979	6.3517	6.3426	6.3468
Exponential	5.97E-05	0.00014	25.375	25.376
Gaussian	9.48E-06	2.65E-05	25.37	25.353
Riesz	1.25E-05	3.63E-05	25.37	25.346
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32558	6.3475	6.3469	6.3523
cosine	6.32156	6.348	6.3451	6.3485
Riemann	6.62E-06	1.51E-05	25.367	25.357
Parzen	0.00011	0.00028	25.373	25.398
Tukey	2.57E-05	5.78E-05	25.373	25.359
Bohman	8.45E-06	2.33E-05	25.37	25.352
Poisson	7.87E-06	2.17E-05	25.368	25.356
Hanning-Poisson	6.69E-06	1.82E-05	25.367	25.355
Cauchy	2.29E-05	5.38E-05	25.373	25.36
Bartlett-Hann	6.32508	6.3484	6.3464	6.3511
NLW	4.75E-06	9.67E-06	25.356	25.362

Table (11). MSE values of Boundary AR(2) spectrum estimation with (n=50)

Windows	α_1=1 α_2=0.5	α_1=-1 α_2=0.5	α_1=1 α_2=1.2	α_1=-1 α_2=1.2
rectangle	1.89E-05	3.81E-05	25.362	25.361
triangle	9.57E-06	1.85E-05	25.361	25.353
Hanning	6.32095	6.34912	6.345	6.3509
Hamming	1.16E-05	2.08E-05	25.362	25.357
Blackman	3.29335	3.27052	46.906	46.885
Exact Blackman	6.23425	6.26211	6.4327	6.4384
Blackman-Harris	6.32027	6.34818	6.3461	6.3516
Blackman-Nuttall	6.27855	6.30637	6.388	6.3936
Flat-Top	6.32636	6.35391	6.3399	6.3456
Exponential	4.07E-05	7.62E-05	25.363	25.371
Gaussian	1.02E-05	1.87E-05	25.362	25.355
Riesz	1.27E-05	2.76E-05	25.36	25.349
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32104	6.34899	6.3453	6.3509
cosine	6.32156	6.348	6.3451	6.3485
Riemann	8.19E-06	1.48E-05	25.366	25.355
Parzen	0.00015	0.00029	25.371	25.397
Tukey	1.73E-05	3.62E-05	25.361	25.359
Bohman	9.67E-06	1.82E-05	25.361	25.353
Poisson	8.78E-06	1.64E-05	25.362	25.356
Hanning-Poisson	8.31E-06	1.48E-05	25.362	25.356
Cauchy	1.61E-05	3.28E-05	25.362	25.359
Bartlett-Hann	6.32118	6.3488	6.345	6.3501
NLW	4.88E-06	9.51E-06	25.356	25.362

Table (12). MSE values of Boundary AR(2) spectrum estimation with (n=75)

Windows	α_1=1 α_2=0.5	α_1=-1 α_2=0.5	α_1=1 α_2=1.2	α_1=-1 α_2=1.2
rectangle	2.00E-05	6.81E-05	25.359	25.367
triangle	1.16E-05	2.31E-05	25.366	25.364
Hanning	6.3212	6.35072	6.3423	6.3504
Hamming	1.17E-05	3.39E-05	25.364	25.364
Blackman	3.29185	3.27025	46.913	46.895
Exact Blackman	6.23452	6.26328	6.4302	6.4377
Blackman-Harris	6.32058	6.34896	6.3438	6.3508
Blackman-Nuttall	6.27885	6.30719	6.3858	6.3928
Flat-Top	6.32681	6.35392	6.3388	6.3449
Exponential	4.61E-05	0.00014	25.353	25.367
Gaussian	1.09E-05	2.67E-05	25.365	25.364
Riesz	1.95E-05	3.10E-05	25.367	25.367
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32132	6.34986	6.3429	6.35
cosine	6.32156	6.348	6.3451	6.3485
Riemann	7.98E-06	1.55E-05	25.365	25.359
Parzen	0.00022	0.00042	25.357	25.353
Tukey	1.89E-05	6.13E-05	25.36	25.368
Bohman	1.09E-05	2.18E-05	25.366	25.364
Poisson	9.00E-06	2.25E-05	25.364	25.363
Hanning-Poisson	8.23E-06	1.85E-05	25.365	25.362
Cauchy	1.72E-05	5.70E-05	25.36	25.367
Bartlett-Hann	6.32138	6.34975	6.3431	6.35
NLW	4.85E-06	9.54E-06	25.356	25.362

Table (13). MSE values of Boundary AR(2) spectrum estimation with (n=100)

Windows	α_1=1 ,α_2=0.5	α_1=-1 , α_2=0.5	α_1=1, α_2=1.2	α_1=-1, α_2=1.2
rectangle	3.36E-05	2.97E-05	25.37	25.334
triangle	1.14E-05	3.28E-05	25.369	25.348
Hanning	6.32325	6.34289	6.3459	6.3442
Hamming	1.50E-05	3.09E-05	25.369	25.342
Blackman	3.2913	3.2673	46.892	46.862
Exact Blackman	6.23601	6.25612	6.4326	6.4325
Blackman-Harris	6.3216	6.34242	6.3451	6.3464
Blackman-Nuttall	6.2799	6.30065	6.3871	6.3883
Flat-Top	6.32695	6.34948	6.3375	6.3418
Exponential	9.26E-05	6.38E-05	25.368	25.321
Gaussian	1.19E-05	3.07E-05	25.368	25.345
Riesz	1.84E-05	5.31E-05	25.372	25.349
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32156	6.348	6.3451	6.3485
cosine	6.32244	6.34303	6.3445	6.3455
cosine	6.32156	6.348	6.3451	6.3485
Riemann	8.59E-06	1.90E-05	25.361	25.351
Parzen	0.00035	0.0005	25.353	25.302
Tukey	2.85E-05	2.91E-05	25.37	25.336
Bohman	1.08E-05	3.11E-05	25.367	25.348
Poisson	9.88E-06	2.31E-05	25.366	25.347
Hanning-Poisson	8.68E-06	2.23E-05	25.365	25.348
Cauchy	2.63E-05	2.89E-05	25.37	25.337
Bartlett-Hann	6.32274	6.34435	6.3458	6.3456
NLW	4.84E-06	9.48E-06	25.356	25.362

Table (14). MSE values of Stationary AR(2) spectrum estimation with (n=10)

