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Sub-Differential Characterizations of Lower Semi-Continuous Quasi-Convex Functions on Infinite-Dimensional Spaces and Optimality Conditions Using Variational Inequalities

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Abstract We study optimizations under a weak condition of convexity, called quasi-convexity in infinite dimensional spaces. Although many theorems involving the characterizations of quasi-convex functions and optimizations in finite dimensional spaces appear in the literature, very few results exist on the characterizations of quasi-convex functions in infinite dimensional spaces which involve a generalized derivatives of quasi-convex functions. Although the condition $0 \in \partial f(x)$ for $x \in X$, is known to be necessary optimality condition for existence of a minimizer in quasi-convex programming for some sub-differentials, it is not a sufficient condition. We extend the study of subdifferential characterization of quasi-convex functions in infinite dimensional spaces by using some variational inequalities approach to obtain a necessary and sufficient condition for $x \in X$ to be either a local minimum or a global minimum.

Keywords Quasi-convexity, Quasi-monotonicity, Sub-differential and Variational Inequalities

1. Introduction

The study of Quasi-convex functions and optimizations, which play important roles in numerous fields including, economics, engineering, management science, operations research, industrial organization, computer vision, curve fitting, and various applied sciences, is several decades old. [1,11,19,22]. The notion of Quasi-convex functions and the characteristics convexity of its level set was first recognized by De Finetti in his work, “Sulle Straficazoni Convesse” in 1949, [9]. Since then, efforts have been focused on this class of functions because of its similar features with convex functions and its wider applications [11,14,13,21]. A quasi-convex optimization problem is a mathematical optimization problem in which the objective is to minimize a quasi-convex function over a convex set. Because every convex function is also quasi-convex, Quasi-convex programs therefore generalize convex programs [1].

The prefix ‘quasi” means “as if”. Thus, we expect quasi-convex functions to possess some special qualities that are similar to those of convex functions. However, while some properties of convex functions and optimizations have analogues equivalence of quasi-convexity some properties do not have. For instance, although the sub-level sets of both convex and quasi-convex functions are convex, quasi-convex functions differ from convex functions in the following ways among others; quasi-convex functions can be discontinuous in the interior of their domain, not every local minimum is a global minimum, local minimum that are not global cannot be strict minima. First order conditions are not sufficient to identify even local optima under quasi-convexity. [11,14].

Many theorems involving the characterizations of quasi-convex functions and optimizations in finite dimensional spaces appear in the literature. One of the most important properties of convex functions is that their level sets are convex. This property is also a fundamental geometric characterization of quasi-convex functions which sometimes is treated as their definition [10,11,14,19,21]. However, the most attractive characterizations of quasi-convex functions are those which involve gradients (a detailed account of the current state of research on the topic can be found in [5]). As to generalized derivatives of quasi-convex functions, very few results exist (see [5,12]). In [12], a study of quasi-convex functions is presented via Clarke's sub-differential, but the authors restricted themselves to the case of Lipschitz functions on a finite dimensional space only.

Interestingly, [21] and independently, [2-4] characterized

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the lower semi-continuous quasi-convex functions in terms of generalized (Clarke-Rockafellar) sub-differentials and directional derivatives in infinite dimensional spaces with the concept of quasi-monotone maps and prove that a lower semi-continuous function on an infinite dimensional space is quasi-convex if and only if its generalized sub-differential or its directional derivative is quasi-monotone. Although [2-4] and [21] studied the subdifferential characterizations of quasi-convex functions in infinite dimensional spaces, none of the studies covered their optimality conditions in infinite dimensional spaces. [20] did a study on a necessary optimality condition for lower semi-continuous quasi-convex functions on closed convex sets but did not cover the sufficient optimality condition of the problems. He didn't adopt the variational inequality approach in his study of optimality conditions but rather adopted the normal cone approach in his minimization of the quasi-convex function. Variational inequalities have found many applications in optimization and in order fields of applied, especially in mechanics [15].

Although the condition $0 \in \partial f(x)$ for $x \in X$, is known to be necessary optimality condition for existence of a minimizer in quasi-convex programming for some sub-differentials, it is not a sufficient condition. We extend the study of [2-4] and [21] by using some variational inequalities approach to obtain a necessary and sufficient condition for $x \in X$ to be either a local minimum or a global minimum.

