

# An M/M/1/N Queueing Model with Retention of Reneged Customers and Balking

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**Abstract** The concept of customer balking and reneging has been exploited to a great extent in recent past by the queuing modelers. Economically, if we see, the customer impatience (due to balking and reneging) leads to the loss of potential customers and thereby results into the loss in the total revenue. Taking into consideration this customers' loss due to impatience, a new queuing model has been developed that deals with retention of reneged customers. According to this model, a reneged customer can be convinced in many cases by employing certain convincing mechanism to stay in the queue for completion of his service. Thus, a reneged customer can be retained in the queuing system with some probability (say,  $q$ ) and it may leave the queue without receiving service with probability  $p$  ( $=1-q$ ). This process is referred to as customer retention. We consider a single server, finite capacity queuing system with customer retention and balking in which the inter-arrival and service times follow negative-exponential distribution. The reneging times are assumed to be exponentially distributed. An arriving customer may not join the queue if there is at least one customer in the system, i.e. the customer may balk. The steady state solution of the model has been obtained. Some performance measures have been computed. The sensitivity analysis of the model has been carried out. The effect of probability of retention on the average system size has been studied. The numerical results show that the average system size increases proportionately and steadily as the probability of retention increases. Some particular cases of the model have been derived and discussed.

**Keywords** Customer Retention, Reneging, Balking, Steady-State Solution, Sensitivity-Analysis, Finite Capacity

## 1. Introduction

In the current scenario of population explosion and globalization of international commerce and trade, the queuing problems have gained a lot of significance in the decision making process. Queuing theory has revolutionized the industry and logistics sector apart from its immense applications in many other areas like city traffic, air traffic, biosciences, population studies, health sector etc. Queuing models have been built according to the prevailing demands or situation. The concept of customer impatience in queuing has been introduced in 1950' and still people are exploiting this concept in various applications[1-11]. A customer is said to be impatient if he tends to join the queue only when a short wait is expected and tends to remain in the line if his wait has been sufficiently small. Impatience generally takes three forms. The first is balking, the reluctance of a customer to join a queue upon arrival, the second reneging, the reluctance to remain in line after joining and waiting, and the third jockeying between lines when each of a number of

parallel lines has its own queue[12]. The notion of customer impatience appeared in the queuing theory in the work of Haight[13] in 1957. He considered a model of balking for M/M/1 queue in which there is a greatest queue length at which an arrival would not balk. This length was a random variable whose distribution was same for all customers. Haight[14] studied a queue with reneging in which he studied the problem like how to make rational decision while waiting in the queue, the probable effect of this decision etc. Ancker et al[15] studied M/M/1 queuing system with balking and reneging and performed its steady state analysis. Ancker et al[16] also obtained results for a pure balking system (no reneging) by setting the reneging parameter equal to zero.

Queuing theory has successfully been applied to various congestion (queuing) situations involving revenue generation through servicing customers. Aforementioned queuing systems deal with the customers' loss due to impatience (balking or reneging) which results into a substantial reduction in the total revenue. Customer impatience has become the burning problem of private as well as government sector enterprises. They are constantly working towards customer retention for better future prospects.

In this paper, we consider a single server, finite capacity queuing system with customer retention and balking, in

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which the inter-arrival and service times follow negative-exponential distribution. Each customer upon arriving in the queue will wait a certain length of time for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting service. This time is a random variable and follows exponential distribution. An impatient customer (due to reneging) can be made to stay in service system for his service by utilizing certain convincing mechanism. Such customers are termed as retained customers. When a customer gets impatient (due to reneging), he may leave the queue with some probability say  $p$  and may remain in the queue with some other probability  $q(=1-p)$ . An arriving customer may not join the queue if there is at least one customer in the system, i.e. the customer may balk. The steady state solution of the model has been obtained. Some queueing models have been obtained as particular cases of the model. The sensitivity of the analysis model has been carried out to show the impact of customer retention probabilities on the measures of performances like expected system size etc. A comparative study of the present model with two other related models has also been carried out.

Rest of the model has been arranged as follows: section 2 deals with the formulation of stochastic queueing model, in section 3, the differential-difference equations of the model have been made, in section 4, the steady-state solution of the model has been derived, section 5 deals with the sensitivity analysis of the model, section 6, deals with the particular cases of the model with comparative analysis and in section 7, the paper has been concluded.