Windows	α_1=0.2 α_2=0.5	α_1=0.5 α_2=0.2	α_1=-0.3 α_2=-0.1	α_1=-0.1 α_2=-0.3	α_1=-0.1 α_2=0.7	α_1=0.7 α_2=-0.1	α_1=0.3 α_2=-0.3	α_1=-0.3 α_2=0.3
rectangle	3.40E-05	1.99E-05	1.63E-05	2.82E-05	3.34E-05	8E-04	2E-04	1.92E-05
triangle	1.03E-05	6.79E-06	6.25E-06	6.92E-06	1.72E-05	8E-04	2E-04	9.77E-06
Hanning	6.34543	6.323	6.34085	6.32653	6.3469	6.187	6.273	6.3419
Hamming	1.30E-05	7.55E-06	7.44E-06	7.56E-06	1.95E-05	8E-04	2E-04	1.12E-05
Blackman	3.27155	3.2876	3.27432	3.28683	3.2687	3.386	3.324	3.2735
Exact Blackman	6.2583	6.2363	6.25435	6.23926	6.2599	6.101	6.187	6.2553
Blackman-Harris	6.34428	6.3224	6.34076	6.3248	6.346	6.186	6.273	6.3417
Blackman-Nuttall	6.30249	6.2807	6.29895	6.28311	6.3042	6.145	6.231	6.2999
Flat-Top	6.3503	6.3285	6.34731	6.32979	6.3521	6.192	6.279	6.3484
Exponential	5.84E-05	4.07E-05	2.70E-05	5.65E-05	5.26E-05	9E-04	2E-04	3.21E-05
Gaussian	1.14E-05	6.86E-06	6.57E-06	6.68E-06	1.81E-05	8E-04	2E-04	1.04E-05
Riesz	1.44E-05	1.01E-05	1.01E-05	1.33E-05	2.24E-05	8E-04	2E-04	1.37E-05
cosine	6.3419	6.3224	6.339	6.32609	6.3465	6.188	6.273	6.3435
cosine	6.3419	6.3224	6.339	6.32609	6.3465	6.188	6.273	6.3435
cosine	6.34503	6.3232	6.34148	6.32569	6.3467	6.187	6.273	6.3424
cosine	6.3419	6.3224	6.339	6.32609	6.3465	6.188	6.273	6.3435
Riemann	5.31E-06	4.13E-06	3.84E-06	1.72E-06	1.20E-05	8E-04	1E-04	7.32E-06
Parzen	3.04E-05	4.92E-05	2.38E-05	5.77E-05	5.07E-05	9E-04	2E-04	3.63E-05
Tukey	3.24E-05	1.85E-05	1.58E-05	2.67E-05	3.18E-05	8E-04	2E-04	1.84E-05
Bohman	9.71E-06	6.26E-06	5.46E-06	5.47E-06	1.63E-05	8E-04	2E-04	9.64E-06
Poisson	9.21E-06	5.70E-06	4.78E-06	4.85E-06	1.48E-05	8E-04	1E-04	8.00E-06
Hanning-Poisson	7.08E-06	4.67E-06	3.61E-06	2.90E-06	1.30E-05	8E-04	1E-04	7.34E-06
Cauchy	2.87E-05	1.62E-05	1.43E-05	2.29E-05	2.98E-05	8E-04	2E-04	1.69E-05
Bartlett-Hann	6.34458	6.3228	6.34039	6.32625	6.3468	6.187	6.273	6.3423
NLW	3.43E-06	3.28E-06	1.63E-06	1.66E-06	7.66E-06	8E-04	1E-04	4.74E-06

Table (15). MSE values of Stationary AR(2) spectrum estimation with (n=20)

Windows	α_1=0.2 α_2=0.5	α_1=0.5 α_2=0.2	α_1=-0.3 α_2=-0.1	α_1=-0.1 α_2=-0.3	α_1=-0.1 α_2=0.7	α_1=0.7 α_2=-0.1	α_1=0.3 α_2=-0.3	α_1=-0.3 α_2=0.3
rectangle	2.95E-05	2.22E-05	2.32E-05	1.29E-05	7.19E-05	8E-04	2E-04	2.00E-05
triangle	1.14E-05	1.19E-05	1.12E-05	6.83E-06	2.08E-05	8E-04	2E-04	1.08E-05
Hanning	6.34314	6.32216	6.33907	6.32228	6.35104	6.189	6.273	6.34321
Hamming	1.39E-05	1.21E-05	1.02E-05	6.25E-06	2.93E-05	8E-04	2E-04	1.21E-05
Blackman	3.27379	3.28939	3.27556	3.28674	3.26992	3.388	3.328	3.27254
Exact Blackman	6.25624	6.23546	6.25221	6.23604	6.26341	6.103	6.187	6.2565
Blackman-Harris	6.34235	6.32149	6.33831	6.3225	6.34896	6.188	6.272	6.34278
Blackman-Nuttall	6.30056	6.27976	6.29653	6.28075	6.30719	6.147	6.231	6.30098
Flat-Top	6.34836	6.32761	6.34451	6.32971	6.35371	6.194	6.278	6.34921
Exponential	5.15E-05	4.20E-05	4.20E-05	3.94E-05	0.00015	9E-04	2E-04	4.16E-05
Gaussian	1.23E-05	1.15E-05	1.02E-05	6.17E-06	2.36E-05	8E-04	2E-04	1.09E-05
Riesz	1.71E-05	1.99E-05	1.98E-05	1.27E-05	2.70E-05	9E-04	2E-04	1.58E-05
cosine	6.3419	6.32235	6.339	6.32609	6.34651	6.188	6.273	6.34354
cosine	6.3419	6.32235	6.339	6.32609	6.34651	6.188	6.273	6.34354
cosine	6.34312	6.32224	6.33905	6.32313	6.34988	6.189	6.273	6.3435
cosine	6.3419	6.32235	6.339	6.32609	6.34651	6.188	6.273	6.34354
Riemann	7.45E-06	5.48E-06	4.42E-06	3.22E-06	1.23E-05	8E-04	2E-04	8.73E-06
Parzen	9.77E-05	8.52E-05	7.34E-05	0.00011	0.00021	9E-04	2E-04	0.00013
Tukey	2.89E-05	2.16E-05	2.40E-05	1.11E-05	6.41E-05	8E-04	2E-04	1.83E-05
Bohman	1.10E-05	1.05E-05	9.45E-06	6.12E-06	1.89E-05	8E-04	2E-04	1.02E-05
Poisson	9.82E-06	8.48E-06	7.27E-06	4.20E-06	2.00E-05	8E-04	2E-04	8.94E-06
Hanning-Poisson	8.14E-06	7.12E-06	5.63E-06	3.75E-06	1.52E-05	8E-04	2E-04	8.23E-06
Cauchy	2.55E-05	1.95E-05	2.03E-05	1.00E-05	5.90E-05	8E-04	2E-04	1.72E-05
Bartlett-Hann	6.3428	6.32232	6.33919	6.32341	6.34976	6.189	6.273	6.34335
NLW	3.43E-06	4.07E-06	1.63E-06	1.66E-06	7.66E-06	8E-04	1E-04	4.74E-06