2. Preliminaries

Let X be a Banach space with norm $\|.\|$, X^* its topological dual $\langle .,. \rangle$ for the duality pairing and $\langle x^*, x\rangle$ the value of $x^* \in X^*$ at $x \in X$ For each $x, y \in X$, we define the closed line segment $[x,y] = \lambda x + (1-\lambda)y$ for some for $\lambda \in [0,1]$ and define $(x,y]$, $[x,y)$ and (x,y) analogously and we denote an open ball centered at x with radius ε by $B_\varepsilon(x) = \{x' \in X: \|x' - x\| < \varepsilon\}$.

Given a lower semi-continuous (l.s.c.) function $f: X \to \mathbb{R} \cup \{+\infty\}$, the effective domain is defined by

$$dom\, f = \{x \in X: f(x) < +\infty\}.$$

For a multivalued operator $T: X \to X^*$, the domain of T is $dom\, T = \{x \in X: T(x) \neq \emptyset$.

Definition 2.1. A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be quasi-convex if for each $x, y \in \mathrm{dom}\, f$,

$$z \in [x,y] \Longrightarrow f(z) \leq \max\{f(x), f(y)\}. \quad (1)$$

This is equivalent to the convexity of the level sets

$$S_\lambda f = \{ x \in X: f(x) \leq \lambda\},\ \forall \lambda \in \mathbb{R}. \quad (2)$$

f is said to be strictly quasi-convex if the inequality (1) is strict when $x \neq y$.

Definition 2.2. [4] A differentiable function f is called quasi-convex if for every $x, y \in dom\, f$

$$\langle \nabla f(x), y - x\rangle > 0 \Longrightarrow f(x) \leq f(y) \quad (3)$$

Definition 2.3 An operator ∂ that associates to any l.s.c. function $f: X \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in X$ a subset $\partial f(x)$ of X^* is a sub-differential if it satisfies the following properties:

(i) $\partial f(x) = \{x^* \in X^*: \langle x^*, y - x\rangle + f(x) \leq f(y), \forall y \in X\}$, whenever f is convex;

(ii) $0 \in \partial f(x)$, whenever $x \in dom\, f$ is a local minimum of f;

(iii) $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$, whenever g is a real a real-valued convex continuous function which is ∂-differentiable at x, where g-differentiable at x means that both $\partial g(x)$ and $\partial(-g)(x)$ are non-empty. We say that f is ∂-differentiable at x when $\partial f(x)$ is non-empty while $\partial f(x)$ are called the sub-gradients of f at x.

The Clarke-Rockafellar general derivative of f at $x_0 \in \mathrm{dom}(f)$ in the direction $d \in X$ is given by

$$f^\uparrow(x_0, d) = \sup_{\varepsilon > 0} \limsup_{\substack{x \to_f x_0 \\ \lambda \searrow 0}} \inf_{d' \in B_\varepsilon(d)} \frac{f(x + \lambda d') - f(x)}{\lambda},$$

where $B_\varepsilon(d) = \{d' \in X: \|d' - d\| < \varepsilon\}$, $\lambda \searrow 0$ indicates the fact that $\lambda > 0$ and $\lambda \to 0$,

and $x \to f^{x_0}$ means that both $x \to x_0$ and $f(x) \to f(x_0)$.

The Clarke-Rockafellar subdifferential of f at x_0 is defined by

$\partial f(x_0) = \{x^* \in X^*: (x^*, d) \leq f^\uparrow(x_0, d), \forall d \in X\}$ if $x_0 \in X \backslash \mathrm{dom}(f)$, then

$\partial f(x_0) = \emptyset$.

We extend the notion of quasi-convexity to less smooth function using the concept of generalized directional derivatives and sub-differential.

Definition 2.4. [4] A l.s.c function $f: X \to \mathbb{R} \cup \{+\infty\}$ is called quasi-convex (with respect to Clarke-Rockerfeller Subdifferentials) if for any $x, y \in X$:

$$\exists x^* \in \partial f(x): \langle x^*, y - x\rangle > 0 \Longrightarrow \forall z \in [x,y],\ f(z) \leq f(y). \quad (4)$$

Definition 2.5. A multivalued operator $T: X \to X^*$ is said to be quasi-monotone if

$$\exists x^* \in T(x): \langle x^*, y - x\rangle > 0 \Longrightarrow \forall y^* \in T(x), \langle y^*, y - x\rangle \geq 0.$$

3. Sub-Differential Characterizations of Quasi-Convex Functions

Our aim is to show that f if quasi-convex if and only $\partial f(x)$ is quasi-monotone. We need the following lemma.

