

# A Maximin Zero Suffix Method for Quadratic Transportation Problem

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**Abstract** In the present paper, a maximin zero suffix method introduced by Sharma *et al* (2014) has been extended to solve the quadratic transportation problem (QTP). The proposed method is better than the existing methods in the literature for solving QTP because it is free from any huge algebraic computation and drawing of graphs of quadratic cost functions to search absolute point. The algorithm of the proposed method has been given and a numerical example has been presented to explain its application. The proposed method is very simple, easy to understand and apply for any practitioners who have not much knowledge of mathematics.

**Keywords** Quadratic transportation problem, Estimated cost matrix, Transformed cost matrix, Reduced cost matrix, Zero suffix method, Optimal solution

## 1. Introduction

The classical transportation problem (CTP) is a particular class of linear programming problem (LPP) and deals with the distribution of goods (or products) from  $m$  sources (plants or suppliers) to  $n$  destinations (warehouses or customers). The main objective in CTP is to determine the amount of goods to be shipped from each source to each destination so as to maintain the demand and supply requirements at the minimum total transportation cost. In CTP, the cost of transporting one unit of goods from a source to a destination is independent of the amount of goods shipped.

The classical transportation problem (CTP) is defined as

$$\text{Min } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\text{Subject to } \sum_{j=1}^n x_{ij} = a_i \text{ for } i = 1, 2, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j \text{ for } j = 1, 2, \dots, n$$

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

$$x_{ij} \geq 0 \text{ for all } (i, j)$$

where  $c_{ij}$  = Unit transportation cost from  $i$  th source to  $j$  th destination

$a_i$  = Amount of goods available at  $i$  th source.

$b_j$  = Amount of goods required at  $j$  th destination

$x_{ij}$  = Amount of goods transported from  $i$  th source to  $j$  th destination.

$m$  = Number of sources.

$n$  = Number of destinations.

CTP, being a well structured problem, has been studied extensively in the literature. It was Hitchcock (1941) who firstly developed the basic transportation problem and suggested constructive method of solution. Later on, Dantzig (1963) formulated the CTP as an LPP and provided the solution. A number of researchers have worked to find suitable methods of solving CTP using different assumptions including Charnes and Cooper (1954), Shih (1987), Arsham and Khan (1989), Adlakha and Kowalski (1999), Ji and Chu (2002), Korukoglu and Balli (2011), Sudhakar *et al* (2012), are some among others. Recently Sharma *et al* (2013) have introduced an easy and interesting method named, "A modified zero suffix method" for finding optimal solution for CTP.

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In the real life situation, per unit transportation cost of goods does not remain always constant. Generally, it decreases with the increase in the volume of transported goods but not always. Sometimes, it has also been seen that it increases with the increase in the volume of the transported goods due to highway congestion (restriction of over loading). This idea leads to a new dimension of the CTP and generally it has been observed that per unit cost  $c_{ij}$  is a function of the amount shipped  $x_{ij}$ . If the per unit transportation cost  $c_{ij}$  in CTP is a quadratic function

$f_{ij}(x_{ij})$  of  $x_{ij}$ , then CTP is termed as the quadratic transportation problem (QTP). In QTP, the cost coefficients are quadratic functions, and in CTP the cost coefficients are constants.

Mathematically, the quadratic transportation problem (QTP) is defined as

$$\begin{aligned} \text{Min } Z &= \sum_{i=1}^m \sum_{j=1}^n f_{ij}(x_{ij}) \\ \text{Subject to } \sum_{j=1}^n x_{ij} &= a_i \text{ for } i = 1, 2, \dots, m \\ \sum_{i=1}^m x_{ij} &= b_j \text{ for } j = 1, 2, \dots, n \\ \sum_{i=1}^m a_i &= \sum_{j=1}^n b_j \\ f_{ij}(x_{ij}) &\geq 0, \quad f_{ij}(0) = 0 \\ x_{ij}, a_i, b_j &\geq 0, \text{ integer} \end{aligned}$$

There is no any efficient direct algorithm to solve QTP in the literature. However, QTP and its indirect solution techniques have been discussed by researchers including Hochbaum *et al* (1992), Megiddo and Tamir (1993), Eduardo *et al* (2003), Cosares and Hochbaum (1994) are some among others. The techniques developed by these researchers are very much complicated and time-consuming. Recently, Adlakha and Kowalski (2013) have introduced an analytical algorithm for QTP which is based on finding absolute point (AP) in QTP by tracing graph for each quadratic cost function which is time-consuming and tedious and needs computer application for tracing graphs of each quadratic cost function.