Table (16). MSE values of Stationary AR(2) spectrum estimation with (n=50)

Windows	α_1=0.2 α_2=0.5	α_1=0.5 α_2=0.2	α_1=-0.3 α_2=-0.1	α_1=-0.1 α_2=-0.3	α_1=-0.1 α_2=0.7	α_1=0.7 α_2=-0.1	α_1=0.3 α_2=-0.3	α_1=-0.3 α_2=0.3
rectangle	2.71E-05	2.91E-05	2.19E-05	2.54E-05	3.29E-05	9E-04	2E-04	1.87E-05
triangle	1.23E-05	1.11E-05	7.08E-06	8.80E-06	2.10E-05	9E-04	2E-04	1.06E-05
Hanning	6.33998	6.32254	6.33959	6.32749	6.34581	6.188	6.271	6.34369
Hamming	1.47E-05	1.29E-05	9.49E-06	1.07E-05	2.01E-05	9E-04	2E-04	1.02E-05
Blackman	3.27507	3.29047	3.27963	3.28868	3.27166	3.389	3.327	3.27341
Exact Blackman	6.25302	6.23559	6.25253	6.24071	6.25877	6.103	6.185	6.25702
Blackman-Harris	6.33907	6.32142	6.33847	6.32672	6.3448	6.188	6.27	6.34337
Blackman-Nuttall	6.2973	6.27971	6.2967	6.28497	6.30302	6.146	6.229	6.30155
Flat-Top	6.34527	6.32721	6.34419	6.33262	6.35074	6.194	6.277	6.34991
Exponential	6.35E-05	6.52E-05	4.68E-05	5.96E-05	6.70E-05	9E-04	2E-04	4.78E-05
Gaussian	1.29E-05	1.11E-05	7.41E-06	8.67E-06	1.92E-05	9E-04	2E-04	9.68E-06
Riesz	1.84E-05	1.91E-05	1.33E-05	1.58E-05	3.43E-05	9E-04	2E-04	1.81E-05
cosine	6.3419	6.32235	6.339	6.32609	6.34651	6.188	6.273	6.34354
cosine	6.3419	6.32235	6.339	6.32609	6.34651	6.188	6.273	6.34354
cosine	6.3398	6.32221	6.33928	6.32751	6.34558	6.188	6.271	6.34407
cosine	6.3419	6.32235	6.339	6.32609	6.34651	6.188	6.273	6.34354
Riemann	7.57E-06	7.06E-06	3.58E-06	5.10E-06	1.26E-05	8E-04	2E-04	8.13E-06
Parzen	0.00017	0.00023	0.00019	0.00021	0.00028	1E-03	3E-04	0.00021
Tukey	2.44E-05	2.69E-05	2.09E-05	2.29E-05	3.18E-05	9E-04	2E-04	1.72E-05
Bohman	1.18E-05	1.04E-05	6.17E-06	7.51E-06	1.86E-05	9E-04	2E-04	1.00E-05
Poisson	9.69E-06	8.97E-06	5.62E-06	6.69E-06	1.56E-05	8E-04	2E-04	7.95E-06
Hanning-Poisson	8.85E-06	7.48E-06	4.29E-06	5.20E-06	1.38E-05	8E-04	2E-04	7.51E-06
Cauchy	2.23E-05	2.40E-05	1.81E-05	2.08E-05	2.92E-05	9E-04	2E-04	1.55E-05
Bartlett-Hann	6.34041	6.32251	6.33934	6.32721	6.34605	6.188	6.271	6.34371
NLW	3.43E-06	4.13E-06	1.63E-06	1.66E-06	7.75E-06	8E-04	1E-04	4.74E-06

Table (17). MSE values of Stationary AR(2) spectrum estimation with (n=75)

Windows	$\alpha_1=0.2$ $\alpha_2=0.5$	$\alpha_1=0.5$ $\alpha_2=0.2$	$\alpha_1=-0.3$ $\alpha_2=-0.1$	$\alpha_1=-0.1$ $\alpha_2=-0.3$	$\alpha_1=-0.1$ $\alpha_2=0.7$	$\alpha_1=0.7$ $\alpha_2=-0.1$	$\alpha_1=0.3$ $\alpha_2=-0.3$	$\alpha_1=-0.3$ $\alpha_2=0.3$
rectangle	1.87E-05	2.18E-05	2.25E-05	1.67E-05	4.47E-05	9E-04	2E-04	2.12E-05
triangle	9.30E-06	9.34E-06	7.70E-06	6.95E-06	1.67E-05	8E-04	2E-04	8.67E-06
Hanning	6.341422	6.323205	6.339873	6.32631	6.34855	6.188	6.272	6.34356
Hamming	9.87E-06	1.10E-05	1.00E-05	8.74E-06	2.32E-05	8E-04	2E-04	1.01E-05
Blackman	3.277671	3.288971	3.276029	3.28914	3.271774	3.388	3.326	3.27464
Exact Blackman	6.254508	6.236425	6.25312	6.23955	6.261199	6.103	6.186	6.25688
Blackman-Harris	6.340625	6.322383	6.339332	6.32557	6.346994	6.188	6.272	6.34321
Blackman-Nuttall	6.298836	6.280658	6.29754	6.28383	6.305213	6.146	6.23	6.3014
Flat-Top	6.346765	6.328363	6.345667	6.33158	6.352393	6.194	6.278	6.34968
Exponential	4.26E-05	5.30E-05	5.75E-05	4.72E-05	0.0001	9E-04	2E-04	5.46E-05
Gaussian	8.89E-06	9.36E-06	8.16E-06	6.97E-06	1.86E-05	8E-04	2E-04	8.60E-06
Riesz	1.63E-05	1.51E-05	1.36E-05	1.28E-05	2.29E-05	8E-04	2E-04	1.39E-05
cosine	6.341897	6.322351	6.339	6.32609	6.346511	6.189	6.273	6.34354
cosine	6.341897	6.322351	6.339	6.32609	6.346511	6.189	6.273	6.34354
cosine	6.34138	6.323156	6.340059	6.32635	6.347832	6.188	6.273	6.34393
cosine	6.341897	6.322351	6.339	6.32609	6.346511	6.189	6.273	6.34354
Riemann	5.17E-06	6.99E-06	4.69E-06	4.55E-06	1.15E-05	8E-04	1E-04	7.32E-06
Parzen	0.000176	0.000186	0.000208	0.00025	0.000334	0.001	3E-04	0.00021
Tukey	1.75E-05	1.96E-05	1.99E-05	1.44E-05	3.97E-05	8E-04	2E-04	1.89E-05
Bohman	8.55E-06	8.63E-06	7.37E-06	6.59E-06	1.59E-05	8E-04	2E-04	8.20E-06
Poisson	6.97E-06	7.61E-06	6.08E-06	5.39E-06	1.58E-05	8E-04	1E-04	7.45E-06
Hanning-Poisson	6.09E-06	6.79E-06	5.09E-06	4.82E-06	1.35E-05	8E-04	1E-04	6.84E-06
Cauchy	1.58E-05	1.78E-05	1.80E-05	1.33E-05	3.68E-05	8E-04	2E-04	1.70E-05
Bartlett-Hann	6.341538	6.322949	6.339587	6.32624	6.347962	6.188	6.272	6.34358
NLW	3.42E-06	4.12E-06	1.65E-06	1.67E-06	7.71E-06	8E-04	1E-04	4.78E-06

