American Journal of Operational Research 2013, 3(2): 28-44
DOI: 10.5923/j.ajor.20130302.02

An Exact Method for Solving the Four Index Transportation Problem and Industrial Application

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Abstract Due to the importance of freight transport in the panel of logistics cost, our research is conducted to optimize transportation on cost criterion. Its aim is to find a solution on the minimum transportation cost for the four index transportation problem (4ITP: origin, destination, goods type, vehicle type). The 4ITP was solved by Ninh's method where the problem is not degenerated and the solution existence condition (SEC) is verified. Given the complexity of such a model, we develop an exact method to solve this problem with real variables in all untreated cases where it is degenerated and the SEC is not satisfied. On the criterion of degeneration treatment capacity and execution time, our method takes more advantages over the simplex. Based on the obtained result, we propose an extension used the elements in FORTRAN 95 library to integer and mix variables, which matche the piece and weight transportation model. This will be a key factor to the implementation of this four index model in industry.

Keywords Degeneration, Four Index Transportation Problem, Potential Method, Integer Variable, Real Variable, Simplex Method

1. Introduction

The study of road freight transport includes all the methods and activities which aim is to coordinate the physical flows by optimizing all intervening in each link of the chain. Since the first classical problem, defined as the two index transportation problem (2ITP), was developed in 1941 by Hitchcock, the research in this area has achieved satisfactory results on various extensions. There are two main directions of research: theoretical and operational research. On the theoretical point of view, the research on transportation problem is developed at the macro level with the n index problem. On the operational angle, the results are still limited because the application of transportation models to the reality is complex. Consequently, the remaining problem is to find methods to solve the particular problems.

A classification of transportation problems can be determined based on the typology of index number and corresponding resolution method. We classify them into four groups: 2 indexes, 3 indexes, 4 indexes and n indexes. In this paper, we are interested in the four index transportation problem, called 4ITP with four indexes: origin, destination, goods types and truck types, one of extension cases of the Hitchcock's problem. The 4ITP includes the exchange of one or several goods types transported by vehicle types in a single link between origins to destinations.

This article is organized as follow: in the first part, we present the economical signification and the formalization of 4ITP. Subsequently, we give a brief literature review and explain the relationship with our problem. In the third part, we propose an exact method, an extension of Ninh's method (Ninh, 1979), to solve the 4ITP in all particular cases. To illustrate this method, we also present the obtained execution results on the artificial database. Based on it, we compare the performance of our method with the simplex. Finally, an approached method used to solve the 4ITP with integer and mix variables will be proposed. We also develop a program on the integer four index model in Fortran 95 by coupling between this algorithm and the main program of 4ITP with real variable. This will be the first test for the implementation of our algorithm with real industrial data.

2. Problem Presentation

2.1. Economical Signification

In the large economic zone, there is a group O with m production sites that manufacture p product types. These products are raw materials for the manufacturing network of its customer D. The group O can possess q vehicle types or use the provider transportation service to transport the goods. The maximum quantity that a vehicle type H_l can load is δ_l (ton, e.g.). The production network of group D uses the flow

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Published online at http://journal.sapub.org/ajor

"just-in-order" to avoid the storage of raw materials in the manufacturers. Therefore, these raw materials are stored in a logistics network that consists of n warehouses. To maintain the production flow, the manufacturing sites send their requests on raw materials to the management centre. Based on this obtained database, the centre can calculate the ordered total quantity of product type S_k is γ_k (ton, e.g.) and then sends the order to its subcontractor, group *O*.

Note that the production site of subcontractor is O_i (i=1...m) and these are called origins. The warehouse of customer is D_j (j=1...n) and these are called destinations. The product type S_k (k=1...p) and the vehicle type H_l (l=1...q). The quantity of goods (all types of existing products to this site) that the site O_i is ready to deliver is α_i (ton, e.g.); the storage quantity of the warehouse D_j is β_j (ton, e.g.); the management centre demands to be supplied enough γ_k tons of product S_k for its total network while $\alpha_i, \beta_j, \gamma_k, \delta_l$ must have the same calculation unit.

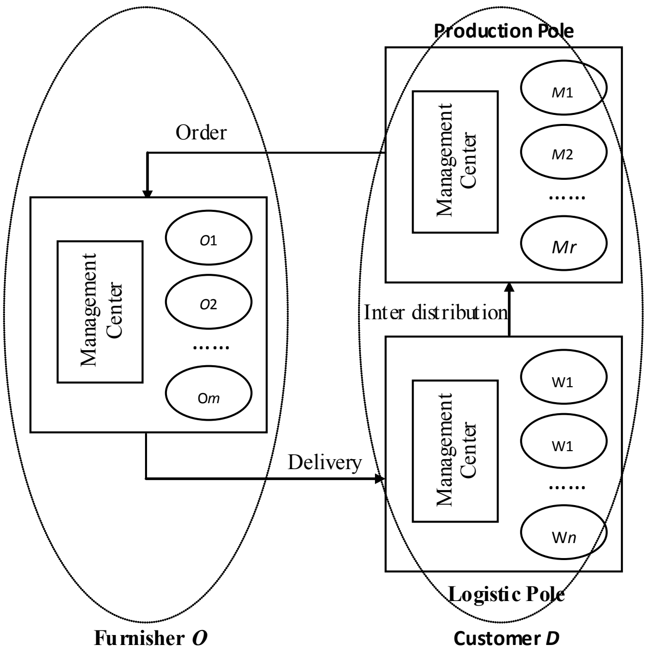

Figure 1. Order and delivery system between Furnisher *O* and Customer *D*

Suppose that the average unit cost for transporting a unit of goods type S_k from origin O_i to destination D_j by vehicle type H_l is c_{ijkl}. This unit cost will be calculated to cover the fixed and variable costs such as: the distance between the production site and the customer's warehouse, toll, amortization costs of vehicles, the reciprocal relationship between the vehicle type and the kind of goods being moved,...

The problem is posed: We must develop a transportation planning that responds to the command of customer. The goods are ready from production sites and delivered to the customer's warehouses by the existing vehicle types with the lowest total transportation cost.

2.2. Model

The 4ITP formalization is elaborated with the following data:

→ m origin nodes $O_i (i = 1 \dots m)$. The goods delivery capacity of this node is $\alpha_i (\alpha_i > 0\ i = 1 \dots m)$.

→ p goods types. The total quantity of goods type S_k $(k = 1 \dots p)$ at all the nodes is $\gamma_k (\gamma_k > 0\ k = 1 \dots p)$.

→ n destination nodes D_j $(j = 1 \dots n)$. The reception capacity of goods at this node is β_j $(\beta_j > 0\ j = 1 \dots n)$.

→ q vehicle types. The total quantity of goods that the vehicle type $H_l (l = 1 \dots q)$ can transport is δ_l $(\delta_l > 0\ l = 1 \dots q)$.

→ $c_{ijkl} \geq 0 (i = 1 \dots m, j = 1 \dots n, k = 1 \dots p, l = 1 \dots q)$ the unit transportation cost for a unit of goods type S_k that is transported from origin node O_i to destination node D_j using the vehicle type H_l .

The hypothesis must be fixed by inhibition for the transport of goods in the direction from destination nodes to origin nodes.

The variables of 4ITP are

$x_{ijkl} \geq 0$ $(i = 1 \dots m, j = 1 \dots n, k = 1 \dots p, l = 1 \dots q)$ the quantity of goods type S_k is transported from node O_i to node D_j by the vehicle type H_l in the solution to establish.

The problem becomes: Determine the variables

$$x_{ijkl} \geq 0\ (i = 1 \dots m, j = 1 \dots n, k = 1 \dots p, l = 1 \dots q)$$

$$\text{for } minL(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} c_{ijkl} . x_{ijkl} \quad (1)$$

subject to

$$\sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} x_{ijkl} = \alpha_i\ (i = 1 \dots m) \quad (2)$$

$$\sum_{i=1}^{m} \sum_{k=1}^{p} \sum_{l=1}^{q} x_{ijkl} = \beta_j\ (j = 1 \dots n) \quad (3)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{q} x_{ijkl} = \gamma_k\ (k = 1 \dots p) \quad (4)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} x_{ijkl} = \delta_l\ (l = 1 \dots q) \quad (5)$$

These constraints concern the offer at origin O_i, reception capacity at destination D_j, quantity of product S_k and transported quantity of vehicle H_l.

3. Literature Review

The first transportation problem, also called the two index problem (2ITP) or the foundation to research the classical transportation problem, was developed in 1941 by Hitchcock. The first resolution method defined by Kantorovich and Gavorin, Russian researchers, is the potential method. Then, G.B. Danzig proposed another method for solving the classical problem based on the simplex method. In 1958, Gleyzal presented a method using the dual simplex algorithm, and in 1963, Kuhn proposed a method for solving the allocation problem, a special case of transportation problem, by developing the idea of a Hungarian mathematician in 1931. Although other methods have been introduced so far, the potential method is the most commonly used method in research and teaching.

A classification of extension on the Hitchcock's problem is based on the typology of index numbers and resolution methods. Four groups are arranged: 2 indexes, 3 indexes, 4 indexes and n indexes.

3.1. Two Indexes Transportation Problem Group

In order to solve the Hitchcock's problem, the most commonly used method is the potential method. They used a rectangle: one side represents origins. They divided this side into m segments and the i^{th} segment represents origin O_i (i=1…m). The other side represents destinations. They also divided it into n segments and j^{th} segment represents destination D_j (j=1…n). Each cell of the table represents an arc (ij) in which, they noted the cost c_{ij} and goods x_{ij}. Then, they calculated the potentials and tested the optimality criterion. If the solution is not optimal, this solution is improved to obtain the other better solution.

The first branch focuses on the way to find a good initial solution. There are several new methods proposed, for example, an exact method DOR[6] and an approached method which obtains a very good initial solution (with a less ecart than 2% on the dual problem) for the primal problem whose execution time is $O(n^3)$[23]. The method DOR is very special. Being different from the previous methods (determine the boxes consisting of distributed goods with the minimal transportation cost), Dubeau and Gueye determined the boxes with a high cost: determine the boxes (i,j) in which the goods are not delivered , i.e. x_{ij}=0, then determine the boxes in which a minimum goods quantity must be assigned and continue until the first solution. On the performance of this method, the researchers found that it is more effective than the Vogel Russel method and in many cases of small size, the DOR method immediately gives the optimal solution.

The second branch consists in a constraint on the limited transportation capacity on the roads. The problem treated the bound on total availabilities at sources and total destination requirements. The application domain was found in a variety of real world problems such as: telecommunication networks, production-distribution system, rail and urban road system, planned automated cargo system with finite capacity of resources, for example, vehicles, docks, parking places, ...[3]. Another extension in this branch was the problem with exclusionary side constraints, a practical distribution and logistics problem: a warehouse may be unable, or may not be allowed, to receive goods simultaneously from some pairs of sources due to the damage or possible deterioration, e.g hazardous materials such as explosives, flammables, oxidizing elements should be separated from the others in order to avoid the accidents[24].

In the third branch, the linear objective function is kept but the unit transportation cost is fuzzy, the offer and the demand are fuzzy and integer. Several algorithms were proposed to solve these transportation problems in a fuzzy environment but until now, no one has used generalized fuzzy numbers for solving them. So, an extension was proposed for solving a special type of fuzzy transportation problem by assuming that a decision maker is uncertain not only about the precise values of transportation cost but also the supply and demand of the product. This is a case in which transportation costs are represented by generalized trapezoidal fuzzy numbers[13].