## 2. Stochastic Queueing Model

The queueing model under consideration is based on following assumptions:

1. The arrivals occur in a Poisson stream one by one with an average arrival rate  $\lambda$ . The inter-arrival times are independently, identically and exponentially distributed with parameter  $\lambda$ .
2. There is only one server and service times are exponentially distributed with parameter  $\mu$ .
3. The queue discipline is first-come, first-served (FCFS).
4. The capacity of the system is taken as finite (say  $N$ ).
5. Each customer upon arriving in the queue will wait a certain length of time (reneging time) for service to begin. If it has not begun by then, he will get impatient and may leave the queue without getting service with probability  $p$  and may remain in the queue for his service with probability  $q(=1-p)$ . The reneging times follow exponential distribution with parameter  $\xi$ .
6. The arriving customers balk with probability  $n/N$ , where  $n$  is the number in system and  $N$  is the maximum number allowed in the system.

## 3. Derivation of Differential-Difference Equations

Define,

$P_n(t)$  = the probability that there are  $n$  customer in the system, that is,  $n-1$  in the queue and one in service.

For  $1 \leq n \leq N$ , in an infinitesimally small interval  $(t, t + \delta t)$

$P_n(t + \delta t) = P_r$  {there are  $n$  customer in the system at time  $(t + \delta t)$ }.

It can happen in the following mutually-exclusive ways:

- i. On arrival a customer either decides to join the queue with probability  $(1-n/N)$  or balks with probability  $n/N$  when  $n$  customers are ahead of him ( $n = 0, 1, \dots, N-1$ ). The system might be in state  $n$  at time  $t$  and during the time interval of length  $\delta t$ , no arrival, no departure and no reneging occurs.
- ii. The system might be in state  $n$  at time  $t$ , an arrival and a departure both occur during the time interval of length  $\delta t$ .
- iii. The system might be in state  $n-1$  at time  $t$  and an arrival takes place during the time interval of length  $\delta t$ .
- iv. The system might be in state  $n+1$  at time  $t$ , a departure takes place and no arrival occurs during the time interval of length  $\delta t$ .
- v. The system might be in state  $n$  at time  $t$  and during the interval of length  $\delta t$ , a reneged customer with probability  $q(=1-p)$  does not abandon the queue i.e. he stays in the system for his service.
- vi. The system might be in state  $n+1$  at time  $t$  and during the interval of length  $\delta t$  an impatient customer with probability  $p$  abandons the queue.

Hence, for  $1 \leq n \leq N-1$

$$\begin{aligned} P_n(t + \delta t) = & P_n(t) \left[ 1 - \left( 1 - \frac{n}{N} \right) \lambda \delta t \right] [1 - \mu \delta t] [1 - (n-1)\xi \delta t] \\ & + P_{n-1}(t) \left[ \left( 1 - \frac{n}{N} \right) \lambda \delta t \right] + P_{n+1}(t) (\mu \delta t) [1 - (1 - \frac{n}{N}) \lambda \delta t] \\ & + P_n(t) [(n-1)\xi \delta t] + P_{n+1}(t) [n \xi p \delta t] + o(\delta t) \end{aligned}$$

Finding the difference  $P_n(t + \delta t) - P_n(t)$ , dividing both sides by  $\delta t$  and taking limit  $\delta t \rightarrow 0$  leads to equation (2).  $o(\delta t)$  approaches to zero as rapidly as  $\delta t \rightarrow 0$

Similarly, other equations can be derived.

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad (1)$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = & - \left[ \left( 1 - \frac{n}{N} \right) \lambda + \mu + (n-1)\xi p \right] P_n(t) \\ & + [\mu + n\xi p] P_{n+1}(t) + \left[ \left( 1 - \frac{n-1}{N} \right) \lambda \right] P_{n-1}(t); \quad (2) \end{aligned}$$

$$1 \leq n \leq N-1$$

$$\begin{aligned} \frac{dP_N(t)}{dt} = & \left[ \left( 1 - \frac{N-1}{N} \right) \lambda \right] P_{N-1}(t) - [\mu + (N-1)\xi p] P_N(t); \quad (3) \\ n = N \end{aligned}$$

## 4. Steady-State Solution

In steady state,  $\lim_{t \rightarrow \infty} P_n(t) = P_n$  and therefore  $\frac{dP_n(t)}{dt} = 0$  as  $t \rightarrow \infty$ .