In the present paper, "A maximin zero suffix method" introduced by Sharma *et al* (2014) for solving assignment problems has been extended to solve QTP. To use this method, firstly estimated cost matrix of the QTP has to be obtained and then "A maximin zero suffix method" can be applied on the estimated cost matrix. This method is very

simple, easy to apply and free from drawing graph of each quadratic cost function. An example of QTP from Adlakha and Kowalski (2013) has been considered to explain the "Maximin zero suffix method" for finding optimal solution.

## 2. Estimated Cost Matrix

Firstly each quadratic cost is estimated to be constant cost. Then idea developed by Sharma *et al* (2014) is used to solve it. Per unit transporting cost of goods from  $i$  th source to  $j$  th destination is  $a_{ij}x_{ij}^2 + b_{ij}x_{ij} + c_{ij}$  where  $a_{ij} \neq 0$ . If  $a_{ij} = 0$  and  $b_{ij} \neq 0$ , the problem is linear transportation problem (LTP). If  $a_{ij} = 0$  and  $b_{ij} = 0$ , the problem is a classical transportation problem (CTP).

The possible allocation to the  $(i, j)$  th cell lies from 0 to  $\min\{a_i, b_j\}$ . The maximum possible allocation to the  $(i, j)$  th cell is  $\min\{a_i, b_j\}$  and the corresponding cost is

$$a_{ij} \left\{ \min(a_i, b_j) \right\}^2 + b_{ij} \left\{ \min(a_i, b_j) \right\} + c_{ij} = d_{ij}$$

(say). The minimum possible allocation cost there is  $c_{ij}$  for the case of non allocation. Corresponding to the quadratic allocated cost matrix we have an  $n \times n$  estimated cost matrix  $[d_{ij}]$ . Note that in the classical transportation problem at most cost is  $c_{ij} \left\{ \min(a_i, b_j) \right\}$  and at least cost is 0 in the case of non - allocation.

## 3. A Maximin Zero Suffix Method

Our goal is to find the minimum possible allocation cost which is possible when we allocate the lowest estimated cost cell.

To obtain the minimum allocated cost, we search the least cost cell in the estimated cost matrix for possible allocation. After subtracting the lowest cost of each row from the corresponding row and then subtracting the lowest cost of each column from the corresponding column we get at least one zero in each row and each column in the transformed estimated cost matrix  $[d_{ij}']$ . The lowest cost cells in each row and each column are detected by these zeros. On minimum cost distribution, zero cells (Lowest cost cells) should be assigned as far as possible satisfying the demand and supply conditions. Our aim is to assign zero cells one by one satisfying corresponding demand and supply conditions in such a manner that over all transporting cost becomes optimal one. The question is to which zero cells should be allocated firstly for getting an optimal transportation schedule.