In the fourth branch, the transportation system must respond to multiple objectives. Thus, the linear objective function is also kept but the mono-objective problem is transformed into multi-objective one. Following this extension, the problems were developed: the possibility coefficient of objective function[9], interval parameters[4] and interval-valued fuzzy parameters[13].The fuzzy programming was essentially used to solve them. The fifth branch consists of the solution on the mass transportation problem, a typical example in the continuous programming, which was proposed at the first time by Monge in 1871. It was discretized by Hitchcock to become the 2ITP. Then, Kantorovich introduced the problem under geometric form and Kangabo used the continuous programming to solve it[12].

3.2. Three Indexes Transportation Problem Group

By increasing the index numbers, the three index transportation problem (3ITP) was developed and solved by an extension of the potential method, under the geometric form. These properties are source, destination and type of product or mode of transportation. Its extensions were less varied: a new approached method, based on adding additional parameters to create a new deterministic problem, was proposed to solve the interval problem. For the fuzzy problem, a simple approached method based on transforming this problem into interval problem and an evolutionary algorithm based parametric approach are presented[11]. In addition, an improved genetic algorithm was proposed for solving multi-objective problem in which the coefficients of objective function were represented as fuzzy numbers[15].

With the use of the parallelepiped, we see that the constraints of the three index problem can be represented under a sequence of rectangles representing the constraints of two indexes when one variable is kept, the other two variables are changed and superpose on each other. Hence, if they continue extending toward this direction, they obtain the development of n index transportation problem.

3.3. Multi-indexes Transportation Problem Group

The general n indexes transportation problem was first proposed by Ninh in 1979[16]. From this general problem, they have $(n-1)$ n index problems (nITP) according to the constrains on summation of $(n-1)$ indexes.[17]. Being different from the precedent thinking, Ninh did not use the n dimensional super box to solve it, he gave an exact method on the plan, an extension of potential method coordinating the resolution of the primal and dual problems. However, this result does not cover the solution of all particular problems of nITP because he proposed only the necessary condition and the sufficient condition so that the general problem has a solution[17].

Among these n index problems, Ninh decided to solve a particular case with the summation on (n-1) indexes which contributes significantly in economic terms. It is presented:

Determine $x_{i_1 i_2 \dots i_n} \geq 0$ $i_j = 1 \dots n_j$ $j = 1 \dots n$ for

$$minL(X) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \dots \sum_{i_n=1}^{n_n} c_{i_1 i_2 \dots i_n} x_{i_1 i_2 \dots i_n} \quad (6)$$

subject to

$$\sum_{i_2=1}^{n_2}\sum_{i_3=1}^{n_3}\dots\sum_{i_n=1}^{n_n} x_{i_1 i_2 \dots i_n} = a_{i_1}^{I_1} \quad i_1=1\dots n_1 \quad (7)$$
$$\sum_{i_1=1}^{n_1}\sum_{i_3=1}^{n_3}\dots\sum_{i_n=1}^{n_n} x_{i_1 i_2 \dots i_n} = a_{i_2}^{I_2} \quad i_2=1\dots n_2 \quad (8)$$
……………………………………………………………………………….
$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\dots\sum_{i_{n-1}=1}^{n_{n-1}} x_{i_1 i_2 \dots i_n} = a_{i_n}^{I_n} \quad i_n=1\dots n_n \quad (9)$$

The parameters are known and deterministic:

$$a_{i_1}^{I_1} > 0, a_{i_2}^{I_2} > 0, \dots, a_{i_n}^{I_n} > 0, c_{i_1 i_2 \dots i_n} \geq 0$$

For solving this problem, Ninh proposed an exact method, an extension of potential method, by coordinating the resolution of primal and dual problems. Ninh also presented and demonstrated the theorem of the necessary and sufficient condition so that the problem has had the solution (solution existence condition - SEC). This problem was also presented under the geometric form with the constraint on decomposable cost parameter. This is a k dimensional super box A in which the r^{th} dimension $A(r)$ is composed of n_r segments. The section A_{ri} is a super box with k-1 dimensions that is attached to the segment i of the edge A_{ri}. It was solved by an approached method[22].

3.4. Four Indexes Transportation Problem Group

Among these problems, the 4ITP is a model that concerns business organizations and attracts researchers. It fits the demands to the shuttle type in the companies that can use various trucks to transport goods from the manufacture (origin) to the deposit (destination).

By replacing n=2 in the theorem SEC of n index problem, Ninh found the SEC for the 2ITP that existed previously. By replacing n=4, he obtained the SEC for the 4ITP as follow:

Theorem 1: The necessary and sufficient condition so that the 4ITP has a solution:

$$\sum_{i=1}^{m}\alpha_i = \sum_{j=1}^{n}\beta_j = \sum_{k=1}^{p}\gamma_k = \sum_{l=1}^{q}\delta_l \quad (10)$$

If the 4ITP has solution, this problem will certainly have the optimal solution.

If the problem (1...5) responds to the condition (10), the constraint system (2...5) can transform into a system consisting of (m+n+p+q-3) equations.

Based on this result, Ninh solved this 4ITP where the SEC was satisfied and the problem was not degenerated[17]. Remark that there are three 4ITP according to the constraints if they take the summation on one, two or three indexes. Following this success, Zitouni proposed an exact method which was built on Ninh's method improvement to solve the capacitated four index transportation problem C4ITP[25,26]. In 2010, based on a large database, Djamel realized a comparative numerical study between two classical methods (simplex method and interior point method) and Zitouni's method. Based on the criterion of the problem treatment capacity and execution time, the numerical result demonstrated that the Zitouni's method is the most favourable[5].

Table 1. Research results of transportation problems

Group	Model	Reference	Method	Size
2 indexes	Classical 2I	[6]	Potential method	40x40
	Search good primal solution	[6]	Exact method (method DOR)	40x40
		[23]	Approached method	250x250
	Exclusionary side constraints	[24]	Brand and bound algorithm	110x120
	Capacitated problem with bounds	[3]	Approached method	2x3
	Fuzzy cost coefficients	[10]	Based on LP with interval coeff and simplex	2x3
	Generalized trapezoidal fuzzy cost	[13]	Approached method	3x3
	Multi-objective, possibility coefficient	[9]	Fuzzy programming	k=2 ; 3x4
	Multi-objective, interval parameters	[4]	Fuzzy programming	k=2 ; 2x3
	Multi-objective, fuzzy parameters	[13]	Approached method	k=2 ; 3x4
	Multi-objective, fuzzy parameters, min t	[2]	Parametric programming	k=6 ; 5x8
	Continuous	[12]	Continuous programming	40x40
3 indexes	Classical 3I		Extension potential method	
	Fuzzy, multi-objective	[11],[15]	Evolutionary algorithm	k=2; 2x2x2
		[14]	Improved genetic algorithm	k=3; 3x3x3
4 indexes	Classical 4I	[17]	Ninh's method (Extension potential method)	2x2x2x2
		[7],[8],[21]	Pham's method (Extension Ninh's method)	10x10x100x10
	Capacitated constraints	[5],[25],[26]	Extension of Ninh's method	Tops: 771 Ridges: 1500
	Integer and mix variables	[19]	Extension Pham's method	10x10x100x10
	Industry application orientation	[8],[19]	Extension Pham's method	10x10x100x10
	Interval cost parameter	[20]	Extension particles swarm optimization PSO	10x10x100x10
Multi-indexes	Classical n indexes	[16],[17]	Ninh's method (extension potential method)	2x2x2x2
	Decomposable cost parameter	[22]	Approached method	3x3x4

In 2011, Pham proposed an exact method to solve all the untreated cases, where the 4ITP is degenerated or not and the SEC is not verified or not[7,18,21]. The resolution algorithm was programmed in FORTRAN 95 language and tested with the large database: 100.000 variables[8]. Based on this model 4ITP with real variables, we also introduced an approached method to solve the 4ITP with integer and mix variables and developed a program in the same language[19]. This is the foundation for implementing our model in the mix transportation system as piece and as weight with the industrial database.

The table 1 is extracted in the previous classification and shows the results of authors. It presents the possible extensions, methods and illustrated sizes.

Thus, before our method is proposed in 2011, the research results on the four index transportation problem (4ITP) are relatively limited with only a solved case where the SEC is verified and the problem is not degenerated. What principle we propose for solving the remaining untreated cases where the 4ITP is degenerated and the SEC is not verified?

4. Resolution Method

We found a sufficient condition so that the 4ITP is degenerated. Thus, we treat the non-validation of the SEC under the theoretical angle and the appearance of degeneration under the practical angle. By coupling with Ninh's method[17], we present a general algorithm solving the 4ITP in all particular cases where the SEC is verified or not and the problem is degenerated or not.

4.1. Degeneration Condition

We elaborate and demonstrate the sufficient condition so that the 4ITP is degenerated with a constraint where the SEC is verified.

4.1.1. General Form

Theorem 2: If there are m_1 values $\alpha_i (0 < m_1 < m)$, n_1 values $\beta_j (0 < n_1 < n)$, p_1 values $\gamma_k (0 < p_1 < p)$, q_1 values $\delta_l (0 < q_1 < q)$ that:

sum of these α_i = sum of these β_j =

sum of these γ_k = sum of these δ_l

then, the 4ITP will be degenerated.

We permute m_1 equations whose right side are these α_i, n_1 equations whose right side are these β_j, p_1 equations whose right side are these γ_k, q_1 equations whose right side are these δ_l at the first lines in each group. Thus, in order to simplify but not to diminish the generality of the problem, we present the sufficient condition under the reduced form.

4.1.2. Reduced Form

Theorem 3: If there are numbers $m_1, n_1, p_1, q_1 (0 < m_1 < m, 0 < n_1 < n, 0 < p_1 < p, 0 < q_1 < q)$ that:

$$\sum_{i=1}^{m_1} \alpha_i = \sum_{j=1}^{n_1} \beta_j = \sum_{k=1}^{p_1} \gamma_k = \sum_{l=1}^{q_1} \delta_l \qquad (11)$$

then, the 4ITP will be degenerated.

Consequently, the general form can be transformed into the reduced form by the permutation of any equations in the same group and visa versa.

4.1.3. Demonstration

We demonstrate the theorem for the reduced form. Suppose that the constraint system (2...5) responds to the theorem (3), we demonstrate that this problem is degenerated. Thus, it is only necessary to find a degenerated solution in order to demonstrate the degenerated 4ITP, i.e. the number of positive variables of this solution is less than $(m+n+p+q-3)$.

For this purpose, we seek a method to transform the initial problem into two sub-problems whose system of constraints are independent linear equation systems.

We choose the variables x_{ijkl} to respond to the constraints: $\sum_{j=1}^{n_1} \sum_{k=1}^{p_1} \sum_{l=1}^{q_1} x_{ijkl} = \alpha_i \quad i = 1...m_1$

This means that m_1 first origin nodes provide p_1 first goods types to n_1 first destination nodes by q_1 first vehicle types.

