

An Inventory Model for Time Dependent Weibull Deterioration with Partial Backlogging

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Abstract This paper deals with developing an inventory model for Weibull deterioration with two parameters. Here shortages are allowed and are partially backlogged. We have derived the optimal order quantity by minimizing the total inventory cost. To illustrate the model a numerical example has been provided and sensitivity analysis has also been carried out to study the effect of parameters on decision variables and the total cost of an inventory of this model.

Keywords Inventory Model, Deterioration, Deterministic Demand, Partial Backlogging

1. Introduction

The effect of deterioration is very important for most of the items which cannot be neglected in inventory model. Deterioration may be defined as decay, change or spoilage that prevents the item from being used for its original purpose. Food items, drugs, pharmaceuticals, photographic film, electronic components and radioactive substances are a few examples of items in which appreciable deterioration can take place during the normal storage period of the units and consequently this loss must be taken into account when analysing the model.

Ghare and Schrader[7] were the pioneer in study of inventory model considering the effect of deterioration. In their paper they have considered constant rate of deterioration in an inventory model with no shortages. There after a lot of research work has been done in this direction. The literature survey by Raafat[9], Shah and Shah[11] and Goyal and Giri[8] give review on deteriorating inventory models. Abad[1, 2] has derived pricing and ordering policy for a variable rate of deterioration and partial backlogging. In his papers the partial backlogging was assumed to be exponential function of waiting time till the next replenishment.

The backorder cost and lost sale have been considered by Dye et al.[4] in their inventory model. Shah and Shukla[12] have developed deterministic inventory model for deteriorating items with shortages. In this case they considered partial backlogging for unsatisfied demand.

Tripathy and Mishra[14] have improved upon their model by considering deterioration to be a linear

function of time instead of being a constant. Some of the important research works in this direction are due to Roy et al.[10], Deng et al.[3], Dye et al.[6], Dye[5] and Teng et al.[13], etc.

In this paper we have developed a deterministic inventory model for two parameter Weibull deterioration. Shortages are allowed and are partially backlogged for this model. The unsatisfied demand is backlogged and is a function of time. The optimal order quantity has derived by minimizing the total inventory cost. A numerical example has been provided to illustrate the model. Sensitivity analysis has also been carried out to study the effect of parameters on decision variables and the total cost of an inventory of this model.

2. Notations

We need the following notations for developing mathematical model.

- i. A : ordering cost per order
- ii. c : purchase cost per unit.
- iii. h : inventory holding cost per unit per time unit.
- iv. π_b : backorder cost per unit short per time unit.
- v. π_l : cost of lost sales per unit.
- vi. t_1 : time at which the inventory level reaches zero, $t_1 \geq 0$
- vii. t_2 : length of period during which shortages are allowed, $t_2 \geq 0$
- viii. T : Total length of cycle time i.e. $T = (t_1 + t_2)$.
- ix. I_m : Maximum inventory level during $[0, T]$.
- x. I_b : Maximum backordered units during stock out period.
- xi. Q : Total order quantity during a cycle of length T i.e. $Q = I_m + I_b$
- xii. $I_1(t)$: level of positive inventory at time t , $0 \leq t \leq t_1$
- xiii. $I_2(t)$: level of negative inventory at time t , $t_1 \leq t \leq T$

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Published online at <http://journal.sapub.org/ajor>

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xiv. TC : total average cost per time unit.

3. Assumptions

The following are the assumptions needed for developing the model:

- The model developed for single item inventory.
- The rate of demand 'D' is known and constant.
- The rate of deterioration, $\theta = \alpha\beta t^{\beta-1}$, follows a two parameter Weibull distribution, where $\alpha(0 < \alpha < 1)$ is the scale parameter, $\beta > 1$ is the shape parameter. It is assumed that the deterioration increases with time $t > 0$.
- Infinite replenishment rate.
- Lead - time is zero or negligible.
- Planning horizon is infinite.
- The rate of backlogging during the shortage period is variable and depends on the length of the waiting time till the next replenishment. The negative inventory backlogging rate will be $B(t) = \frac{1}{1+\delta(T-t)}$; where $\delta > 0$ denotes the backlogging parameter and $t_1 \leq t \leq T$

4. Mathematical Modeling

Considering the effect of demand and deterioration during $[0, t_1]$ the inventory level at any instant of time during $[0, t_1]$ can be described by the following differential equation

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -D \quad 0 \leq t \leq t_1$$

Now putting the value of θ in above equation we get,

$$\frac{dI_1(t)}{dt} + \alpha\beta t^{\beta-1} I_1(t) = -D \quad 0 \leq t \leq t_1 \quad (1)$$