Lemma 3.1. Let $a, b \in X$ with $f(a) < f(b)$. Then, exist $c \in [a,b)$, and sequence $x_n \to f^c$ and $x_n^* \in \partial f(x_n)$ with $\langle x_n^*, x - x_n\rangle > 0$ for every $x = c + \lambda(b - a)$ with $\lambda > 0$.

Proof. By Approximate mean value inequality theorem [3], we can find an $x_0 \in [a,b)$ and a sequence $x_n \to f^C$ and $x_n^* \in \partial f(x_n)$ verifying

$$\lim_{n\to+\infty}\inf\langle x_n^*, c-x_n\rangle \geq 0 \text{ and } \lim_{n\to+\infty}\inf\langle x_n^*, b-a\rangle > 0. \quad (5)$$

Letting $x = c + \lambda(b-a)$ with $\lambda > 0$ it holds

$$\langle x_n^*, c-x_n\rangle = \langle x_n^*, c-x_n\rangle + \lambda\langle x_n^*, b-a\rangle > 0, \quad (6)$$

for n sufficiently large.

Theorem 3.2. (Quasi-convexity). f is quasi-convex if and only if ∂f is quasi-monotone.

Proof. We show that if f is not quasi-convex then, ∂f is not quasi-monotone.

Suppose that there exist some x, y, z in X with $z \in [x, y]$ and $f(z) > \max\{f(x), f(y)\}$. According Lemma 3.1 applied with $a = x$ and $b = z$, there exists a sequence $y_n \in \mathrm{dom}\partial f$ and $y_n^* \in \partial f(y_n)$ such that

$$y_n \to \bar{y} \in [x, z],\ \bar{y} \neq z \text{ and } \langle y_n^*, y - y_n\rangle > 0. \quad (7)$$

Let $0 < \lambda \leq 1$ be such that $z = \bar{y} + \lambda(y - \bar{y})$ and set $z_n = y_n + \lambda(y - y_n)$, so that $z_n \to z$. Since f is lower semi-continuous, we may pick $n \in \mathbb{N}$ very large with $f(z_n) > f(y)$. Apply Lemma 3.1 again with $a = y$ and $b = z_n$ to find sequences $x_k \in \mathrm{dom}\partial f$, $x_k^* \in \partial f(x_k)$ such that

$$x_k \to \bar{x} \in [y, z_n],\ \bar{x} \neq z_n \text{ and } \langle x_k^*, y_n - x_k\rangle > 0. \quad (8)$$

In particular, $\bar{x} \neq y_n$ and

$$\langle y_n^*, \bar{x} - y_n\rangle = \frac{\|\bar{x}-y_n\|}{\|y-y_n\|}\langle y_n^*, y - y_n\rangle > 0; \quad (9)$$

hence, $\langle y_n^*, x_k - y_n\rangle > 0$ for k sufficiently large. But $\langle y_n^*, y_n - x_k\rangle > 0$, showing that ∂f is not quasi-monotone.

Conversely, we suppose that f is quasi-convex and show that ∂f is quasi-monotone. Let $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ with $\langle x^*, y - x\rangle > 0$. We need to verify that $f^{\uparrow}(y, x - y) \leq 0$. We fix $\varepsilon > 0$ and $\omega \in (0, \varepsilon)$ such that

$\langle x^*, v - x\rangle > 0$ for all $v \in B_\omega(y)$.

We fix $v \in B_\omega(y)$. Since $f^{\uparrow}(y, x - y) > 0$ we can find $\varepsilon' \in (0, \varepsilon - \omega)$, $u \in B_{\varepsilon'}(x)$ and $t \in (0,1)$ such that $f(u + t(v - u) > f(u)$. From the quasi-convexity of f we deduce that $f(u) < f(v)$, whence,

$$f(v + \lambda(u - v)) \leq f(v) \text{ for all } \lambda \in (0,1),$$

so that

$\inf_{\mu\in B_\varepsilon(x-y)} \frac{f(v+\lambda\mu)-f(v)}{\lambda} \leq \frac{f(v+\lambda(u-v))-f(v)}{\lambda} \leq 0$ for all $\lambda \in (0,1)$.