Table (18). MSE values of Stationary AR(2) spectrum estimation with (n=100)

Windows	$\alpha_1=0.2$ $\alpha_2=0.5$	$\alpha_1=0.5$ $\alpha_2=0.2$	$\alpha_1=-0.3$ $\alpha_2=-0.1$	$\alpha_1=-0.1$ $\alpha_2=-0.3$	$\alpha_1=-0.1$ $\alpha_2=0.7$	$\alpha_1=0.7$ $\alpha_2=-0.1$	$\alpha_1=0.3$ $\alpha_2=-0.3$	$\alpha_1=-0.3$ $\alpha_2=0.3$
rectangle	2.79E-05	1.98E-05	2.99E-05	2.92E-05	3.06E-05	9E-04	2E-04	2.88E-05
triangle	1.04E-05	1.02E-05	1.17E-05	1.03E-05	1.37E-05	8E-04	1E-04	1.09E-05
Hanning	6.342771	6.323235	6.339062	6.32584	6.346371	6.187	6.273	6.34488
Hamming	1.27E-05	1.05E-05	1.46E-05	1.18E-05	1.71E-05	8E-04	2E-04	1.33E-05
Blackman	3.274899	3.28956	3.277997	3.28662	3.273348	3.388	3.326	3.27394
Exact Blackman	6.255906	6.236522	6.251975	6.23898	6.259398	6.101	6.186	6.25794
Blackman-Harris	6.342071	6.322573	6.337898	6.32493	6.345509	6.186	6.272	6.34403
Blackman-Nuttall	6.300274	6.280838	6.296136	6.28321	6.303708	6.145	6.23	6.30223
Flat-Top	6.348315	6.328841	6.34372	6.33102	6.351547	6.192	6.278	6.34994
Exponential	6.61E-05	4.85E-05	6.51E-05	6.83E-05	7.54E-05	9E-04	2E-04	5.98E-05
Gaussian	1.06E-05	9.63E-06	1.21E-05	1.00E-05	1.44E-05	8E-04	1E-04	1.13E-05
Riesz	1.69E-05	1.72E-05	2.08E-05	1.98E-05	1.97E-05	9E-04	2E-04	1.67E-05
cosine	6.341897	6.322351	6.339	6.32609	6.346511	6.189	6.273	6.34354
cosine	6.341897	6.322351	6.339	6.32609	6.346511	6.189	6.273	6.34354
cosine	6.342808	6.323308	6.338681	6.32569	6.346277	6.187	6.273	6.34481
cosine	6.341897	6.322351	6.339	6.32609	6.346511	6.189	6.273	6.34354
Riemann	6.93E-06	7.57E-06	4.42E-06	4.70E-06	1.12E-05	8E-04	1E-04	7.72E-06
Parzen	0.000223	0.000188	0.000252	0.00024	0.000304	9E-04	4E-04	0.00017
Tukey	2.50E-05	1.81E-05	2.80E-05	2.75E-05	2.66E-05	9E-04	2E-04	2.69E-05
Bohman	9.43E-06	9.42E-06	1.05E-05	9.12E-06	1.33E-05	8E-04	1E-04	1.02E-05
Poisson	8.54E-06	7.80E-06	8.70E-06	7.44E-06	1.25E-05	8E-04	1E-04	9.53E-06
Hanning-Poisson	7.09E-06	7.14E-06	6.76E-06	5.77E-06	1.15E-05	8E-04	1E-04	8.00E-06
Cauchy	2.28E-05	1.65E-05	2.51E-05	2.38E-05	2.52E-05	9E-04	2E-04	2.41E-05
Bartlett-Hann	6.342546	6.323109	6.339062	6.32584	6.346384	6.187	6.273	6.34448
NLW	3.42E-06	4.13E-06	1.63E-06	1.66E-06	7.69E-06	8E-04	1E-04	4.77E-06

2 - If AR(2) process is stationary (tables from (14) to (18)), there are some important notifications,

(a) Generally, the values of MSE are very small, close to zero.

(b) The performance of the suggested kernel (NLW) was the best from all other kernels for all series sizes.

(c) If one used group (A) kernels, then the MSE values were smaller than matching values if one used group (B) or group (C) kernels.

(d) If one used Blackman kernel, then the MSE values were smaller than matching values if one used group (B) kernels.

(e) If one used group (B) kernels with $\alpha_1 = 0.2$ and $\alpha_2 = 0.5$ then the MSE values were bigger than matching values with $\alpha_1 = 0.5$ and $\alpha_2 = 0.2$.

(f) If one used group (B) kernels with $\alpha_1 = -0.3$ and $\alpha_2 = -0.1$ then the MSE values were bigger than (in simple differences) matching values with $\alpha_1 = -0.1$ and $\alpha_2 = -0.3$.

(g) If one used group (A) kernels with $\alpha_1 = -0.1$ and $\alpha_2 = 0.7$ then the MSE values were smaller than matching values with $\alpha_1 = 0.7$ and $\alpha_2 = 0.2$.

(h) If one used group (B) kernels with $\alpha_1 = -0.1$ and $\alpha_2 = 0.7$ then the MSE values were bigger than matching values with $\alpha_1 = 0.7$ and $\alpha_2 = -0.1$.