We choose the variables x_{ijkl} to respond to the constraint: $\sum_{j=n_1+1}^{n} \sum_{k=p_1+1}^{p} \sum_{l=q_1+1}^{q} x_{ijkl} = \alpha_i \quad i = (m_1+1)...m$

This means that $(m - m_1)$ last origin nodes provide $(p - p_1)$ last goods types to $(n - n_1)$ last destination nodes by $(q - q_1)$ last vehicle types.

In order to it, we represent the constraints (2…5) as follow:

$$\begin{cases} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} x_{ijkl} = \alpha_i & i = 1 \dots m_1 \\ \sum_{i=1}^{m} \sum_{k=1}^{p} \sum_{l=1}^{q} x_{ijkl} = \beta_j & j = 1 \dots n_1 \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{q} x_{ijkl} = \gamma_k & k = 1 \dots p_1 \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} x_{ijkl} = \delta_l & l = 1 \dots q_1 \end{cases} \qquad (12)$$

$$\begin{cases} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} x_{ijkl} = \alpha_i & i = (m_1+1) \dots m \\ \sum_{i=1}^{m} \sum_{k=1}^{p} \sum_{l=1}^{q} x_{ijkl} = \beta_j & j = (n_1+1) \dots n \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{q} x_{ijkl} = \gamma_k & k = (p_1+1) \dots p \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} x_{ijkl} = \delta_l & l = (q_1+1) \dots q \end{cases} \qquad (13)$$

We study two independent linear equation systems:

$$\begin{cases} \sum_{j=1}^{n_1} \sum_{k=1}^{p_1} \sum_{l=1}^{q_1} x_{ijkl} = \alpha_i & i = 1 \dots m_1 \\ \sum_{i=1}^{m_1} \sum_{k=1}^{p_1} \sum_{l=1}^{q_1} x_{ijkl} = \beta_j & j = 1 \dots n_1 \\ \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} \sum_{l=1}^{q_1} x_{ijkl} = \gamma_k & k = 1 \dots p_1 \\ \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} \sum_{k=1}^{p_1} x_{ijkl} = \delta_l & l = 1 \dots q_1 \end{cases} \qquad (14)$$

$$\begin{cases} \sum_{j=n_1+1}^{n} \sum_{k=p_1+1}^{p} \sum_{l=q_1+1}^{q} x_{ijkl} = \alpha_i & i = (m_1+1) \dots m \\ \sum_{i=m_1+1}^{m} \sum_{k=p_1+1}^{p} \sum_{l=q_1+1}^{q} x_{ijkl} = \beta_j & j = (n_1+1) \dots n \\ \sum_{i=m_1+1}^{m} \sum_{j=n_1+1}^{n} \sum_{l=q_1+1}^{q} x_{ijkl} = \gamma_k & k = (p_1+1) \dots p \\ \sum_{i=m_1+1}^{m} \sum_{j=n_1+1}^{n} \sum_{k=p_1+1}^{p} x_{ijkl} = \delta_l & l = (q_1+1) \dots q \end{cases} \qquad (15)$$

The problem (1...5) can be divided into two sub-problems as follows:

- For the sub-problem 1:

Determine the variables x_{ijkl} $(i = 1 \dots m_1, j = 1 \dots n_1, k = 1 \dots p_1, l = 1 \dots q_1)$ for $minL(X) = \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} \sum_{k=1}^{k_1} \sum_{l=1}^{q_1} x_{ijkl}$

subject to:

$$\begin{cases}\sum_{j=1}^{n_1}\sum_{k=1}^{p_1}\sum_{l=1}^{q_1} x_{ijkl} = \alpha_i \quad i = 1 \dots m_1 \\ \sum_{i=1}^{m_1}\sum_{k=1}^{p_1}\sum_{l=1}^{q_1} x_{ijkl} = \beta_j \quad j = 1 \dots n_1 \\ \sum_{i=1}^{m_1}\sum_{j=1}^{n_1}\sum_{l=1}^{q_1} x_{ijkl} = \gamma_k \quad k = 1 \dots p_1 \\ \sum_{i=1}^{m_1}\sum_{j=1}^{n_1}\sum_{k=1}^{p_1} x_{ijkl} = \delta_l \quad l = 1 \dots q_1\end{cases}$$

→ The constraints system (14) of this problem responds to its SEC (the relationship 10). Therefore, we determine the maximum $(m_1 + n_1 + p_1 + q_1 - 3)$ of positive variables. (condition 16)

→ We determine the null variables that belong to the constraint system (12) but not belong to the constraint system (14). (condition 17)

→ Thus, all of positive and null variables responding to the condition (16) and null variables responding to the condition (17) are the roots of the constraint system (12). The number of positive variables is maximum $(m_1 + n_1 + p_1 + q1-3.$

• For the sub-problem 2:

Determine les variables x_{ijkl} $(i = m_1 + 1 \dots m, j = n_1 + 1 \dots n, k = p_1 + 1 \dots p, l = q1+1 \dots q$ for:

$$minL(X) = \sum_{m_1+1}^{m}\sum_{n_1+1}^{n}\sum_{p_1+1}^{p}\sum_{q_1+1}^{q} x_{ijkl}$$

subject to:

$$\begin{cases}\sum_{j=n_1+1}^{n}\sum_{k=p_1+1}^{p}\sum_{l=q_1+1}^{q} x_{ijkl} = \alpha_i \quad i = (m_1 + 1) \dots m \\ \sum_{i=m_1+1}^{m}\sum_{k=p_1+1}^{p}\sum_{l=q_1+1}^{q} x_{ijkl} = \beta_j \quad j = (n_1 + 1) \dots n \\ \sum_{i=m_1+1}^{m}\sum_{j=n_1+1}^{n}\sum_{l=q_1+1}^{q} x_{ijkl} = \gamma_k \quad k = (p_1 + 1) \dots p \\ \sum_{i=m_1+1}^{m}\sum_{j=n_1+1}^{n}\sum_{k=p_1+1}^{p} x_{ijkl} = \delta_l \quad l = (q_1 + 1) \dots q\end{cases}$$

→ From (10) and (11), we have:

$$\sum_{i=m_1+1}^{m} \alpha_i = \sum_{j=n_1+1}^{n} \beta_j = \sum_{k=p_1+1}^{p} \gamma_k = \sum_{l=q_1+1}^{q} \delta_l \quad (18)$$

→ The constraints system (15) of this problem responds to its SEC (the relationship 18). Therefore, we determine the maximum $((m - m_1) + (n - n_1) + (p - p_1) + (q - q_1) - 3$ of positive variables (condition 19).

→ We determine the null variables that belong to the constraint system (13) but not belong to the constraint system (15) (condition 20).

→ Thus, all of positive and null variables responding to the condition (19) and null variables responding to the condition (20) are the roots of the constraint system (13). The number of positive variables is maximum $((m - m_1) + n-n1+p-p1+q-q1-3.$

• For the initial problem:

→ The set of variables $\{x_{ijkl} \geq 0\}$ of the systems (12) and (13) being determined above-mentioned responds to the constraints (2…5). So it is a solution for the 4ITP.

→ The number of positives variables $(x_{ijkl} > 0)$ of this solution is at most: $(m_1 + n_1 + p_1 + q_1 - 3) + ((m - m_1) + (n - n_1) + (p - p_1) + (q - q_1) - 3) = (m + n + p + q - 6) < (m + n + p + q - 3)$.

This solution is degenerated. Therefore, the initial problem is degenerated.

4.2. Degeneration, General Resolution Principle

Based on the result of the sufficient condition so that the 4ITP is degenerated, we notice that the cause of degeneration of 4ITP can be defined as the transportation problem divided into two independent sub-problems and each sub-problem responds to the SEC. In order to eliminate the degeneration, we must modify the initial problem so that it cannot be divided into two independent sub-problems, i.e. each sub-problem does not respond to SEC. For this purpose, we add parameters at right side in the constraints of this problem so that each sub problem does not respond to the SEC but the initial transportation problem always responds to the SEC.

Thus, the main idea of the resolution method is that we must add to the sub problem 1 and subtract from the sub problem 2 the same ε. We will at once add ε to the right side of one constraint equation that belongs to the group i of sub problem 1 and subtract ε from the right side of one constraint equation that belongs to the group i of sub problem 2. This means that the SEC is not cancelled. Thus, we demonstrate that after modifying the initial transportation problem, we obtain a new problem that can be solved and un-degenerated.

However, in fact the research of numbers m_1, n_1, p_1, q_1 is relatively difficult even if we know they exist. Therefore, we propose a practical algorithm to recognize a degenerated problem and eliminate the degeneration for two cases:

→ Case 1: The degeneration appears in the elaboration process of the first solution

→ Case 2: The degeneration appears in the process of solution improvement to obtain the better solution.

4.3. Degeneration, Resolution Method (Case 1)

4.3.1. General Principle

For the case 1, during the process of determining a variable x_{ijkl}>0 of the first solution, if we lose more than one equation in the constraints, the solution is degenerated.

Supposing that we lose two equations in the constraints, there will be two equations of which value at right side becomes zero. To remove the degeneration, we keep the value zero on the right side of one equation and for the value zero on the right side of the other equation, we add ε (ε>0 and infinitesimal) and we simultaneously subtract ε from the right side of one equation that belongs to this group to respond to the SEC.

We have four particular large groups for determining a first positive variable $x_{i_0 j_0 k_0 l_0}$ of the first solution. Among them, the first case illustrates an un-degenerated solution, the other cases illustrate the degeneration.

$x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \alpha_{i_0}$
$x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \alpha_{i_0} = \beta_{j_0}$
$x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \alpha_{i_0} = \beta_{j_0} = \gamma_{k_0}$
$x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \alpha_{i_0} = \beta_{j_0} = \gamma_{k_0} = \delta_{l_0}$

4.3.2. Method

Suppose that we determine the variable $x_{i_0j_0k_0l_0}$.

This variable represents only in the equations whose right side is $\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}$. The equations are:

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{i_0jkl} = \alpha_{i_0} \quad (21)$$
$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ij_0kl} = \beta_{j_0} \quad (22)$$
$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijk_0l} = \gamma_{k_0} \quad (23)$$
$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl_0} = \delta_{l_0} \quad (24)$$

We have $x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}]$

4.3.2.1. Case 1

$x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \alpha_{i_0}$

The equation (21) is written:

$$x_{i_0j_0k_0l_0} + \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{i_0jkl} = \alpha_{i_0} \quad (25)$$

$(jkl) \neq (j_0k_0l_0)$

Car $x_{i_0j_0k_0l_0} = \alpha_{i_0}$ the equation (25) becomes:

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{i_0jkl} = 0 \quad (26)$$

$(jkl) \neq (j_0k_0l_0)$

$x_{i_0jkl} \geq 0\ \forall(jkl)$, we have $x_{i_0jkl} = 0\ \forall(jkl) \neq (j_0k_0l_0)$.

Thus, we determine the variables $x_{i_0j_0k_0l_0} = \alpha_{i_0}$ and $x_{i_0jkl} = 0\ \forall(jkl) \neq (j_0k_0l_0)$, and the constraint system (2...5) loses an equation.