To overcome this critical situation we focus on two zero cells  $(p, q)$  and  $(r, s)$  of the transformed estimated cost matrix. Let  $(p, q)$  cell is assigned and  $(r, s)$  cell is not assigned. It means that  $r$  th supplying source is not used for  $s$  th demand destination. Then  $r$  th supply must be assigned to other demand. To assign  $r$  th supply for minimum cost we select a cell in the  $r$  th row whose cost is the next lowest cost in it's row. Let  $d_{rv}'$  is the next minimum in the  $r$  th row. Then  $r$  th supply must be assigned to  $v$  th demand. So the excess cost in the absence of not assigning to  $(r, s)$  cell is  $d_{rv}'$ . Also in the absence of not assigning to the  $(r, s)$  cell,  $s$  th demand is not taken by the  $r$  th supply. To assign a value to the  $s$  th demand for minimum cost, we select a cell in the  $s$  th column whose cost is the next smallest in its column. Let  $d_{us}'$  is the next minimum in the  $s$  th column. Then  $s$  th demand must be fulfilled by the  $u$  th supply. So the excess cost in the absence of not assigning to the  $(r, s)$  cell, the assignment cost may be increased at most by maximum of  $(d_{rv}', d_{us}') = \max\{\text{lowest cost in its row, lowest cost in its column excluding itself}\} = \text{suffix value of zero of the } (r, s) \text{ cell}$ . Similarly, in case of not assigning to the  $(p, q)$  cell we may have to face excess cost which is equal to the suffix value of zero of the  $(p, q)$  cell. Note that  $d_{ij}'$  is the value of  $d_{ij}$  in the transformed estimated cost matrix. We conclude that to get minimum transportation cost it is better to assign that zero cell whose suffix value is the greatest one.

The methodology of the paper is to block the path of exceeding cost above the optimal one in the process of assigning the zero cells. This is done by finding suffix value of all zeros of the transformed table and assignment is done to the cell having greatest suffix value. Then we have to delete the row/column of assigned cell to get the reduced

table. The process is continued in the reduced table till all the demand and supply are exhausted.

### 4. Algorithm

The algorithm of maximin zero suffix method for finding an optimal solution of quadratic transportation problem (QTP) consists of the following steps:

Step 1. Construct the estimated cost matrix  $[d_{ij}]$  where

$$d_{ij} = a_{ij} \left\{ \min(a_i, b_j) \right\}^2 + b_{ij} \left\{ \min(a_i, b_j) \right\} + c_{ij}$$

Step 2. Subtract the least element of each row from all the elements of the corresponding row of the estimated cost matrix

Step 3. Subtract the least element of each column from all the elements of the corresponding column obtained in step 2 to get the transformed estimated cost matrix  $[d_{ij}']$

Step 4. Find the suffix value of each zero cell  $(p, q)$  using the following formula. Suffix value of the cell

$$(p, q) = \max \left\{ \min_{j \neq q} d_{pj}', \min_{i \neq p} d_{iq}' \right\}$$

Step 5. Assign the cell having the greatest suffix value and delete the exhausted row/column to get the reduced estimated cost matrix

Step 6. In the reduced table update  $c_{ij}$  for those cells whose  $\min\{a_i, b_j\}$  is reduced by  $k$  as

$$d_{ij}' - \{a_{ij}\{k\} + b_{ij}\}$$

Step 7. Repeat steps 2 to 6 till all the demands and supplies are exhausted.

### 5. A Numerical Example

Let us consider the cost matrix  $f_{ij}(x_{ij})$  for the QTP given in Adlakha and Kowalski (2013).

Table 1.

	$D_1$	$D_2$	$D_3$	Supply
$O_1$	$2x^2 + x$	$3x^2 + 2x$	$x^2 + 4x$	2
$O_2$	$x^2 + 3x$	$2x^2 + 2x$	$3x^2 + x$	2
$O_3$	$3x^2 + 3x$	$2x^2 + 3x$	$4x^2 + x$	2
Demand	1	4	1	

Step 1. Estimated cost matrix of the given QTP is

**Table 2.**

	$D_1$	$D_2$	$D_3$	Supply
$O_1$	$2 \times 1^2 + 1 = 3$	$3 \times 2^2 + 2 \times 2 = 16$	$1^2 + 4 \times 1 = 5$	2
$O_2$	$1^2 + 3 \times 1 = 4$	$2 \times 2^2 + 2 \times 2 = 12$	$3 \times 1^2 + 1 = 4$	2
$O_3$	$3 \times 1^2 + 3 \times 1 = 6$	$2 \times 2^2 + 3 \times 2 = 14$	$4 \times 1^2 + 1 = 5$	2
Demand	1	4	1	