(i) If one used group (B) kernels or group (A) kernels with $\alpha_1 = 0.3$ and $\alpha_2 = -0.3$ then the MSE values were bigger than matching values with $\alpha_1 = -0.3$ and $\alpha_2 = 0.3$.

3 - If AR(2) process is nonstationary (tables from (19) to (23)), there are different cases,

(a) If one used group (B) kernels with the following pairs of parameter values,

α_1	0.6	0.9	1.1	1.2	-1.1
α_2	1.2	1.2	1.2	1.1	1.2

Then

(i) The MSE values were smaller than matching values if one used group (A) or group (C) kernels.

(ii) The performance of Blackman kernel (group (C)) was the worst from all kernels according to the MSE criterion.

(iii) The performance of The Flat-Top kernel was the best from all other kernels for all series sizes n.

(b) If one used group (A) kernels with the following pairs of parameter values,

α_1	1.2	1.2	-1.2
α_2	0.6	0.9	0.7

Then,

(i) The MSE values were smaller than matching values if one used group (B) or group (C) kernels

(ii) The performance of group (B) kernels was the worst from all kernels according to the MSE criterion.

(iii) The performance of the suggested kernel (NLW) was the best from all other kernels for all series sizes n.

Generally, the MSE values were very small in all cases.

(c) If one used Blackman kernel with the following pairs of parameter values,

α_1	-0.9	-1.1	1.1	**0.7**
α_2	-1.2	-1.2	-1.2	-1.2

Then,

(i) The MSE values were smaller than matching values if one used group (B) or group (A) kernels

(ii) The performance of group (B) kernels was the worst from all kernels according to the MSE criterion.

(iii) The performance of Blackman kernel was the best from all other kernels for all series sizes n.

Relatively, the MSE values were large in all cases.

(d) If one used group (B) kernels with the following pairs of parameter values,

α_1	-1.2	-1.2
α_2	-0.9	-1.1

Then,

(i) The MSE values were smaller than matching values if one used group (A) or group (C) kernels

(ii) The performance of Blackman kernel was the worst from all kernels according to the MSE criterion.

Relatively, the MSE values were very large in all cases.

Table (19). MSE values of nonstationary AR(2) spectrum estimation with (n=10)

Windows	α_1=0.6 α_2=1.2	α_1=1.2 α_2=0.6	α_1=-0.9 α_2=-1.2	α_1=-1.2 α_2=-0.9	α_1=0.9 α_2=1.2	α_1=1.2 α_2=0.9	α_1=-1.2 α_2=0.7	α_1=0.7 α_2=-1.2	α_1=1.1 α_2=1.2	α_1=1.2 α_2=1.1	α_1=-1.1 α_2=-1.2	α_1=-1.2 α_2=-1.1	α_1=-1.1 α_2=1.2	α_1=1.1 α_2=-1.2
rectangle	25.37	1.85E-05	63.34	787.04	25.37	9.21E-06	3.04E-05	34.78	25.37	25.36	97.38	560.63	25.35	97.92
triangle	25.36	1.19E-05	63.34	787.03	25.36	8.66E-06	1.96E-05	34.8	25.36	25.36	97.38	560.63	25.35	97.92
Hanning	6.347	6.31911	109.7	652.18	6.347	6.3404	6.34623	70.8	6.347	6.346	153.4	686.11	6.348	154.1
Hamming	25.37	1.33E-05	63.34	787.03	25.36	7.84E-06	2.23E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Blackman	46.88	3.29452	37.79	891.94	46.89	3.27347	3.26789	16.71	46.89	46.89	64.92	478.15	46.87	65.35
Exact Blackman	6.434	6.23266	109.4	653.06	6.434	6.25408	6.25904	70.51	6.435	6.433	153	685.21	6.435	153.6
Blackman-Harris	6.347	6.3188	109.7	652.18	6.347	6.3407	6.34497	70.81	6.347	6.346	153.4	686.11	6.349	154.1
Blackman-Nuttall	6.389	6.27707	109.6	652.6	6.389	6.29888	6.3032	70.67	6.389	6.388	153.2	685.67	6.391	153.8
Flat-Top	6.339	6.32481	109.8	652.11	6.34	6.34798	6.35094	70.83	6.341	6.34	153.4	686.18	6.343	154.1
Exponential	25.37	2.76E-05	63.34	787.05	25.37	1.58E-05	5.21E-05	34.77	25.37	25.36	97.39	560.62	25.34	97.92

Gaussian	25.36	1.27E-05	63.34	787.03	25.36	8.36E-06	2.10E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Riesz	25.36	1.47E-05	63.34	787.03	25.36	1.16E-05	2.33E-05	34.8	25.37	25.36	97.38	560.63	25.35	97.92
cosine	6.347	6.32019	109.7	652.17	6.346	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.347	6.32019	109.7	652.17	6.346	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.346	6.31961	109.7	652.17	6.346	6.34132	6.34572	70.81	6.347	6.345	153.4	686.12	6.348	154.1
cosine	6.347	6.32019	109.7	652.17	6.346	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
Riemann	25.36	6.62E-06	63.35	787	25.36	8.54E-06	1.50E-05	34.81	25.35	25.36	97.39	560.65	25.35	97.92
Parzen	25.38	3.43E-05	63.35	787.05	25.36	3.09E-05	6.85E-05	34.79	25.35	25.36	97.4	560.66	25.34	97.92
Tukey	25.37	1.78E-05	63.34	787.04	25.37	9.62E-06	2.81E-05	34.79	25.37	25.36	97.38	560.62	25.35	97.93
Bohman	25.36	1.23E-05	63.34	787.03	25.36	8.93E-06	1.99E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Poisson	25.36	1.03E-05	63.34	787.03	25.36	6.99E-06	1.72E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Hanning-Poisson	25.36	9.54E-06	63.34	787.02	25.36	7.52E-06	1.67E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Cauchy	25.37	1.69E-05	63.34	787.04	25.37	8.52E-06	2.71E-05	34.79	25.37	25.36	97.38	560.63	25.35	97.92
Bartlett-Hann	6.347	6.31939	109.7	652.17	6.346	6.34169	6.34669	70.81	6.347	6.346	153.4	686.12	6.348	154.1
NLW	25.36	6.19E-06	63.34	787.03	25.36	6.50E-06	1.07E-05	34.8	25.35	25.36	97.38	560.63	25.36	97.92

Table (20). MSE values of nonstationary AR(2) spectrum estimation with (n=10)