For the equation (22), it is written:

$$x_{i_0j_0k_0l_0} + \sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ij_0kl} = \beta_{j_0} \quad (27)$$
$$(ikl) \neq (i_0k_0l_0)$$

Because $x_{i_0j_0k_0l_0} = \alpha_{i_0}$ and $x_{i_0jkl} = 0\ \forall(jkl) \neq (j_0k_0l_0)$,

the equation (27) is written:

$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ij_0kl} = \beta_{j_0} - \alpha_{i_0} \quad (28)$$

Note $\beta_{j_0}^{(1)} = \beta_{j_0} - \alpha_{i_0}$, the equation (28) is written:

$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ij_0kl} = \beta_{j_0}^{(1)} \quad \beta_{j_0}^{(1)} = \beta_{j_0} - \alpha_{i_0} \quad (29)$$

Similarly, the equation (23) is written:

$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijk_0l} = \gamma_{k_0}^{(1)} \quad \gamma_{k_0}^{(1)} = \gamma_{k_0} - \alpha_{i_0} \quad (30)$$

The equation (24) becomes:

$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl_0} = \delta_{l_0}^{(1)} \quad \delta_{l_0}^{(1)} = \delta_{l_0} - \alpha_{i_0} \quad (31)$$

The right sides of the other equations in the constraint system (2…5) does not change. The constraint system (2...5) becomes:

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i^{(1)} \quad \alpha_i^{(1)} = \alpha_i \quad i \in [1,i_0[\cup]i_0,m]$$
$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ij_0kl} = \beta_{j_0}^{(1)} \quad \beta_{j_0}^{(1)} = \beta_{j_0} - \alpha_{i_0}$$
$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j^{(1)} \quad \beta_j^{(1)} = \beta_j \quad j \in [1,j_0[\cup]j_0,n]$$
$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijk_0l} = \gamma_{k_0}^{(1)} \quad \gamma_{k_0}^{(1)} = \gamma_{k_0} - \alpha_{i_0}$$
$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k^{(1)}$$
$$\gamma_k^{(1)} = \gamma_k \quad k \in [1,k_0[\cup]k_0,p]$$
$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl_0} = \delta_{l_0}^{(1)} \quad \delta_{l_0}^{(1)} = \delta_{l_0} - \alpha_{i_0}$$
$$\sum_{\substack{i=1\\ i\neq i_0}}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l^{(1)} \quad \delta_l^{(1)} = \delta_l \quad l \in [1,l_0[\cup]l_0,q]$$

We see that after determining the variables $x_{i_0j_0k_0l_0} = \alpha_{i_0}$ and $x_{i_0jkl} = 0\ \forall(jkl) \neq (j_0k_0l_0)$, the new constraint system always responds to the SEC:

$$\sum_{\substack{i=1\\ i\neq i_0}}^{m} \alpha_i^{(1)} = \sum_{i=1}^{m}\alpha_i - \alpha_{i_0} = \sum_{j=1}^{n}\beta_j - \alpha_{i_0}$$
$$= \sum_{\substack{j=1\\ j\neq j_0}}^{n}\beta_j + \beta_{j_0} - \alpha_{i_0}$$
$$= \sum_{\substack{j=1\\ j\neq j_0}}^{n}\beta_j^{(1)} + \beta_{j_0}^{(1)} = \sum_{j=1}^{n}\beta_j^{(1)}$$
$$= \cdots = \sum_{k=1}^{p}\gamma_k^{(1)} = \cdots = \sum_{l=1}^{q}\delta_l^{(1)}$$

We continues determining the second variable, third variables,…until when we obtain the first solution.

We similarly execute for the other cases:

If $x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \beta_{j_0}$
or $x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \gamma_{k_0}$
or $x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \delta_{l_0}$

4.3.2.2. Case 2

$x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \alpha_{i_0} = \beta_{j_0}$

Similarly to case 1, we have:

$\alpha_{i_0}^{(1)} = 0$ and $\beta_{j_0}^{(1)} = \beta_{j_0} - \alpha_{i_0} = 0$

Thus, the constraint system (2…5) loses two equations, the solution obtained will be degenerated.

To eliminate the degeneration, we keep: $\alpha_{i_0}^{(1)} = 0$

and we give: $\beta_{j_0}^{(1)} = \varepsilon$

Thus, we must replace β_{j_0} with $\beta_{j_0}^{'} = \beta_{j_0} + \varepsilon$ and β_j with $\beta_j^{'} = \beta_j - \varepsilon$ (with one $j \in [1,j_0[\cup]j_0,n]$) so that the constraint system always responds to the SEC.

We continue determining the second positive variable.

4.3.2.3. Case 3

$x_{i_0j_0k_0l_0} = min[\alpha_{i_0},\beta_{j_0},\gamma_{k_0},\delta_{l_0}] = \alpha_{i_0} = \beta_{j_0} = \gamma_{k_0}$

Similarly to case 1, we have:

$\alpha_{i_0}^{(1)} = 0$; $\beta_{j_0}^{(1)} = 0$ and $\gamma_{k_0}^{(1)} = 0$

Thus, the constraint system (2…5) loses three equations, the solution obtained will be degenerated.

To eliminate the degeneration, we keep: $\alpha_{i_0}^{(1)} = 0$

and we give: $\beta_{j_0}^{(1)} = \varepsilon_1$

$\gamma_{k_0}^{(1)} = \varepsilon_2$

Thus, we must replace: β_{j_0} with $\beta_{j_0}^{'} = \beta_{j_0} + \varepsilon_1$

γ_{k_0} with $\gamma_{k_0}^{'} = \gamma_{k_0} + \varepsilon_2$

In order that the constraint system always responds to the SEC, we must replace:

β_j with $\beta_j^{'} = \beta_j - \varepsilon_1$ (with one $j \in [1, j_0[\cup]j_0, n]$)
γ_k with $\gamma_k^{'} = \gamma_k - \varepsilon_2$ (with one $k \in [1, k_0[\cup]k_0, p]$)

We continue determining the second positive variable.

4.3.2.4. Case 4

$x_{i_0 j_0 k_0 l_0} = min[\alpha_{i_0}, \beta_{j_0}, \gamma_{k_0}, \delta_{l_0}] = \alpha_{i_0} = \beta_{j_0} = \gamma_{k_0} = \delta_{l_0}$

Similarly to case 1, we have:

$\alpha_{i_0}^{(1)} = 0$; $\beta_{j_0}^{(1)} = 0$; $\gamma_{k_0}^{(1)} = 0$ and $\delta_{l_0}^{(1)} = 0$

Thus, the constraint system (2…5) loses four equations, the solution obtained will be degenerated.

To eliminate the degeneration, we keep: $\alpha_{i_0}^{(1)} = 0$

and we give: $\beta_{j_0}^{(1)} = \varepsilon_1$

$\gamma_{k_0}^{(1)} = \varepsilon_2$

$\delta_{l_0}^{(1)} = \varepsilon_3$

Thus, we must replace: β_{j_0} with $\beta_{j_0}^{'} = \beta_{j_0} + \varepsilon_1$
γ_{k_0} with $\gamma_{k_0}^{'} = \gamma_{k_0} + \varepsilon_2$
δ_{l_0} with $\delta_{l_0}^{'} = \delta_{l_0} + \varepsilon_3$

In order that the constraint systemalways responds to SEC, we must replace:

β_j with $\beta_j^{'} = \beta_j - \varepsilon_1$ (with one $j \in [1, j_0[\cup]j_0, n]$)
γ_k with $\gamma_k^{'} = \gamma_k - \varepsilon_2$ (with one $k \in [1, k_0[\cup]k_0, p]$)
δ_l with $\delta_l^{'} = \delta_l - \varepsilon_3$ (with one $l \in [1, l_0[\cup]l_0, q]$)

We continue determining the second positive variable.

Thus, once we determine a positive variable, the constraint system (2…5) loses an equation and continue...., finally, we obtain a solution that consists of ($m+n+p+q-3$) positive variables. This is the solution un-degenerated.

4.4. Degeneration, Resolution Method (Case 2)

4.4.1. General Principle

The transformation process of an un-optimal solution for searching a better solution is presented as follows:

→ Determine the parameters: $u_i^I, u_j^J, u_k^K, u_l^L, \Delta_{ijkl}$

$u_i^I + u_j^J + u_k^K + u_l^L = c_{ijkl}$ with all $x_{ijkl} > 0$

$\Delta_{ijkl} = c_{ijkl} - (u_i^I + u_j^J + u_k^K + u_l^L)$ with all $x_{ijkl} = 0$

If $\forall \Delta_{ijkl} \geq 0$, the solution is optimal.

If $\exists \Delta_{ijkl} < 0$, the solution is not optimal. We must execute the next steps:

→ Determine $\min \Delta_{ijkl} < 0$,

Suppose that $\min \Delta_{ijkl} < 0 = \Delta_{i^*j^*k^*l^*}$

→ Determine the parameters t_{ijkl} corresponding to the variables $x_{ijkl} > 0$ for:

$\sum_{(ijkl):x_{ijkl}>0} t_{ijkl} . P_{ijkl} + P_{i^*j^*k^*l^*} = 0$

P_{ijkl} is the coefficient vector of variables x_{ijkl} in the constraint system (2…5). It is the column vector with four digits 1 at the line i, $m+j$, $m+n+k$, $m+n+p+l$, the others are zero.

→ Determine: $\theta = min \frac{x_{ijkl}}{|t_{ijkl}|}$ with $t_{ijkl} < 0$

and suppose that $\theta = \frac{x_{i^{**}j^{**}k^{**}l^{**}}}{|t_{i^{**}j^{**}k^{**}l^{**}}|}$

→ Elaborate the new solution:

$x_{ijkl}^{'} = x_{ijkl} + \theta . t_{ijkl}$ with $x_{ijkl} > 0$

$x_{i^*j^*k^*l^*}^{'} = \theta$

$x_{ijkl}^{'} = x_{ijkl}$ with $x_{ijkl} = 0$ et $(ijkl) \neq (i^*j^*k^*l^*)$

And we see that:

$$x_{i^{**}j^{**}k^{**}l^{**}}^{'} = x_{i^{**}j^{**}k^{**}l^{**}} + \theta . t_{i^{**}j^{**}k^{**}l^{**}} = x_{i^{**}j^{**}k^{**}l^{**}} + \frac{x_{i^{**}j^{**}k^{**}l^{**}}}{|t_{i^{**}j^{**}k^{**}l^{**}}|} t_{i^{**}j^{**}k^{**}l^{**}} = 0$$

The new solution X' has maximum ($m+n+p+q-3$) positive variables.

Supposing that during this transformation process, we find a solution with two variables $x_{ijkl}^{'}$ that are determined from positive variables ($x_{ijkl}>0$) becoming zero. This means that the number of positive variables x_{ijkl} of this solution is less than ($m+n+p+q-3$). It is a degenerated solution.

To eliminate the degeneration, we add ε ($\varepsilon>0$ and infinitesimal) in the second variable zero. Thus, the SEC always fits but the amount of four members α_i, β_j, γ_k, δ_l is increased by ε. In mathematical term, the new transportation problem is different from the initial one because the total amount of transported goods increases more ε.

4.4.2. Method

For example, suppose that $x_{i^+j^+k^+l^+}^{'}$ is the second variable becoming zero in the solution improvement process.