Thus, the steady-state equations corresponding to equations (1) - (3) are as follows:

$$0 = -\lambda P_0 + \mu P_1 \quad (4)$$

$$0 = -\left[\left(1 - \frac{n}{N}\right)\lambda + \mu + (n-1)\xi p\right]P_n + [\mu + n\xi p]P_{n+1} + \left[\left(1 - \frac{n-1}{N}\right)\lambda\right]P_{n-1} \quad (5)$$

$$; 1 \leq n \leq N-1$$

$$0 = \left[\left(1 - \frac{N-1}{N}\right)\lambda\right]P_{N-1} - [\mu + (N-1)\xi p]P_N; n = N \quad (6)$$

Solving recursively equations (4) - (6), we get

$$P_n = \prod_{k=1}^n \frac{N-(k-1)}{N} \frac{\lambda}{\mu + (k-1)\xi p} P_0; 1 \leq n \leq N-1 \quad (7)$$

Also for  $n=N$ , we get

$$P_N = \prod_{k=1}^N \frac{N-(k-1)}{N} \frac{\lambda}{\mu + (k-1)\xi p} P_0; n = N \quad (8)$$

Using the normalization condition,  $\sum_{n=1}^N P_n = 1$ , we get

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^N \prod_{k=1}^n \frac{N-(k-1)}{N} \frac{1}{\mu + (k-1)\xi p}\right)} \quad (9)$$

### Measures of Performance

(1) The Expected System Size

$$L_s = \sum_{n=0}^N n P_n$$

$$L_s = \sum_{n=1}^N n \prod_{k=1}^n \frac{N-(k-1)}{N} \frac{\lambda}{\mu + (k-1)\xi p} P_0$$

(2) The Expected Queue Length

$$L_q = \lambda W_q$$

(3) The Expected Waiting Time in the System

$$W_s = \frac{L_s}{\lambda}$$

(4) The Expected Waiting Time in the queue

$$W_q = W_s - \frac{1}{\mu}$$

## 5. Sensitivity Analysis

For  $N = 4$  and different value of  $\mu, \lambda$  &  $p$ , we have

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^4 \prod_{k=1}^n \frac{4-(k-1)}{4} \frac{1}{\mu + (k-1)\xi p}\right)}$$

$$L_s = \sum_{n=1}^4 n \prod_{k=1}^n \frac{4-(k-1)}{4} \frac{\lambda}{\mu + (k-1)\xi p} P_0$$

### CASE-1: Variation of Expected System Size with the variation in Average Arrival Rate

When

$$N = 4, \lambda = 2.0, 2.1, 2.2, \dots, 2.9, \mu = 3, \xi = 0.1 \text{ \& } q = 0.6$$

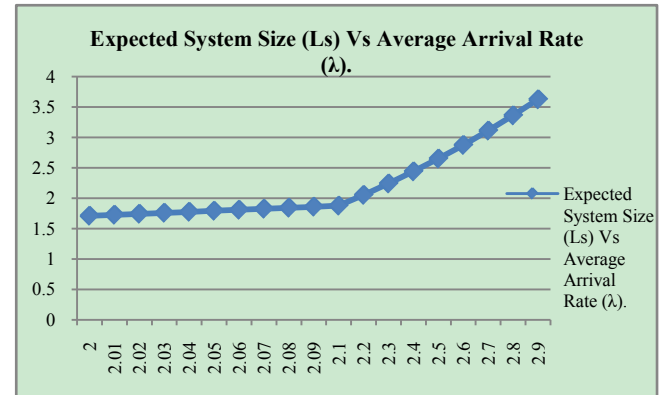


Figure 1.

### CASE-2: Variation of Expected System Size with the variation in Average Service Rate

When

$$N = 4, \lambda = 2, \mu = 2.0, 2.1, \dots, 2.9, \xi = 0.1 \text{ \& } q = 0.6$$

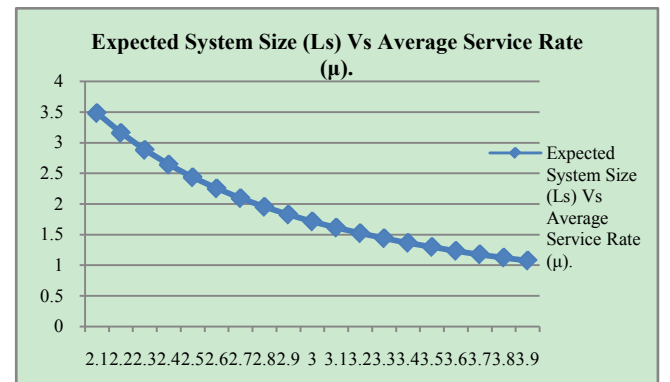


Figure 2.