Where  $x_{ij} = \min \{a_i, b_j\}$

Step 2. Subtracting the row minimum from the respective row in table 2, we get

**Table 3.**

	$D_1$	$D_2$	$D_3$	Supply
$O_1$	0	13	2	2
$O_2$	0	8	0	2
$O_3$	1	9	0	2
Demand	1	4	1	

Step 3. Subtracting Column minimum from the respective column in table 3, we get

**Table 4.**

	$D_1$	$D_2$	$D_3$	Supply
$O_1$	0	5	2	2
$O_2$	0	0	0	2
$O_3$	1	1	0	2
Demand	1	4	1	

Step 4. Suffix Value of each zero in table 4 is shown in table 5 along with the allocated amount to be shipped.

**Table 5.**

	$D_1$	$D_2$	$D_3$	Supply
$O_1$	1 $O_2$	5	2	1
$O_2$	$O_0$	$O_1$	$O_0$	2
$O_3$	1	1	$O_1$	2
Demand	4	4	1	

Step 5. Reduced estimated cost matrix with updated  $d_{12}' = 5 - (3 \times 1 + 2) = 0$  has been obtained and shown in table 6 along with suffix value of zeros and allocated amount to be shipped

**Table 6.**

	$D_2$	$D_3$	Supply
$O_1$	1 $0_2$	2	
$O_2$	$0_0$	$0_0$	2
$O_3$	1	$0_1$	2
Demand	4-3	1	

Now repeating steps 2 to 6 we get

**Table 7.**

	$D_2$	$D_3$	Supply
$O_2$	2 $0_1$	2	
$O_3$	1	$0_1$	2
Demand	1	1	

**Table 8.**

	$D_2$	$D_3$	Supply
$O_3$	1 1	1 0	2
Demand	1	1	

Finally an optimal transportation table is as follows:

	$D_1$	$D_2$	$D_3$	Supply
$O_1$	1	1		2
$O_2$		2		2
$O_3$		1	1	2
	1	4	1	

The minimum cost for the given QTP can be obtained as

$$\begin{aligned}
 \text{Minimum cost} &= (2 \times 1^2 + 1) + (3 \times 1^2 + 2 \times 1) + (2 \times 2^2 + 2 \times 2) + (2 \times 1^2 + 3 \times 1) + (4 \times 1^2 + 1) \\
 &= 3 + 5 + 12 + 5 + 5 \\
 &= 30
 \end{aligned}$$

It would be recalled that the total minimum transportation cost for the above QTP obtained by Adlakha and Kowalski (2013) is also 30.

## 6. Optimality Criteria and Convergence of the Algorithm

Since we have used the method of minimizing the excess transporting cost corresponding to a cell by assigning the lowest cost cell in each row and each column, there is no chance that the cost obtained according to this process is higher than the optimal cost.

As the number of rows and the number of columns are finite and at each iteration one row or one column is deleted, the algorithm will terminate in a finite number of steps. The number of iterations will be at most  $m + n$  and at least  $\max(m, n)$ .

## 7. Concluding Remarks

A maximin zero suffix method has been introduced to solve the quadratic transportation problem (QTP). The proposed method is better than all existing methods in the literature for solving QTP because it is free from any huge algebraic computation and drawing of graphs of quadratic cost functions to search absolute point. The algorithm of the proposed method has been given and the application of the proposed method has been explained by a numerical example. The proposed method is very simple, easy to understand and apply for any practitioners who have not much knowledge of Mathematics and therefore useful for managerial decisions. The proposed maximin zero suffix method for solving QTP can be extended to the polynomial transportation problem (PTP) in which the cost coefficients are higher than second degree polynomials.

The future research on the proposed method is to apply the method for solving real-world problem. For using this method to solve real-world problems a computer program must be developed because the real-world problems consist of several sources and destinations which is difficult to handle manually. Since the proposed method is easy to apply and understand, it can be easily applied to solve real-world problems.

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