Windows	$\alpha_1=0.6$ $\alpha_2=1.2$	$\alpha_1=1.2$ $\alpha_2=0.6$	$\alpha_1=-0.9$ $\alpha_2=-1.2$	$\alpha_1=-1.2$ $\alpha_2=-0.9$	$\alpha_1=0.9$ $\alpha_2=1.2$	$\alpha_1=1.2$ $\alpha_2=0.9$	$\alpha_1=-1.2$ $\alpha_2=0.7$	$\alpha_1=0.7$ $\alpha_2=-1.2$	$\alpha_1=1.1$ $\alpha_2=1.2$	$\alpha_1=1.2$ $\alpha_2=1.1$	$\alpha_1=-1.1$ $\alpha_2=-1.2$	$\alpha_1=-1.2$ $\alpha_2=-1.1$	$\alpha_1=-1.1$ $\alpha_2=1.2$	$\alpha_1=1.1$ $\alpha_2=-1.2$
rectangle	25.37	1.85E-05	63.34	787.04	25.37	9.21E-06	3.04E-05	34.78	25.37	25.36	97.38	560.63	25.35	97.92
triangle	25.36	1.19E-05	63.34	787.03	25.36	8.66E-06	1.96E-05	34.8	25.36	25.36	97.38	560.63	25.35	97.92
Hanning	6.347	6.31911	109.7	652.18	6.347	6.3404	6.34623	70.8	6.347	6.346	153.4	686.11	6.348	154.1
Hamming	25.37	1.33E-05	63.34	787.03	25.36	7.84E-06	2.23E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Blackman	46.88	3.29452	37.79	891.94	46.89	3.27347	3.26789	16.71	46.89	46.89	64.92	478.15	46.87	65.35
Exact Blackman	6.434	6.23266	109.4	653.06	6.434	6.25408	6.25904	70.51	6.435	6.433	153	685.21	6.435	153.6
Blackman-Harris	6.347	6.3188	109.7	652.18	6.347	6.3407	6.34497	70.81	6.347	6.346	153.4	686.11	6.349	154.1
Blackman-Nuttall	6.389	6.27707	109.6	652.6	6.389	6.29888	6.3032	70.67	6.389	6.388	153.2	685.67	6.391	153.8
Flat-Top	6.339	6.32481	109.8	652.11	6.34	6.34798	6.35094	70.83	6.341	6.34	153.4	686.18	6.343	154.1
Exponential	25.37	2.76E-05	63.34	787.05	25.37	1.58E-05	5.21E-05	34.77	25.37	25.36	97.39	560.62	25.34	97.92
Gaussian	25.36	1.27E-05	63.34	787.03	25.36	8.36E-06	2.10E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Riesz	25.36	1.47E-05	63.34	787.03	25.37	1.16E-05	2.33E-05	34.8	25.37	25.36	97.38	560.63	25.35	97.92
cosine	6.347	6.32019	109.7	652.17	6.346	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.347	6.32019	109.7	652.17	6.346	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.346	6.31961	109.7	652.17	6.346	6.34132	6.34572	70.81	6.347	6.345	153.4	686.12	6.348	154.1
cosine	6.347	6.32019	109.7	652.17	6.346	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
Riemann	25.36	6.62E-06	63.35	787	25.36	8.54E-06	1.50E-05	34.81	25.35	25.36	97.39	560.65	25.35	97.92
Parzen	25.38	3.43E-05	63.35	787.05	25.36	3.09E-05	6.85E-05	34.79	25.35	25.36	97.4	560.66	25.34	97.92
Tukey	25.37	1.78E-05	63.34	787.04	25.37	9.62E-06	2.81E-05	34.79	25.37	25.36	97.38	560.62	25.35	97.93
Bohman	25.36	1.23E-05	63.34	787.03	25.36	8.93E-06	1.99E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Poisson	25.36	1.03E-05	63.34	787.03	25.36	6.99E-06	1.72E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Hanning-Poisson	25.36	9.54E-06	63.34	787.02	25.36	7.52E-06	1.67E-05	34.8	25.36	25.36	97.38	560.64	25.35	97.92
Cauchy	25.37	1.69E-05	63.34	787.04	25.37	8.52E-06	2.71E-05	34.79	25.37	25.36	97.38	560.63	25.35	97.92
Bartlett-Hann	6.347	6.31939	109.7	652.17	6.346	6.34169	6.34669	70.81	6.347	6.346	153.4	686.12	6.348	154.1
NLW	25.36	6.19E-06	63.34	787.03	25.36	6.50E-06	1.07E-05	34.8	25.35	25.36	97.38	560.63	25.36	97.92

Table (21). MSE values of nonstationary AR(2) spectrum estimation with (n=10)