### CASE-3: Variation of Expected System Size with the variation in Average Reneging Rate

When

$$N = 4, \lambda = 2, \mu = 3, q = 0.6, \xi = 0.01, 0.02, \dots, 0.09, 0.1, 0.2, \dots, 0.8$$

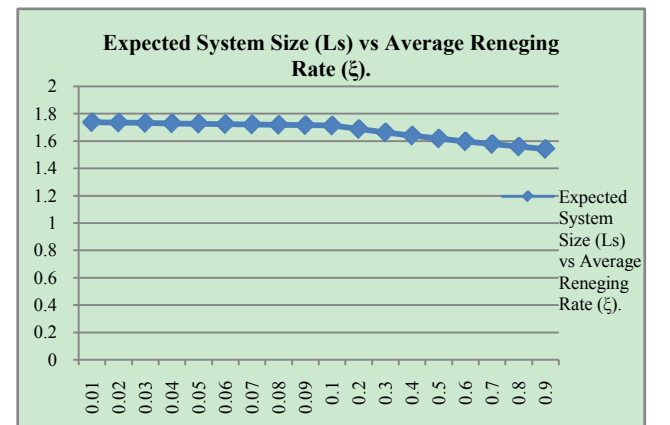


Figure 3.

#### CASE-4: Variation of Expected System Size with the variation in Probability of Retention

When

$N = 4, \lambda = 2, \mu = 3, \xi = 0.1$  &  $q = 0.1, 0.2, \dots, 0.9, 0.91, 0.92, \dots, 0.99$ .

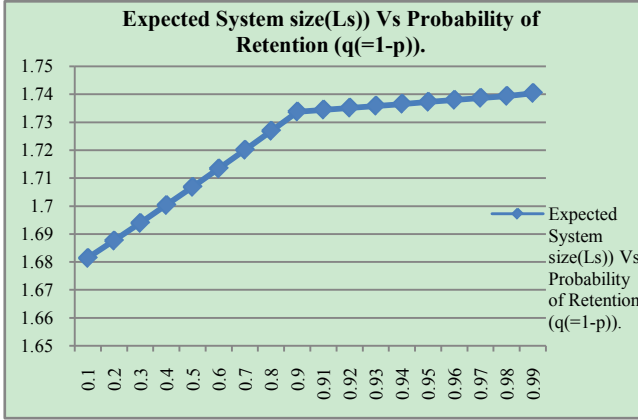


Figure 4.

From the figures 1 and 2, we see that with the increase in average arrival and average service rates, there is exponential increase and decrease in the expected system size respectively. From fig.3 one can see a regular decrease in average system size with the increase in average reneging rate. The most important is the effect of probability of retention on the system size. From fig.4 we see that as we increase the probability of retention there is a steady and proportional increase in the average system size.

## 6. Particular Cases

### (i) When the probability of retention, $q=0$

The model reduces to M/M/1/N queueing model with reneging and balking with

$$P_n = \prod_{k=1}^n \frac{N-(k-1)}{N} \frac{\lambda}{\mu + (k-1)\xi} P_0. \quad (10)$$

Also for  $n = N$  we get

$$P_N = \prod_{k=1}^N \frac{N-(k-1)}{N} \frac{\lambda}{\mu + (k-1)\xi} P_0. \quad (11)$$

Using the normalization condition,  $\sum_{n=1}^N P_n = 1$ , we get

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^N \prod_{k=1}^n \frac{N-(k-1)}{N} \frac{1}{\mu + (k-1)\xi}\right)} \quad (12)$$

### Expected System Size

$$L_s = \sum_{n=0}^N n P_n$$

$$L_s = \sum_{n=1}^N n \prod_{k=1}^n \frac{N-(k-1)}{N} \frac{\lambda}{\mu + (k-1)\xi} P_0$$

### (ii) When the probability of reneging, $p=0$ , implies $\xi = 0$ :

The model reduces to a simple M/M/1/N queue with balking with

$$P_n = \prod_{k=1}^n \frac{N-(k-1)}{N} \left(\frac{\lambda}{\mu}\right)^n P_0; 1 \leq n \leq N \quad (13)$$

Using the normalization condition,  $\sum_{n=1}^N P_n = 1$ , we get

$$P_0 = \frac{1}{\left(1 + \sum_{n=1}^N \prod_{k=1}^n \frac{N-(k-1)}{N} \left(\frac{\lambda}{\mu}\right)^n\right)} \quad (14)$$

### Expected System Size

$$L_s = \sum_{n=0}^N n P_n$$

$$L_s = \sum_{n=1}^N n \prod_{k=1}^n \frac{N-(k-1)}{N} \left(\frac{\lambda}{\mu}\right)^n P_0 \quad (15)$$

## 7. Conclusions

This paper discussed a Markovian single server finite capacity queueing model with balking and possibilities of retaining reneged customers. The introduction of this concept of customer retention in queueing models has a lot of significance in the revenue generating queueing systems. The steady-state probabilities of system size have been obtained and some measures of performance have also been computed. The effect of parameters like average arrival rate, average service rate etc. have been studied numerically and it has been found that with the increase in probability of retention, the expected system size also increases proportionately and steadily. Some particular cases of the model have been discussed.

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