→ We give $x_{i^+j^+k^+l^+}^{'} = \varepsilon$ ($\varepsilon>0$ and infinitesimal)

→ Because the variable $x_{i^+j^+k^+l^+}^{'}$ represents in the equations whose right side is $\alpha_{i^+}, \beta_{j^+}, \gamma_{k^+}, \delta_{l^+}$, we have:

The right side α_{i^+} must become $\alpha_{i^+} + \varepsilon$
The right side β_{j^+} must become $\beta_{j^+} + \varepsilon$
The right side γ_{k^+} must become $\gamma_{k^+} + \varepsilon$
The right side δ_{l^+} must become $\delta_{l^+} + \varepsilon$

In the case there are more than two variables $x_{ijkl}=0$, we add ε_1, ε_2… to these variables based on the above principle to eliminate the degeneration. After obtaining the optimal solution, we replace all these ε with zero and we obtain the result for the initial 4ITP.

4.5. SEC not Verified, Resolution Method

4.5.1. General Principle

If the transportation problem does not respond to the SEC, it is impossible to solve a transportation problem. Thus, we modify this problem to create a new problem coming up to our expectation.

The modification principle is to add fictitious nodes and parameters (fictitious node $m+1$; fictitious node $n+1$; fictitious parameter $p+1$; fictitious parameter $q+1$) to create equality between the four amounts.

The transportation cost is considered as null: from a fictitious node to all destination nodes, from all origin nodes to a fictitious destination node, to transport the fictitious goods or by using a fictitious vehicle.

Note that: $A = max(\sum_{i=1}^{m} \alpha_i, \sum_{j=1}^{n} \beta_j, \sum_{k=1}^{p} \gamma_k, \sum_{l=1}^{q} \delta_l)$

→ For the index i:

We add the node m+1 and note: $\alpha_{m+1} = A - \sum_{i=1}^{m} \alpha_i$

→ For the index j:

We add the node n+1 and note: $\beta_{n+1} = A - \sum_{j=1}^{n} \beta_j$

→ For the index k:

We add the parameter p+1 and note: $\gamma_{p+1} = A - \sum_{k=1}^{p} \gamma_k$

→ For the index l:

We add the parameter q+1 and note: $\delta_{q+1} = A - \sum_{l=1}^{q} \delta_l$

4.5.2. Method

Based on this principle, we elaborate the new problem according to three main particular cases.

4.5.2.1. Case 1: An Amount in the Relationship (10) is Superior to Other

E.g. $\sum_{i=1}^{m} \alpha_i > max(\sum_{j=1}^{n} \beta_j, \sum_{k=1}^{p} \gamma_k, \sum_{l=1}^{q} \delta_l)$

Note $A = max(\sum_{i=1}^{m} \alpha_i, \sum_{j=1}^{n} \beta_j, \sum_{k=1}^{p} \gamma_k, \sum_{l=1}^{q} \delta_l)$

We have $A = \sum_{i=1}^{m} \alpha_i$

$$\alpha_{m+1} = A - \sum_{i=1}^{m} \alpha_i = 0$$

$$\beta_{n+1} = A - \sum_{j=1}^{n} \beta_j > 0$$

$$\gamma_{p+1} = A - \sum_{k=1}^{p} \gamma_k > 0$$

$$\delta_{q+1} = A - \sum_{l=1}^{q} \delta_l > 0$$

The initial problem becomes:

Determine the variables $x_{ijkl} \geq 0$ i=1...m, j=1...(n+1), k=1...(p+1), l=1...(q+1) for

$$minL(X) = \sum_{i=1}^{m} \sum_{j=1}^{n+1} \sum_{k=1}^{p+1} \sum_{l=1}^{q+1} x_{ijkl} . c_{ijkl}$$

subject to:

$\sum_{j=1}^{n+1} \sum_{k=1}^{p+1} \sum_{l=1}^{q+1} x_{ijkl} = \alpha_i$ i=1...m

$\sum_{i=1}^{m} \sum_{k=1}^{p+1} \sum_{l=1}^{q+1} x_{ijkl} = \beta_j$ j=1...(n+1)

$\sum_{i=1}^{m} \sum_{j=1}^{n+1} \sum_{l=1}^{q+1} x_{ijkl} = \gamma_k$ k=1...(p+1)

$\sum_{i=1}^{m} \sum_{j=1}^{n+1} \sum_{k=1}^{p+1} x_{ijkl} = \delta_l$ l=1...(q+1)

The SEC: $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n+1} \beta_j = \sum_{k=1}^{p+1} \gamma_k = \sum_{l=1}^{q+1} \delta_l = A$

So the SEC is verified. The new problem is solvable.

4.5.2.2. Case 2: Two Amounts in the Relationship (10) are Equal and Superior to Other

E.g. $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j > max(\sum_{k=1}^{p} \gamma_k, \sum_{l=1}^{q} \delta_l)$

Note $A = max(\sum_{i=1}^{m} \alpha_i, \sum_{j=1}^{n} \beta_j, \sum_{k=1}^{p} \gamma_k, \sum_{l=1}^{q} \delta_l)$

We have $A = \sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j$

$$\alpha_{m+1} = A - \sum_{i=1}^{m} \alpha_i = 0$$

$$\beta_{n+1} = A - \sum_{j=1}^{n} \beta_j = 0$$

$$\gamma_{p+1} = A - \sum_{k=1}^{p} \gamma_k > 0$$

$$\delta_{q+1} = A - \sum_{l=1}^{q} \delta_l > 0$$

The initial problem becomes:

Determine the variables $x_{ijkl} \geq 0$ i=1...m, j=1...n, k=1...(p+1), l=1...(q+1) for

$$minL(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p+1} \sum_{l=1}^{q+1} x_{ijkl} . c_{ijkl}$$

subject to:

$\sum_{j=1}^{n} \sum_{k=1}^{p+1} \sum_{l=1}^{q+1} x_{ijkl} = \alpha_i$ i=1...m

$\sum_{i=1}^{m} \sum_{k=1}^{p+1} \sum_{l=1}^{q+1} x_{ijkl} = \beta_j$ j=1...n

$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{q+1} x_{ijkl} = \gamma_k$ k=1...(p+1)

$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p+1} x_{ijkl} = \delta_l$ l=1...(q+1)

The SEC: $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{p+1} \gamma_k = \sum_{l=1}^{q+1} \delta_l = A$

So the SEC is verified. The new problem is solvable.

4.5.2.3. Case 3: Three Amounts in the Relationship (10) are Equal and Superior to the Last

E.g. $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{p} \gamma_k > \sum_{l=1}^{q} \delta_l$

Note $A = max(\sum_{i=1}^{m} \alpha_i, \sum_{j=1}^{n} \beta_j, \sum_{k=1}^{p} \gamma_k, \sum_{l=1}^{q} \delta_l)$

We have $A = \sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{p} \gamma_k$

$$\alpha_{m+1} = A - \sum_{i=1}^{m} \alpha_i = 0$$

$$\beta_{n+1} = A - \sum_{j=1}^{n} \beta_j = 0$$

$$\gamma_{p+1} = A - \sum_{k=1}^{p} \gamma_k = 0$$

$$\delta_{q+1} = A - \sum_{l=1}^{q} \delta_l > 0$$

The initial problem becomes:

Determine the variables $x_{ijkl} \geq 0$ i=1...m, j=1...n, k=1...p, l=1...(q+1) for

$$minL(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q+1} x_{ijkl} . c_{ijkl}$$

subject to:

$\sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q+1} x_{ijkl} = \alpha_i$ i=1...m

$\sum_{i=1}^{m} \sum_{k=1}^{p} \sum_{l=1}^{q+1} x_{ijkl} = \beta_j$ j=1...n

$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{q+1} x_{ijkl} = \gamma_k$ k=1...p

$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} x_{ijkl} = \delta_l$ l=1...(q+1)

The SEC: $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{p} \gamma_k = \sum_{l=1}^{q+1} \delta_l = A$

So the SEC is verified.

The new problem is solvable.

4.5.3. Fictive Variable Elimination

4.5.3.1. Elimination Principle

We have four main groups of fictive variables corresponding to four fictive parameters: m+1, n+1, p+1, q+1.

x^*_{ijkl} is the variables of the found optimal solution. We give:

→ $x^*_{(m+1)jkl} = 0$ for all j, k, l if $\alpha_{m+1} > 0$. Contrary, $\alpha_{m+1} = 0 \Rightarrow$ there are not the variables $x_{(m+1)jkl} \Rightarrow$ this leads to having no variable $x^*_{(m+1)jkl}$

→ $x^*_{i(n+1)kl} = 0$ for all i, k, l if $\beta_{n+1} > 0$. Contrary, $\beta_{n+1} = 0 \Rightarrow$ there are not the variables $x_{i(n+1)kl} \Rightarrow$ no variables $x^*_{i(n+1)kl}$

→ $x^*_{ij(p+1)l} = 0$ for all i, j, l if $\gamma_{p+1} > 0$. Contrary, $\gamma_{p+1} = 0 \Rightarrow$ there are not the variables $x_{ij(p+1)l} \Rightarrow$ no variables $x^*_{ij(p+1)l}$

→ $x^*_{ijk(q+1)} = 0$ for all i, j, k if $\delta_{q+1} > 0$. Contrary, $\delta_{q+1} = 0 \Rightarrow$ there are not the variables $x_{ijk(q+1)} \Rightarrow$ no variables $x^*_{ijk(q+1)}$

4.5.3.2. Economical Signification

The origin node (m+1), the destination node (n+1), the goods type parameter (p+1) and the vehicle type parameter (q+1) are fictives and are added in the constraint system so that the new transportation problem responds to the SEC. Thus, if we use:

→ the goods quantity at the origin (m+1) to respond to the command of customer system, i.e. actually, we do not carry anything because this origin does not exist →the transportation cost for carrying the goods from this node must be equal to zero.

→ the destination node (n+1) does not actually exist, i.e. there is not any order at the node (n+1) → the transportation cost for shipping the goods to this node must be equal to zero.

→ the goods type (p+1) does not actually exist →the transportation cost for shipping this goods type must be equal to zero.

→ the vehicle type (q+1) does not actually exist → there is not the transportation cost corresponding to this vehicle type → the transportation cost is equal to zero.

In the found optimal solution, if there are the variables with the fictive indexes, suppose that $x_{(m+1)547} = 4$ tons, we must give $x_{(m+1)547} = 0$ to obtain the reel variables in the optimal solution of the initial problem. Theoretically, this elimination means that these four tons of goods type 4 are shipped from the origin node(m+1) to the destination node 5 by the truck type 7 but actually, the truck type 7 is under four tons loaded or this truck is not filled.

4.6. General Resolution Algorithm, Real Variables

There are two main cases:

→ If the 4ITP is not degenerated and the SEC is verified, we use Ninh's method[17]

→ If not, our method will be used to transform the problem in the first case.

The steps of our general algorithm solving the 4ITP in all particular cases are presented as follow:

4.6.1. Step 1(Test the SEC)

We test the SEC (the relationship 10)

→ If yes then go to the step 3

→ else go to the step 2

4.6.2. Step 2 (Transform the initial problem)

Search $max(\sum_{i=1}^{m}\alpha_i, \sum_{j=1}^{n}\beta_j, \sum_{k=1}^{p}\gamma_k, \sum_{l=1}^{q}\delta_l)$

Supposing that

$max\left(\sum_{i=1}^{m}\alpha_i, \sum_{j=1}^{n}\beta_j, \sum_{k=1}^{p}\gamma_k, \sum_{l=1}^{q}\delta_l\right) = \sum_{l=1}^{q}\delta_l = A$

Add a fictitious origin node (m+1) with the quantity is

$\alpha_{m+1} = A - \sum_{i=1}^{m}\alpha_i \Rightarrow \sum_{i=1}^{m+1}\alpha_i = A = \sum_{l=1}^{q}\delta_l$.