Windows	α_1=0.6 α_2=1.2	α_1=1.2 α_2=0.6	α_1=-0.9 α_2=-1.2	α_1=-1.2 α_2=-0.9	α_1=0.9 α_2=1.2	α_1=1.2 α_2=0.9	α_1=-1.2 α_2=0.7	α_1=0.7 α_2=-1.2	α_1=1.1 α_2=1.2	α_1=1.2 α_2=1.1	α_1=-1.1 α_2=-1.2	α_1=-1.2 α_2=-1.1	α_1=-1.1 α_2=1.2	α_1=1.1 α_2=-1.2
rectangle	25.362	2.54E-05	63.34	787.02	25.362	2.82E-05	4.18E-05	34.81	25.36	25.37	97.36	560.58	25.36	97.9
triangle	25.364	9.81E-06	63.34	787	25.361	1.26E-05	2.17E-05	34.81	25.36	25.36	97.38	560.64	25.36	97.93
Hanning	6.3456	6.32027	109.7	652.17	6.3462	6.34659	6.35025	70.83	6.345	6.348	153.4	686.1	6.35	154
Hamming	25.363	1.23E-05	63.34	787.01	25.361	1.25E-05	2.32E-05	34.81	25.36	25.37	97.37	560.6	25.36	97.93
Blackman	46.903	3.29101	37.78	891.93	46.9	3.27627	3.2698	16.71	46.89	46.9	64.9	478.08	46.88	65.34
Exact Blackman	6.4332	6.23342	109.4	653.06	6.4339	6.25945	6.26313	70.54	6.433	6.435	152.9	685.19	6.438	153.6
Blackman-Harris	6.3466	6.3193	109.7	652.18	6.3472	6.3454	6.34911	70.82	6.346	6.348	153.4	686.09	6.351	154
Blackman-Nuttall	6.3885	6.2776	109.6	652.6	6.3891	6.3036	6.30731	70.68	6.388	6.39	153.2	685.66	6.393	153.8
Flat-Top	6.3408	6.32524	109.8	652.11	6.3412	6.35088	6.35468	70.84	6.339	6.34	153.4	686.16	6.345	154.1
Exponential	25.358	6.04E-05	63.34	787.05	25.362	5.83E-05	7.75E-05	34.81	25.36	25.38	97.35	560.5	25.36	97.87
Gaussian	25.364	1.02E-05	63.34	787	25.361	1.16E-05	2.13E-05	34.81	25.36	25.37	97.38	560.62	25.36	97.93
Riesz	25.366	1.51E-05	63.33	786.98	25.361	2.16E-05	3.27E-05	34.82	25.36	25.36	97.38	560.67	25.36	97.94
cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.3458	6.32008	109.7	652.17	6.3464	6.34624	6.34994	70.83	6.345	6.347	153.4	686.1	6.35	154
cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
Riemann	25.363	8.10E-06	63.34	787.02	25.361	7.00E-06	1.46E-05	34.81	25.36	25.36	97.38	560.62	25.35	97.92
Parzen	25.353	0.00021	63.36	787.13	25.359	0.00018	0.00026	34.77	25.35	25.39	97.34	560.29	25.34	97.85
Tukey	25.362	2.29E-05	63.34	787.01	25.362	2.73E-05	4.02E-05	34.81	25.36	25.37	97.36	560.6	25.36	97.91
Bohman	25.364	9.49E-06	63.34	787	25.361	1.11E-05	2.00E-05	34.81	25.36	25.36	97.38	560.64	25.36	97.93
Poisson	25.363	8.79E-06	63.34	787.01	25.361	9.77E-06	1.84E-05	34.81	25.36	25.36	97.38	560.62	25.36	97.92
Hanning-Poisson	25.363	7.99E-06	63.34	787.01	25.361	7.93E-06	1.59E-05	34.81	25.36	25.36	97.38	560.62	25.35	97.93
Cauchy	25.362	2.06E-05	63.34	787.01	25.362	2.39E-05	3.68E-05	34.81	25.36	25.37	97.36	560.59	25.36	97.91
Bartlett-Hann	6.3459	6.32033	109.7	652.17	6.3461	6.34623	6.3498	70.83	6.345	6.347	153.4	686.11	6.35	154
NLW	25.358	6.06E-06	63.34	787.03	25.357	6.22E-06	1.02E-05	34.8	25.36	25.36	97.38	560.63	25.36	97.92

Table (22). MSE values of nonstationary AR(2) spectrum estimation with (n=10)

Windows	α_1=0.6 α_2=1.2	α_1=1.2 α_2=0.6	α_1=-0.9 α_2=-1.2	α_1=-1.2 α_2=-0.9	α_1=0.9 α_2=1.2	α_1=1.2 α_2=0.9	α_1=-1.2 α_2=0.7	α_1=0.7 α_2=-1.2	α_1=1.1 α_2=1.2	α_1=1.2 α_2=1.1	α_1=-1.1 α_2=-1.2	α_1=-1.2 α_2=-1.1	α_1=-1.1 α_2=1.2	α_1=1.1 α_2=-1.2
rectangle	25.361	3.04E-05	63.34	787.01	25.364	2.20E-05	4.33E-05	34.79	25.37	25.36	97.41	560.66	25.36	97.89
triangle	25.359	1.43E-05	63.34	787.02	25.361	9.02E-06	2.13E-05	34.8	25.36	25.36	97.39	560.65	25.36	97.9
Hanning	6.3465	6.32091	109.7	652.17	6.3458	6.34621	6.34978	70.82	6.347	6.345	153.4	686.16	6.35	154
Hamming	25.361	1.70E-05	63.34	787.01	25.363	1.07E-05	2.54E-05	34.8	25.36	25.36	97.39	560.63	25.36	97.91
Blackman	46.895	3.29346	37.79	891.97	46.896	3.27724	3.26918	16.7	46.9	46.9	64.9	478.11	46.88	65.34
Exact Blackman	6.4339	6.23408	109.4	653.05	6.4331	6.25926	6.26268	70.53	6.434	6.433	153	685.25	6.438	153.6
Blackman-Harris	6.3472	6.31998	109.7	652.18	6.3461	6.34542	6.34868	70.81	6.347	6.346	153.4	686.15	6.351	154
Blackman-Nuttall	6.3891	6.27826	109.6	652.6	6.3881	6.3036	6.30688	70.68	6.389	6.388	153.2	685.71	6.393	153.8
Flat-Top	6.3411	6.32581	109.8	652.12	6.3396	6.35147	6.35442	70.84	6.341	6.341	153.4	686.21	6.344	154.1
Exponential	25.365	6.66E-05	63.34	786.99	25.366	4.64E-05	8.66E-05	34.78	25.37	25.36	97.4	560.65	25.35	97.9
Gaussian	25.36	1.48E-05	63.34	787.01	25.362	9.36E-06	2.22E-05	34.8	25.36	25.36	97.39	560.64	25.36	97.91
Riesz	25.357	2.06E-05	63.34	787.02	25.361	1.34E-05	2.94E-05	34.8	25.36	25.36	97.41	560.67	25.37	97.89

cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.3464	6.32077	109.7	652.17	6.3454	6.34619	6.34949	70.82	6.347	6.345	153.4	686.16	6.35	154
cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
Riemann	25.36	1.04E-05	63.34	787.03	25.361	6.63E-06	1.53E-05	34.8	25.36	25.36	97.37	560.62	25.36	97.92
Parzen	25.38	0.00024	63.33	786.94	25.375	0.00014	0.00031	34.76	25.36	25.37	97.32	560.52	25.32	97.96
Tukey	25.36	2.78E-05	63.34	787.01	25.363	2.06E-05	3.99E-05	34.8	25.37	25.36	97.41	560.67	25.36	97.89
Bohman	25.359	1.40E-05	63.34	787.02	25.361	8.87E-06	2.02E-05	34.8	25.36	25.36	97.39	560.64	25.36	97.91
Poisson	25.36	1.24E-05	63.34	787.02	25.362	8.00E-06	1.92E-05	34.8	25.36	25.36	97.39	560.64	25.36	97.91
Hanning-Poisson	25.36	1.13E-05	63.34	787.02	25.361	7.04E-06	1.69E-05	34.8	25.36	25.36	97.38	560.63	25.36	97.91
Cauchy	25.36	2.56E-05	63.34	787.01	25.363	1.84E-05	3.75E-05	34.79	25.36	25.36	97.41	560.66	25.36	97.89
Bartlett-Hann	6.3463	6.3207	109.7	652.17	6.3457	6.34598	6.34941	70.82	6.346	6.345	153.4	686.16	6.35	154
NLW	25.358	6.15E-06	63.34	787.03	25.357	6.15E-06	1.03E-05	34.8	25.35	25.36	97.38	560.63	25.36	97.92