Further the inventory level reaches zero at time $t = t_1$. Using this boundary condition we have $I_1(t_1) = 0$. Again since α is very small, using series expansion and ignoring second and higher powers of α , from (1) we get,

$$I_1(t) = D \left[t_1 - t + \frac{\alpha t_1^{\beta+1}}{\beta+1} - \frac{\alpha t^{\beta+1}}{\beta+1} - \alpha t^\beta t_1 + \alpha t^{\beta+1} \right], 0 \leq t \leq t_1 \quad (2)$$

Since inventory level reaches to zero at time t_1 and there after shortages occur. During the interval $[t_1, T]$, the inventory level depends on demand and a fraction of the demand is backlogged. The state of inventory during $[t_1, T]$ can be represented by the differential equation,

$$\frac{dI_2(t)}{dt} = -\frac{D}{1+\delta(T-t)} \quad t_1 \leq t \leq T \quad (3)$$

Since inventory level reaches to zero at time t_1 , the boundary condition $I_2(t_1) = 0$ and the solution of the differential equation (3) is

$$I_2(t) = \frac{D}{\delta} [\ln(1+\delta(t_1+t_2-t)) - \ln(1+\delta t_2)], t_1 \leq t \leq T \quad (4)$$

Thus the maximum inventory will be

$$I_m = I_1(0) = D \left[t_1 + \frac{\alpha t_1^{\beta+1}}{\beta+1} \right], \quad (5)$$

and the maximum backordered units will be

$$I_b = -I_2(t_1+t_2) = \frac{D}{\delta} \ln(1+\delta t_2) \quad (6)$$

So, the order size Q during $[0, T]$ will be $Q = I_m + I_b$

$$\text{i.e. } Q = D \left[t_1 + \frac{\alpha t_1^{\beta+1}}{\beta+1} + \frac{1}{\delta} \ln(1+\delta t_2) \right] \quad (7)$$

The following cost components need to be considered to calculate total cost per cycle:

i. Ordering cost per cycle is A

ii. Holding cost per cycle is

$$h \int_0^{t_1} I_1(t) dt = hD \left[\frac{t_1^2}{2} - \frac{\alpha t_1^{\beta+2}}{(\beta+1)(\beta+2)} + \frac{\alpha t_1^{\beta+2}}{\beta+1} \right]$$

iii. Backorder cost per cycle

$$= \pi_b \int_{t_1}^{t_1+t_2} -I_2(t) dt = \frac{\pi_b D}{\delta^2} [\delta t_2 - \ln(1+\delta t_2)]$$

iv. Cost of lost sales per cycle

$$= \pi_l D \int_{t_1}^{t_1+t_2} \left(1 - \frac{1}{1+\delta(t_1+t_2-t)} \right) dt \\ = \frac{\pi_l D}{\delta} [\delta t_2 - \ln(1+\delta t_2)]$$

v. Purchase cost per cycle

$$= c \times Q = cD \left[t_1 + \frac{\alpha t_1^{\beta+1}}{\beta+1} + \frac{1}{\delta} \ln(1+\delta t_2) \right]$$

Therefore, the total average cost per unit time is

$$TC = \frac{1}{t_1+t_2} \left[\begin{array}{l} \text{ordering cost} + \text{holding cost} \\ + \text{backorder cost} + \text{cost of lost sales} \\ + \text{purchase cost} \end{array} \right] \\ = \frac{1}{t_1+t_2} \left[A + hD \left[\frac{t_1^2}{2} - \frac{\alpha t_1^{\beta+2}}{(\beta+1)(\beta+2)} + \frac{\alpha t_1^{\beta+2}}{\beta+1} \right] \right. \\ \left. + \frac{\pi_b D}{\delta^2} [\delta t_2 - \ln(1+\delta t_2)] + \frac{\pi_l D}{\delta} [\delta t_2 - \ln(1+\delta t_2)] \right. \\ \left. + cD \left[t_1 + \frac{\alpha t_1^{\beta+1}}{\beta+1} + \frac{1}{\delta} \ln(1+\delta t_2) \right] \right] \quad (8)$$

The necessary conditions for minimizing the total average cost per unit will be

$$\frac{\partial TC}{\partial t_1} = 0 \quad (9)$$

and

$$\frac{\partial TC}{\partial t_2} = 0 \quad (10)$$

The pair of (t_1, t_2) calculated from equations (9) and (10), subject to the condition

$$\frac{\partial^2 TC}{\partial t_1^2} \times \frac{\partial^2 TC}{\partial t_2^2} - \frac{\partial^2 TC}{\partial t_1 \partial t_2} > 0 \text{ will minimize } TC.$$