Give $c_{m+1,jkl} = 0, \forall j, \forall k, \forall l$.

Similarly, execute for any amount that is less than A to obtain the relationship $\sum\alpha_i = \sum\beta_j = \sum\gamma_k = \sum\delta_l$

Go to the step 3.

4.6.3. Step 3(Search the first solution)

4.6.3.1. Loop 1 (Search the first variable)

Determine $x_{i_0j_0k_0l_0} = min\,(\alpha_{i_0}, \beta_{j_0}, \gamma_{k_0}, \delta_{l_0})$

◆ If $\min(\alpha_{i_0}, \beta_{j_0}, \gamma_{k_0}, \delta_{l_0}) = \alpha_{i_0}$

Give $x_{i_0j_0k_0l_0} = \alpha_{i_0}$

Thus, $x_{i_0jkl} = 0$ with $j = 1 \dots n, k = 1 \dots p, l = 1 \dots q$, $(jkl) \neq (j_0k_0l_0)$

The right part in the constraints becomes:

$\alpha_i^{(1)} = \alpha_i$ with $i = 1 \dots m, i \neq i_0$ and $\alpha_{i_0}^{(1)} = 0$

$\beta_j^{(1)} = \beta_j$ with $j = 1 \dots n, j \neq j_0$ and $\beta_{j_0}^{(1)} = \beta_{j_0} - \alpha_{i_0}$

$\gamma_k^{(1)} = \gamma_k$ with $k = 1 \dots p, k \neq k_0$ and $\gamma_{k_0}^{(1)} = \gamma_{k_0} - \alpha_{i_0}$

$\delta_l^{(1)} = \delta_l$ with $l = 1 \dots q, l \neq l_0$ and $\delta_{l_0}^{(1)} = \delta_{l_0} - \alpha_{i_0}$

◆ If $\min(\alpha_{i_0}, \beta_{j_0}, \gamma_{k_0}, \delta_{l_0}) = \alpha_{i_0} = \beta_{j_0}$

Replace β_{j_0} with $\beta'_{j_0} = \beta_{j_0} + \varepsilon$, β_j with $\beta'_j = \beta_j - \varepsilon$ with one $j \in [1, j_0[\cup]j_0, n]$

◆ If $\min(\alpha_{i_0}, \beta_{j_0}, \gamma_{k_0}, \delta_{l_0}) = \alpha_{i_0} = \beta_{j_0} = \gamma_{k_0}$

Replace β_{j_0} with $\beta'_{j_0} = \beta_{j_0} + \varepsilon_1$, β_j with $\beta'_j = \beta_j - \varepsilon_1$ with one $j \in [1, j_0[\cup]j_0, n]$

Replace γ_{k_0} with $\gamma'_{k_0} = \gamma_{k_0} + \varepsilon_2$, γ_k with $\gamma'_k = \gamma_k - \varepsilon_2$ with one $k \in [1, k_0[\cup]k_0, p]$

◆ If $\min(\alpha_{i_0}, \beta_{j_0}, \gamma_{k_0}, \delta_{l_0}) = \alpha_{i_0} = \beta_{j_0} = \gamma_{k_0} = \delta_{l_0}$

Replace β_{j_0} with $\beta'_{j_0} = \beta_{j_0} + \varepsilon_1$, β_j with $\beta'_j = \beta_j - \varepsilon_1$ with one $j \in [1, j_0[\cup]j_0, n]$

Replace γ_{k_0} with $\gamma'_{k_0} = \gamma_{k_0} + \varepsilon_2$, γ_k with $\gamma'_k = \gamma_k - \varepsilon_2$ with one $k \in [1, k_0[\cup]k_0, p]$

Replace δ_{l_0} with $\delta'_{l_0} = \delta_{l_0} + \varepsilon_3$, δ_l with $\delta'_l = \delta_l - \varepsilon_3$ with one $l \in [1, l_0[\cup]l_0, q]$

Thus, with $i \neq i_0$ we have the relationship

$$\sum_{\substack{i=1\\i\neq i_0}}^{m}\alpha_i^{(1)} = \sum_{j=1}^{n}\beta_j^{(1)} = \sum_{k=1}^{p}\gamma_k^{(1)} = \sum_{l=1}^{q}\delta_l^{(1)} = A - \alpha_{i_0}$$

We similarly execute to the other cases.

Go to the loop 2 to determine the next variable.

4.6.3.2. Loop 2 (Search the next variable)

Determine $x_{i_1j_1k_1l_1} = \min(\alpha_{i_1}^{(1)}, \beta_{j_1}^{(1)}, \gamma_{k_1}^{(1)}, \delta_{l_1}^{(1)})$

Return to the loop 1.

When all the right sides of the constraint systems (2...5) are equal to zero, stop the loop. We obtain the first solution.

Go to the step 4.

Remarking that the variables $\{x_{m+1,jkl}$ with $\forall(jkl), x_{i,n+1,kl}$ $with\ \forall(ikl), x_{ij,p+1,l}$ with $\forall(ijl), x_{ijk,q+1}$ with $\forall(ijk)\}$ will be determined in the final sequence. .

4.6.4. Step 4 (Control the optimality of the solution X)

4.6.4.1. Determine the Potentials $u_i^I, u_j^J, u_k^K, u_l^L$

Determine the potentials $u_i^I, u_j^J, u_k^K, u_l^L$ $(i = 1 \dots m, j = 1 \dots n, k = 1 \dots p, l = 1 \dots q)$ for $u_i^I + u_j^J + u_k^K + u_l^L = c_{ijkl}$ with $\forall x_{ijkl} > 0$

4.6.4.2. Compute the Elements Δ_{ijkl}

$$\Delta_{ijkl} = c_{ijkl} - \left(u_i^I + u_j^J + u_k^K + u_l^L\right) \; with \; \forall x_{ijkl} = 0$$

$$\text{Test } \forall \Delta_{ijkl} \geq 0 \qquad (32)$$

→ If the condition (32) is satisfied, the solution X is optimal. Go to the step 6.

→ else the solution X is not optimal. Go to the step 5.

4.6.5. Step 5 (Modify the solution X to find the other better solution)

4.6.5.1. Determine the Element $\Delta_{i^*j^*k^*l^*}$

Determine the element $\Delta_{i^*j^*k^*l^*} = min\Delta_{ijkl} < 0$

If there are several Δ_{ijkl} that get the minimum value, give any Δ_{ijkl} of these Δ_{ijkl}.

Determine the element t_{ijkl} for

$\sum_{(ijkl):x_{ijkl}>0} t_{ijkl} \cdot P_{ijkl} + P_{i^*j^*k^*l^*} = 0$

The right part of this constraint is the vector 0.

P_{ijkl} is the coefficient vector of variables x_{ijkl} in the constraint. It is the column vector which has four digits 1 at the line i, m+j, m+n+k, m+n+p+l, the others are zero.

4.6.5.2. Elaborate a New Solution

Determine θ for $\theta = min \frac{x_{ijkl}}{|t_{ijkl}|}$ with $t_{ijkl} < 0$

$x'_{ijkl} = x_{ijkl} + \theta . t_{ijkl}$ with $x_{ijkl} > 0$

$x'_{i^*j^*k^*l^*} = \theta$

$x'_{ijkl} = x_{ijkl}$ with $x_{ijkl} = 0$ and $(ijkl) \neq (i^*j^*k^*l^*)$

We test, among these found variables x'_{ijkl} that are calculated from $x_{ijkl} > 0$, there is just one being zero.

→ If yes then the solution is un-degenerated.

→ else keep the value 0 of a variable and suppose that the value 0 of the others are respectively $\varepsilon_1, \varepsilon_2, \dots$

Return to the step 4.

4.6.6. Step 6 (Find the optimal solution of the initial problem)

→ Replace ε with zero in the found variables.

→ Remove the variables that correspond to the fictitious nodes added and the fictitious parameters in the step 2.

Finally, we obtain the optimal solution of the initial problem.

5. Results

5.1. 4ITP, Numerical Examples

Based on the resolution method in the exact angle, we give numerical examples on small artificial database. The aim of this section is to test the method before an implementation in industry. We examine three cases:

Case 1: SEC is satisfied and the degeneration appears in the first solution.

Case 2: SEC is satisfied and the degeneration appears on the solutions modification process.

Case 3: SEC is not satisfied.

In this simple example, the hypothesis is composed of 8 parameters, 16 unit costs and 16 variables for case 1 and 2; 7 parameters, 8 unit costs and 8 variables for case 3.

The table 2 shows the parameters of the origins, destinations, the types of goods and the types of trucks. The table 3 presents the unit transportation cost. Specially, in the case 3, because of the weakness of the origin 2, all unit transportation costs for goods that are left at the origin 2 are considered as null.

Table 2. Offer, demand, goods types and truck types

Unit: ton

	α			β			γ			δ		
	C1	C2	C3	C1	C2	C3	C1	C2	C3	C1	C2	C3
1	30	30	30	30	35	35	38	38	38	40	40	40
2	40	13	-	40	8	35	32	5	32	30	3	30

Table 3. Unit transportation cost

Unit: €/ton

				C1	C2	C3
Ori 1	Des1	Gds 1	Tru1	8	8	8
			Tru2	10	2	3
		Gds 2	Tru1	17	4	4
			Tru2	14	6	6
	Des2	Gds 1	Tru1	3	9	3
			Tru2	1	1	1
		Gds 2	Tru1	10	7	7
			Tru2	5	5	5
Ori 2	Des1	Gds 1	Tru1	7	7	-
			Tru2	9	5	-
		Gds 2	Tru1	16	3	-
			Tru2	13	4	-
	Des2	Gds 1	Tru1	8	8	-
			Tru2	5	5	-
		Gds 2	Tru1	2	2	-
			Tru2	4	4	-

By using our algorithm for the 4ITP, we obtained the intermediate results (table 4). For case 1, we remark that the first solution is degenerated. The degeneration appears in the intermediate solution S1 for case 2. According to the "database" of case 3, the total quantity of goods at origin 1 is not sufficient to fill all existing trucks and respond to the demand of all destinations. For the cases 1 and 2, the solution S2 verifies the optimal condition.