Table (23). MSE values of nonstationary AR(2) spectrum estimation with (n=100)

Windows	α_1=0.6 α_2=1.2	α_1=1.2 α_2=0.6	α_1=-0.9 α_2=-1.2	α_1=-1.2 α_2=-0.9	α_1=0.9 α_2=1.2	α_1=1.2 α_2=0.9	α_1=-1.2 α_2=0.7	α_1=0.7 α_2=-1.2	α_1=1.1 α_2=1.2	α_1=1.2 α_2=1.1	α_1=-1.1 α_2=-1.2	α_1=-1.2 α_2=-1.1	α_1=-1.1 α_2=1.2	α_1=1.1 α_2=-1.2
rectangle	25.362	2.38E-05	63.35	787.03	25.363	7.53E-05	7.57E-05	34.8	25.36	25.36	97.37	560.61	25.35	97.92
triangle	25.362	1.47E-05	63.34	787.05	25.363	1.58E-05	2.73E-05	34.8	25.36	25.36	97.38	560.64	25.36	97.92
Hanning	6.3464	6.32022	109.7	652.16	6.346	6.35087	6.35121	70.82	6.347	6.346	153.4	686.11	6.344	154
Hamming	25.362	1.55E-05	63.34	787.05	25.363	2.79E-05	3.51E-05	34.8	25.35	25.36	97.38	560.62	25.36	97.93
Blackman	46.896	3.29407	37.78	891.99	46.899	3.27635	3.26925	16.71	46.89	46.9	64.89	478.18	46.89	65.35
Exact Blackman	6.4338	6.23357	109.4	653.04	6.4334	6.26307	6.2639	70.53	6.435	6.433	152.9	685.22	6.432	153.6
Blackman-Harris	6.347	6.31965	109.7	652.16	6.3465	6.34851	6.34974	70.82	6.348	6.346	153.4	686.12	6.346	154
Blackman-Nuttall	6.3889	6.27792	109.6	652.59	6.3885	6.30674	6.30795	70.68	6.39	6.388	153.2	685.69	6.388	153.8
Flat-Top	6.3408	6.32589	109.8	652.1	6.3402	6.35301	6.35509	70.84	6.342	6.34	153.4	686.21	6.341	154.1
Exponential	25.362	4.96E-05	63.35	787.02	25.363	0.00018	0.00016	34.8	25.35	25.36	97.36	560.58	25.34	97.93
Gaussian	25.362	1.46E-05	63.34	787.05	25.363	1.95E-05	2.89E-05	34.8	25.36	25.36	97.38	560.64	25.36	97.92
Riesz	25.362	2.06E-05	63.34	787.05	25.364	2.10E-05	3.91E-05	34.8	25.36	25.36	97.39	560.65	25.36	97.91
cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
cosine	6.3462	6.3204	109.7	652.16	6.3458	6.34946	6.35059	70.82	6.347	6.346	153.4	686.13	6.345	154
cosine	6.3465	6.32019	109.7	652.17	6.3455	6.34527	6.3486	70.82	6.345	6.346	153.4	686.13	6.349	154.1
Riemann	25.361	1.16E-05	63.34	787.04	25.362	6.30E-06	1.64E-05	34.8	25.36	25.36	97.37	560.65	25.36	97.92
Parzen	25.363	0.00017	63.35	787.01	25.36	0.00038	0.00046	34.81	25.31	25.36	97.33	560.5	25.34	97.99
Tukey	25.362	2.22E-05	63.35	787.03	25.364	6.42E-05	6.82E-05	34.8	25.36	25.36	97.37	560.62	25.35	97.92
Bohman	25.362	1.45E-05	63.34	787.05	25.362	1.35E-05	2.45E-05	34.8	25.36	25.36	97.38	560.65	25.36	97.92
Poisson	25.362	1.26E-05	63.34	787.04	25.362	1.61E-05	2.45E-05	34.8	25.36	25.36	97.38	560.63	25.36	97.92
Hanning-Poisson	25.362	1.19E-05	63.34	787.04	25.362	1.07E-05	1.95E-05	34.8	25.36	25.36	97.38	560.64	25.36	97.93
Cauchy	25.362	2.08E-05	63.35	787.04	25.364	5.94E-05	6.31E-05	34.8	25.36	25.36	97.37	560.62	25.35	97.92
Bartlett-Hann	6.3464	6.32019	109.7	652.16	6.3459	6.34937	6.35056	70.82	6.347	6.346	153.4	686.12	6.345	154
NLW	25.358	6.22E-06	63.34	787.03	25.357	6.12E-06	1.02E-05	34.8	25.35	25.36	97.38	560.63	25.36	97.92

5. Conclusions and Recommendations

Several important conclusions and recommendations were reached in our thesis.

5.1. Conclusions

The work conclusions can be stated as follows,

(a) For the AR(1) process, the **Flat-Top** kernel was the best for the nonstationary cases, the **Cosine** kernel was the best for the boundary values cases and the **Suggested** kernel was the best for all stationary values except in the case $n = 10$ and $\alpha = -0.5, -0.3$ where the **Riemann** kernel was the best.

(b) For the AR(2) process, the kernel vantage were,

(i) The **suggested** kernel was the best for all stationary cases.

(ii) For the nonstationary cases, the kernels vantage were as in the following,

α_1	0.6	-1.2	0.9	1.1	1.2	-1.1	**Flat-Top** kernel
α_2	1.2	-0.9	1.2	1.2	1.1	1.2	

α_1	1.2	1.2	-1.2	**Suggested** kernel
α_2	0.6	0.9	0.7	

α_1	-0.9	0.7	-1.1	-1.2	1.1	**Blackman** kernel
α_2	-1.2	-1.2	-1.2	-1.1	-1.2	

(iii) For the boundary values cases, the kernels vantage were as in the following,

α_2	α_1	
	1	-1
0.5	**Suggested** kernel	
1.2	**Flat-Top** kernel	

(c) In general, the performance of suggested kernel is very excellent, since it is possible to convert any nonstationary process to stationary process under suitable transformation.

5.2. Recommendations

There are some recommendations can be stated as follows,

(a) Use the suitable kernel according to the vantage situations stated in the conclusions section above.

(b) Study the performance of the suggested kernel in other cases of ARMA(p,q) process.

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