Now we have

$$\frac{\partial TC}{\partial t_1} = \frac{1}{(t_1 + t_2)^2} \left[-A + hD \left[\frac{t_1^2 + 2t_1 t_2}{2} - \frac{\alpha(\beta+2)t_1^{\beta+1}(t_1 + t_2) - \alpha t_1^{\beta+2}}{(\beta+1)(\beta+2)} + \frac{\alpha(\beta+2)t_1^{\beta+1}(t_1 + t_2) - \alpha t_1^{\beta+2}}{(\beta+1)} \right] - \frac{D(\pi_b + \pi_l \delta)(\delta t_2 - \ln(1 + \delta t_2))}{\delta^2} + cD \left[\frac{t_2 + \frac{\alpha(\beta+1)t_1^\beta(t_1 + t_2) - \alpha t_1^{\beta+1}}{(\beta+1)}}{\ln(1 + \delta t_2)} - \frac{\delta}{\delta} \right] \right] = 0 \quad (11)$$

$$\frac{\partial TC}{\partial t_2} = \frac{1}{(t_1 + t_2)^2} \left[-A - hD \left[\frac{t_1^2}{2} - \frac{\alpha t_1^{\beta+2}}{(\beta+1)(\beta+2)} + \frac{\alpha \beta t_1^{\beta+2}}{\beta+1} \right] + \frac{D(\pi_b + \delta \pi_l)}{\delta^2} \left[\delta t_1 - \frac{\delta(t_1 + t_2) - \ln(1 + \delta t_2)(1 + \delta t_2)}{1 + \delta t_2} \right] + cD \left[\frac{-t_1 - \frac{\alpha t_1^{\beta+1}}{\beta+1} + \frac{\delta(t_1 + t_2) - \ln(1 + \delta t_2)(1 + \delta t_2)}{\delta(1 + \delta t_2)}}{\delta} \right] \right] = 0 \quad (12)$$

The above set of parametric equations (11) and (12) can be solved for t_1^* and t_2^* using mathematica-5.1 software. Now we will consider a numerical example to illustrate and validate the proposed model in the following section.

5. Numerical Example

Example-1: Let an inventory system have the following parametric values in proper units.

$[A, c, h, \pi_b, \pi_l, D, \delta, \alpha, \beta] = [200, 20, 0.7, 12, 10, 20, 2, 0.3, 0.2]$. Using these values for the above model, we get optimum value of $t_1^* = 5.13906$, $t_2^* = 1.23672$ putting the optimum values of t_1^* and t_2^* in equation (7) and (8) we get the optimum values of $Q^* = 150.88$ and minimum total average cost per unit time $TC^* = 566.309$ respectively.

6. Sensitivity Analysis

Now we will perform sensitivity analysis by increasing parameters α , β , δ and D , one at a time and keeping the remaining parameters at their original values. We will be using the values of the numerical example given in the previous section for performing the sensitivity analysis. The

results are given in tabulated form in Table-1 to Table-4.

In table-1, it is observed that increase in demand parameter D results in increase in TC of an inventory system and also increase ordering quantity.

Table 1. Variation in demand D

Parameter D	Change in t_1^*	Change in t_2^*	Change in Q^*	Change in TC^*
20	5.13906	1.23672	150.880	566.309
21	5.14103	1.19682	158.239	594.546
22	5.14294	1.16004	165.592	622.800

Table 2. Variation in deterioration rate ' α '

Parameter α	Change in t_1^*	Change in t_2^*	Change in Q^*	Change in TC^*
0.3	5.13906	1.23672	150.880	566.309
0.4	4.66427	1.88792	151.232	571.327
0.5	4.77792	2.72937	168.651	576.014

In table-2, it is observed that increase in deterioration rate ' α ' results in increase TC of an inventory system and also increase ordering quantity.

Table 3. Variation in deterioration rate ' β '

Parameter β	Change in t_1^*	Change in t_2^*	Change in Q^*	Change in TC^*
0.2	5.13906	1.23672	150.880	566.309
0.3	5.70450	1.83300	173.884	566.327
0.4	6.81755	3.07721	218.994	568.501

In table-3, it is observed that increase in deterioration rate ' β ' results in increase TC of an inventory system and also increase ordering quantity.

Table 4. Variation in backlogging parameter ' δ '

Parameter δ	Change in t_1^*	Change in t_2^*	Change in Q^*	Change in TC^*
2	5.13906	1.23672	150.880	566.309
3	2.74150	1.17100	81.6473	530.534
4	1.26729	1.00699	40.0647	514.411

In table-4, it is observed that increase in backlogging parameter δ results in decrease in TC of an inventory system and decrease ordering quantity.