Table 4. Intermediate solutions

Solution Unit: ton; Cost Unit: €

	1[rt] Solution		S1		S2	
	Solution	Cost	Solution	Cost	Solution	Cost
C1	x_{1111}=30	428	x_{1111}=30	428-ε	x_{1111}=22+ε	348+9ε
	x_{2211}=8		x_{2111}=ε		x_{2111}=8	
	x_{2221}=2		x_{2211}=8-ε		x_{2221}=10-ε	
	x_{2222}=30		x_{2221}=2		x_{2222}=22+ε	
	degenerate		x_{2222}=30		x_{1212}=8-ε	
C2	x_{1111}=30	315	x_{1111}=27	285	x_{1111}=27	285+8ε
	x_{2111}=5		x_{2111}=8		x_{2111}=8	
	x_{2211}=3		x_{2221}=5		x_{2211}=ε	
	x_{2221}=2		x_{1212}=3		x_{2221}=5	
	x_{2222}=3		degenerate		x_{1212}=3	
C3	x_{1111}=30	240	x_{1111}=27	219	$x_{2111}=\frac{43}{2}$	70,5
	x_{2111}=5		x_{2111}=8		x_{2221}=5	
	x_{2211}=3		x_{2221}=5		$x_{1212}=\frac{33}{2}$	
	x_{2221}=2		x_{2222}=27		$x_{2222}=\frac{27}{2}$	
	x_{2222}=30		x_{1212}=3		$x_{1121}=\frac{27}{2}$	

Table 5. Intermediate and Optimal solutions

Solution Unit: ton; Cost Unit: €

	S3		S4		OS	
	Solution	**Cost**	**Solution**	**Cost**	**Solution**	**Cost**
C1	-	-	-	-	x_{1111}=22	348
	-		-		x_{2111}=8	
	-		-		x_{2221}=10	
	-		-		x_{2222}=22	
	-		-		x_{1212}=8	
C2	-	-	-	-	x_{1111}=27	285
	-		-		x_{2111}=8	
	-		-		x_{2221}=5	
	-		-		x_{1212}=3	
	-		-		-	
C3	x_{2111}=8	30	x_{2111}=8	30	x_{1212}=30	30
	x_{2221}=5		x_{2221}=5		-	
	x_{1212}=30		x_{1212}=30		-	
	x_{2121}=27		x_{2121}=27		-	
	degenerate		x_{2222}=ε		-	

The table 5 shows the results of optimal solutions. In the cases 1 and 2, we obtain the optimal solutions of the initial transportation problems, by replacing ε in these variables with zero, because there is ε in the obtained variables. In the case 3, we replace ε with zero and remove these variables to obtain the optimal solution of the initial problem because there are ε and the fictitious variables in the found solution.

5.2. 4ITP, Optimal Planning

Table 6. Optimal planning

Origin	Destination	References	Truck	Quantity (t)
1	1	3	2	3,714
1	1	6	1	6,285
2	1	2	3	5,857
2	1	6	2	4,142
3	2	1	3	9,571
3	2	2	2	0,142
3	2	3	1	13,285
3	2	4	2	17,000
4	1	5	1	5,428
4	1	6	3	4,571
4	2	1	4	10,428
4	2	5	4	9,571
4	2	6	3	9,999
Transportation cost: 68,6099€ Iteration number : 23 Execution time: 0,109375s				

The obtained results in the table 6 is an extract of an optimal planning with the hypothesis of products in bulk, based on the data of a very simple model with 4 origins, 2 destinations, 6 goods types, 4 truck types, 144 unit transportation costs and the SEC is verified.

5.3. 4ITP, Performances

Our algorithm (or Pham's method) is programmed in Fortran 95 language with Gfortran compiler running on a computer equipped with a CPU Intel Core 2 at 1,97Ghz. To test two methods (our method and simplex method), we use the real series which are not degenerated and they are divided into two groups: SEC is verified and SEC it not verified.

The table 7 presents some performances of our program with real variables:

→ In the case of SEC verified, both methods get the optimal value of the objective function: the minimum transportation cost. However, the execution time of Pham's method is fast enough to handle middle sized problems and large ones. It treats up to 100,000 variables in 2 seconds compared to 6.6 seconds for the simplex method.

→ In the case where the SEC is not satisfied, Pham's method always gets the optimal solution and the execution time is very fast. It treats up to 100,000 variables in 3.8 seconds while the simplex method stops immediately without solution.

We found the degeneration series and tested them by two methods: Pham's method and simplex method. The table 8 presents an extract of the execution results by both methods for the degeneration series with the condition of SEC verified. The simplex and its options in Cplex cannot avoid the degeneration for all cases in the 4ITP resolution field. The numerical results show that, for the case where the simplex method cannot find the optimal solution, Pham's method eliminated the degeneration and easily obtained the optimal solution.

Table 7. Execution time, comparison results

Files	SEC	m	n	p	q	Variable number	Cost		CPU-time (s)	
							Simplex method	Pham's Method	Simplex method	Pham's method
HG20	SEC satisfied	4	2	20	4	640	7654.9867	7654.9867	0.3876	0.0075
HG40		4	2	40	4	1280	10042.2389	10042.2389	0.8624	0.0059
HG55		4	5	55	7	7700	13942.7067	13942.7036	1.7564	0.6752
HG95		10	10	95	7	66500	18472.9956	18472.9956	3.5674	1.4100
HG100		10	10	100	10	100000	79354.9456	79354.9456	6.6285	2.0952
HG1	CES no satisfied	1	2	2	2	8	NO SOLUTION	30	NO SOLUTION	0.0000
HG20		4	2	20	4	640		7655.9746		0.0094
HG55		4	5	55	7	7700		13950.6734		0.7439
HG95		10	10	95	7	66500		4532.6856		2.3857
HG100		10	10	100	10	100000		667837.9878		3.8452

Hence, under the theoretical angle, our method has solved the 4ITP in all the cases and obtained the exact results. Under the angle of numerical calculation, the results are pertinent because the parameters ε as well as the calculation errors are eliminated

Table 8. Degeneration, comparison results

Problem size	Simplex Cplex	Pham's method	
		Cost	CPU-time (s)
4x7x40x7	NO SOLUTION	10042.703638	0.0060376
1x5x83x6		54982.9690714	5.6298564
1x4x66x4		36987.6429843	3.8376102

5.4. 4ITP, Extension on Integer and Mix Variables

Building on the foundation of resolution method for 4ITP, we treat the integer model by using the elements in FORTRAN 95 library. We also develop a program coupled with the main program of the initial problem.

The table 8 illustrates our method on the data of a simple model with $m = 2, n = 4, p = 3, q = 3$ and the SEC is not satisfied.

Table 9. Integer optimal solution

Origin	Destination	Product	Truck	Quantity (piece)
1	1	1	2	1785
1	1	1	1	1678
1	1	2	3	1003
1	2	3	1	1067
1	3	1	2	1453
2	3	3	3	1237
2	4	2	4	990
2	4	2	1	564
2	4	1	3	1026

Transportation cost: 378.0945€
Iteration number : 37
Execution time: 0,08763276s

Based on this obtained result, we illustrate this method in a fictitious industrial case. At the same time this enterprise uses the system as pieces and weight unit. The weight unit is especially justified in the packaging and transport that provide the total weight to carry. The piece number is primarily used for the need of production management. The table 10 presents an allocation planning based on the mix model with $m = 2, n = 4, p = 5, q = 4$ and the SEC is not satisfied.

Table 10. Allocation planning

Origin	Destination	Product	Truck	Q *ton*	Q *piece*
1	1	4	2	1.31	145
1	3	5	4	3.393	7
1	1	1	1	0.012	2
1	1	3	4	0.185	23
1	2	3	3	1.45	185
1	3	4	1	0.15	17
2	3	3	3	1.12	143
2	4	2	4	0.0835	334
2	2	3	1	0.096	12
2	3	1	3	0.387	77
2	4	2	2	0.019	76
2	4	5	4	1.29	3

Transportation cost: 451.8765
Iteration number: 21
Execution time: 0.34876129s

This result gives the available weight charge planning as follows:

Table 11. Remaining available planning

Unit: ton

Truck	Maxi load	Total weight	Remaining available weight
1	0.65	0.258	0.392
2	1.5	1.329	0.171
3	3	2.959	0.043
4	5	4.951	0.049

The obtained transportation cost by this model is 451,8765€. The charge rate of used trucks is favourable, 93.56%.

6. Implementation

In this section, we present work in progress. From the industrial database, we develop an algorithm to determine the input database for the model 4ITP. This is implementing in the logistics management system of Airbus's subcontractor in France.

6.1. Industrial Case Study

The Hitchcock's problem was elaborated by the request of the Aéropostale de nuit company and fully resolved in the 50's of last century. Currently, there are at least three companies in France using Hitchcock model in their logistics system. These are leading groups in the field of mass transport: Fedex, Aéropostale de nuit and Air-France cargo. These companies must deliver several types of goods and often use several vehicle types. Therefore, Fedex applies Hitchcock model by using an approximate model to unify all types of product in one common product type and all vehicle types in one common type.

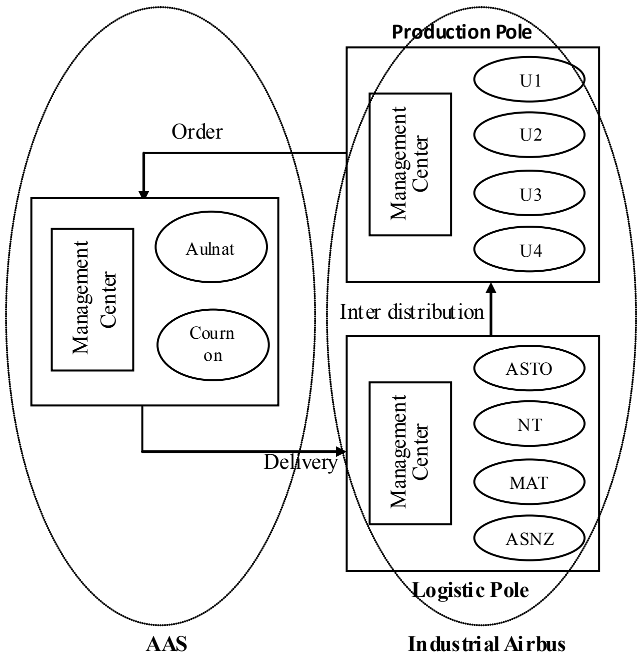

Figure 2. Order and delivery system (AAS and Airbus)

After obtaining the transportation planning and its cost, on one hand, the logistics system uses software to affect the vehicles according to the load transported to each delivery site. The choice of vehicles also depends on the type of transported goods. On the other hand, the accounting system uses two specific models: one takes the different product families to adjust the additional charges; the other adjusts the additional transportation cost by taking the type of vehicles. This is to obtain a final transportation cost. The optimization toward this direction, DHL and TNT, large transportation groups, use the three index transportation model with the third index representing the vehicle type. An open question is asked: these groups must always carry several types of goods and use several types of vehicles at the same time. Are there any models that can adapt to this situation? Can they optimize over the transportation cost?

There is the same question in an industrial case: Auvergne Aéronautique Slicom AAS Company and its customer Airbus Industry. In recent years, the strategy of Airbus in France is to entrust the responsibility for a part of manufacturing and assembly to several subcontractors. AAS, the leading supplier of elementary pieces and sub-assemblies for Airbus in France, has two production sites: Aulnat and Cournon d'Auvergne. AAS manufactures five large families for Airbus: cover, doors, sheet metal, pipes and machined pieces.

The Airbus's production pole delivers a weekly command planning to AAS, in which there are always three information columns to indicate: the quantity of pieces sent (piece unit), weight unit and total weight. The piece unit is primarily used for the production management. The use of the weight unit system is justified by the needs of logistics and packaging pole. Due to the package system has special boxes and small "containers", the total weight transported becomes a single "key" factor for the operations of logistics pole by ignoring the volume of pieces. Airbus has four warehouses located in different areas in France: Saint Eloi, Nantes, Méaulte and Saint Nazaire. Airbus uses the supply system "just-in-order" to avoid the storage on production sites. Therefore, on receiving the receipt of goods at four large warehouses from AAS, the manager of Airbus's central logistics pole redistributes parts based on the weekly production planning at different manufacturing sites.