The results observed above can also be represented graphically in form of the following Figures.

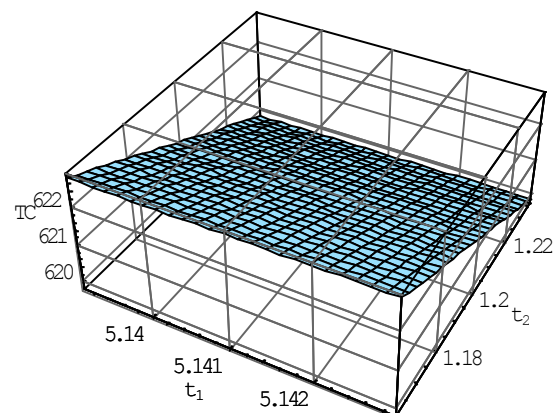


Figure 1. Total cost TC versus t_1 and t_2 for table-1

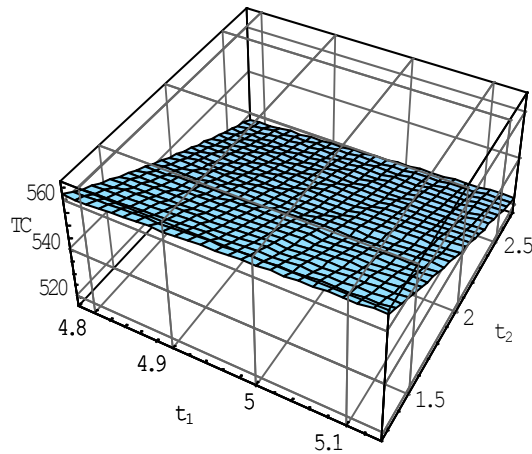


Figure 2. Total cost TC versus t_1 and t_2 for table-2

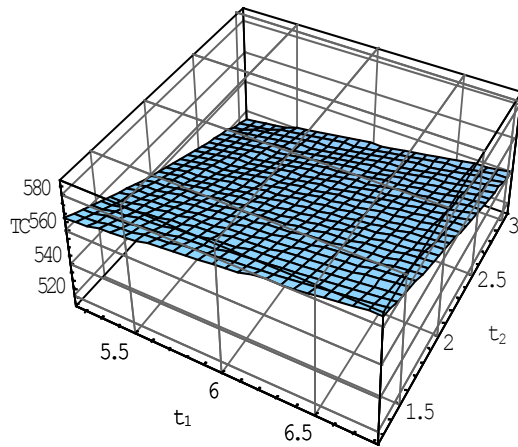


Figure 3. Total cost TC versus t_1 and t_2 for table-3

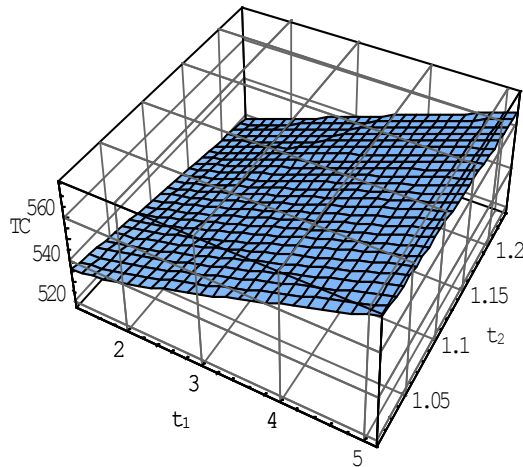


Figure 4. Total cost TC versus t_1 and t_2 for table-4

7. Conclusions

In the present scenario of inventory system, large amount of capital is invested for purchase of inventory. It is important to consider deterioration in inventory decision making. In this paper a deterministic inventory model has been developed for Weibull deterioration. Shortages are allowed and are partially backlogged for this model. The unsatisfied

demand is backlogged and is a function of time. The optimal order quantity has been derived by minimizing the total inventory cost. This will help in inventory decision making under similar condition. A numerical example and sensitivity analysis are presented to illustrate the proposed model. It is observed that increase in demand parameter or scale or shape parameters of deterioration function results in increase in total cost of an inventory system and also increase in ordering quantity. Further studies in this direction can be carried out incorporating more realistic assumption such as stochastic demand and finite replenishment rate.

ACKNOWLEDGEMENTS

We are highly indebted to the anonymous referees for their valuable suggestion and comments to improve upon the earlier version of this article to the present form.

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