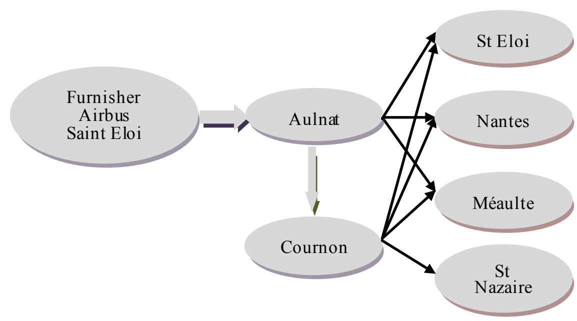

Figure 3. AAS transportation diagram

Thus, the AAS's logistics system should be developed a transportation planning based on important parameters: the delivery capacity of two production sites of AAS, the command of manufacturing network of Airbus, the storage capacity of four warehouses in Airbus's logistics network and the maximum load for trucks used. For the transport, AAS uses four truck types with the usual load: 0.65-ton, 1.5-ton, 3-ton and 5-ton.

→ 2 Origins: Aulnat and Cournon

→ 4 Destinations: Saint Eloi (ASTO), Nantes (NT), Méaulte (MAT), Saint Nazaire (ASNZ)

→ 5 Product types : Cover, Door, Sheet metal, Pipe, Machined Pieces

→ 4 Truck types: 0.65-ton, 1.5-ton, 3-ton, 5-ton

Currently, AAS uses software simulating different solutions to get an acceptable solution. Nevertheless, due to the increasing demand sent by Airbus and the increased complexity of certain parts, the problem is posed: how can the logistics system go directly to the solution with the smallest cost or an acceptable solution in an uncertain environment on the cost value? Is there a model fitting this question?

6.2. Industrial Database

→ AAS consists of *m* production sites O_i (i=1…*m*).

→ They manufacture *p* product types S_k (k=1…*p*).

→ The production capacity of site O_i for the product type S_k in period *t* is α_{ik} (ton).

→ At the site O_i, the quantity of product type S_k in inventory at the end of period *t*-1 is a_{ik} (ton).

→ At the site O_i, the average unit cost for storing a unit of product type S_k in period *t* is c_{ki} (€/ton).

→ This group consists of *q* vehicle types H_l (l=1…*q*) to deliver goods to its customers. The load of vehicle type H_l is δ_l (ton).

→ These products are delivered to the customer, Airbus Industry. It has a production network that consists of *r* manufacturing sites D_b (b=1…*r*).

→ The demand of site D_b for the product type S_k in period *t* is D_{bk} (ton).

→ The raw materials are stored in a logistics network of *n* warehouses D_j (j=1…*n*). The storage capacity of D_j in period *t* is β_j (ton).

→ The average unit cost for transporting a unit of goods type S_k from origin O_i to destination D_j by vehicle type H_l in period *t* is c_{ijkl} (€/ton).

6.3. Database of 4ITP model

The steps are performed to determine the parameters alpha, beta, gamma and delta of model as follow:

Step 1: Calculate the parameter

→ $\alpha_{ikA} = \alpha_{ik} + a_{ik}$

→ $\alpha_{iA} = \sum_{k=1}^{p} \alpha_{ikA}$

→ $\gamma_{kO} = \sum_{i=1}^{m}(\alpha_{ik} + a_{ik})$

→ $\gamma_{kD} = \sum_{b=1}^{r} D_{bk}$

Step 2: Determine γ_k and α_i

→ Case 1: If $\gamma_{kO} < \gamma_{kD}$, we give $\gamma_k = \gamma_{kO}$

$\alpha_i = \alpha_{iA} = \sum_{k=1}^{p} \alpha_{ikA}$. Go to the step 4.

→ Case 2: If $\gamma_{kO} = \gamma_{kD}$, we give $\gamma_k = \gamma_{kO} = \gamma_{kD}$

$\alpha_i = \alpha_{iA} = \sum_{k=1}^{p} \alpha_{ikA}$.

Go to the step 4.

→ Case 3: If $\gamma_{kO} > \gamma_{kD}$, we give $\gamma_{kO} = \gamma_{kD} = \gamma_k$

It means that we adjust γ_k of manufacturer to γ_k of customer. Go to the step 3.

Step 3: Adjustment

If we adjust γ_k of manufacturer to γ_k of customer, i.e. we reduce the total quantity of product type S_k at the manufacturer so that it is equal to the quantity of this type at the customer. This leads to reducing the total delivery quantity of manufacturer. So the following questions must be answered: which production site will stock the surplus quantity of product S_k and how many tons? The aim is to avoid delivering the goods on surplus to customers.

We adjust α_{ikA} to γ_k by using the ant colony algorithm, an extension of PSO method (particle swarm optimization).

→ The general principle of adjustment method is presented in the section 6.4.

→ The criterion of adjustment:

● minimize the total average storage cost at all production sites in period *t*.

● product type *k*: total offer = total demand.

After adjusting, we obtain α_i (It should obtain that $\alpha_i < \alpha_{iA}$). Go to the step 4.

Step 4: Determine the parameters of 4ITP model

→ Offer at origin: Delivery capacity = α_i

→ Demand at destination: Storage capacity (known in the industrial database) = β_j

→ Goods types: the quantity of product type $S_k = \gamma_k$

→ Vehicle types: load vehicle type H_l (known in the industrial database) = δ_l

6.4. Adjustment Method

The adjustment method is based on the principle of the ant movement to build new solutions and determine the vectors carrying adjustment variables.

Each ant builds a solution from the origin node by choosing isomorphic arcs to probabilistically un-fuzzy treat. In an arc, it typically selects the next arc by using wefts (line from origin to destination) or capacity direction arcs *C-i*. The "blind" choices are needed to explore new solutions and determine costs. In fact, the ant is blind with a probability of selected iteration, then *j* is chosen randomly among the *k* closest arcs that remain to be treated. With the probability 1-*pr*, we consider the chosen paths as well as direction. This last formula is classic and at the same time, it considers the visibilities of arcs and wefts weighted by chosen powers.

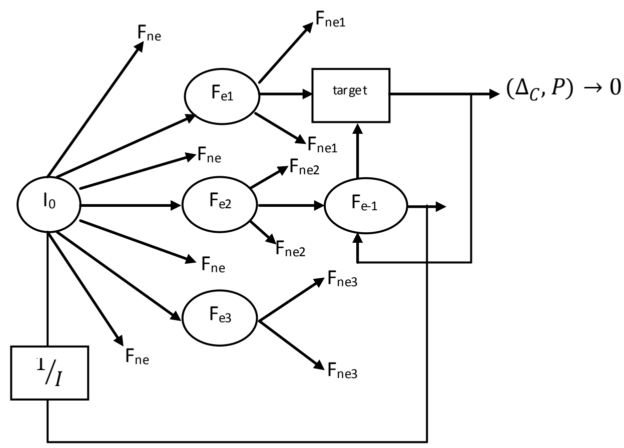

Figure 4. General principle of parameter adjustment method by ant colony algorithm

Each solution is calculated by an ant being under a local search (improvement process) with a type of characterized probability. The local search used is the proposed one in the genetic algorithm of Pham's method. Its purpose is to examine the following transformation in the incapacitated transportation problem.

According to the positive variation of costs (well-known problem is called the travelling problem), an edge is moved to be treated at the extremities as well as two consecutive edges to process and transform the wefts. By moving the extreme edges that respectively represent the origin and destination poles, we optionally change the crossing-direction if the treatment provides a larger gain. The local solution gives the cost-variance n-1.

This iteration runs until it finds the transformations that improve the solution causing the cost variation to be positive. As the genetic algorithm, the local search is essential because it allows to increase the research effort by "drawing" the solutions to the bottom. At the same time, a too high value of the set will train the longer calculation time without getting the significant improvement solutions and increasing the risks of blocking the algorithm in a minimum local. The research, however, must not be intensified enough with a too small value, the convergence will be slower.

The population of f ants has fe "elitist" and f-fe ants "not-elitist". The elitists ensure the convergence of algorithm, while the not-elitists explore the search space in order to maintain the solution diverse and prevent a premature convergence. In the table of f solutions, we replace the last solution of an ant-elitist with the new solution in only improvement case; and we always replace the last solution of an ant-not elitist with the new solution even if there is an improvement or degradation.

```
Program Code
Do the product Kroner-Johnson
  do i=1,n
    do j=1,p
      c(i,j) = 0.
      do k=1,m
         c(i+1),(J+1),(K+1),(l+1)
 1: DECA +(n),3,1
 n::DECA -(p),1,1
      end do
    end do
  end do
  do i=1,n
    print*,c(i,:)
  end do
   Import A
  open(unit=1,file="exo6.Edelta", &
        status="replace", form="unformatted", &
        action="write")
  write( 1 ) c
  close( unit = 1)
Adjustment Product
```

6.5. Result

From the real industrial database provided by AAS, we use the algorithm to determine the input data of 4ITP model. Then, we use the mixed model to determine the optimal transportation schedule. An extract of an optimal transportation planning for the week 12/2012 is presented in the table 12. This is the first results of the implementation process.

Table 12. Implementation result

Origin	Destination	Product Type	Truck Load	Q (ton)	Q (piece)
Cournon	St Eloi	Door	3	0.0037 5	15
Aulnat	St Eloi	Pipe	1.5	0.679	35
Cournon	St Nazaire	Door	5	0.809	327
Cournon	Nantes	Sheet metal	0.65	0.196	156
Cournon	Méaulte	Cover	3	1.006	93
Cournon	Méaulte	Pieces	5	2.628	4
Aulnat	St Eloi	Cover	0.65	0.015	3
Cournon	St Nazaire	Door	1.5	0.105	83
Aulnat	St Eloi	Sheet metal	5	0.150	19
Aulnat	Nantes	Sheet metal	3	1.78	201
Cournon	St Nazaire	Pieces	5	0.994	2
Cournon	Méaulte	Pipe	0.65	0.19	25
Transportation cost: 491.09267€ Execution time: 6.034568767s Charge rate of used trucks: 77.11%					

7. Conclusions

Our work has resulted from the resolution of all previously untreated cases of 4ITP under exact mathematical angle. In the domain of transportation problem, a particular case of linear programming, our method takes more advantages over the simplex. Indeed, the simplex stops immediately without solution in the case where the problem is degenerated and (or) the SEC is not verified. Our method (Pham's method) directly eliminates the degeneration in the same iteration, the execution time is fast (three times as fast as the simplex) and the obtained solution is optimal. Thus, we have proposed a mathematical tool for solving optimization problem of weight transportation.

In operational angle, we obtain the first results in implementation process. We foresee to develop transportation management software user-friendly interface that allows the logistician to entry ordered items. This will be coupled with GPAO, production software of AAS's enterprise, which automatically receives the ended reference markers. Thus, before making the delivery of parts, the logistics management will interrogate this management software that edits a delivery plan after optimization. This plan includes the number of used trucks for a list of part from origins to destinations.

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