Combining the inequalities and for any $\varepsilon > 0$ there exists $\omega > 0$ such that

$$\sup_{\substack{v\in B_\omega(y)\\ \lambda\in(0,1)}} \left[\inf_{\mu\in B_\varepsilon(x-y)} \frac{f(v+\lambda\mu)-f(v)}{\lambda}\right] \leq 0,$$

which shows that $f^{\uparrow}(y, x - y) \leq 0$.

4. Optimality Conditions and Variational Inequalities

Let $\Gamma: X \to X^*$ be a multivalued operator, $S \subset X$ and $\bar{x} \in S$. Recall from [17,18] that, Γ satisfies the variational inequality (10) if and only if

$$\forall x \in S,\ \langle \gamma(x), x - \bar{x}\rangle \geq 0,\ \forall \gamma(x) \in \Gamma(x). \quad (10)$$

Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) function and consider the minimization problem

$$minimize\, f(x), \text{ subject to } x \in S. \quad (11)$$

Then, if N is a convex open neighborhood of $\bar{x}$, we have the following

Lemma 4.1. If ∂f satisfies (10), the following assertions hold.

(i) If $S = X$, then $\bar{x}$ is a global minimum of f.
(ii) If $S = N$, then $\bar{x}$ is a local minimum of f.

Proof. It suffices to prove (ii) Suppose by contradiction that $\bar{x}$ is not a solution of (11), then there exist $x \in S$ such that $f(x) < f(\bar{x})$. By Lemma 3.1, there exist $c \in [x, \bar{x})$ and two sequences $c_n \to f^c$, $c_n^* \in \partial f(c_n)$ with

$$\langle c_n^*, d - c_n\rangle > 0,$$

for any $d = c + \lambda(\bar{x} - x)$ where $\lambda > 0$.

Since S is a convex open neighborhood of $\bar{x}$, then $[x, \bar{x}] \subset S$. Furthermore, for n large enough $c_n \in S$.

For $d = \bar{x}$, we have

$$\langle c_n^*, \bar{x} - c_n\rangle > 0,$$

which contradicts (10). Thus, $\bar{x}$ is a local minimum of f.

Consider now the quasi-convex minimization problem (11) again,

$$minimize\, f(x), \text{ subject to } x \in S. \quad (12)$$

where $f: X \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. and quasi-convex, then we have:

Theorem 4.2. If $S = X$ or $S = N$, then the following assertions are equivalent:

(i) $\bar{x}$ is an optimal solution of (12).
(ii) ∂f satisfies (10).

Proof. (i) $\Longrightarrow$(ii). Suppose $\bar{x}$ is a strict minimum of (12). Then for all $x \in S$ such that $x \neq \bar{x}$, we have $f(x) > f(\bar{x})$. By Lemma 3.1, there exist $c \in [x, \bar{x})$, and two sequences $c_n \to f^c$, $c_n^* \in \partial f(c_n)$ with

$$\langle c_n^*, d - c_n\rangle > 0,$$

for any $d = c + \lambda(\bar{x} - x)$ where $\lambda > 0$.

For $d = x$, we have

$$\langle c_n^*, x - c_n\rangle > 0.$$

Since f is quasi-convex, by **Theorem 3.2.,** ∂f is quasi-monotone. This implies that

$$\langle x^*, x - \bar{x}\rangle \geq 0, \quad \forall x^* \in \partial f(x).$$

Thus, ∂f satisfies the variational inequality (10)

Suppose that $\bar{x}$ is not is a strict minimum of (12) and consider the function g defined by

$$g(x) = \begin{cases} \max\{(f(x), f(\bar{x})\}, & for\ x\neq\bar{x}, \\ v, & for\ x=\bar{x}, \end{cases} \quad (13)$$

where $v < f(\bar{x})$. it is obvious that g is l.s.c., quasi-convex and $\bar{x}$ is a strict local minimum of g.

Then, we have

$$\forall x \neq \bar{x},\ \langle x^*, x - \bar{x}\rangle \geq 0,\ \forall x^* \in \partial g(x).$$

Since $\partial f(x)$ depends only on the values of f in the neighborhood of x, $\partial f(x) = \partial g(x)$.

When $0 \in int(\partial f(\bar{x}))$, i.e. the interior of $\partial f(\bar{x})$, we obtain a more precise result

Lemma 4.3. If $0 \in int(\partial f(\bar{x}))$, then ∂f satisfies the variational inequality (10) on the whole space X and $\bar{x}$ is an optimal solution of (12) with $S = X$. Moreover, $\bar{x}$ is a global minimum of f.

Proof. Suppose that $0 \in int(\partial f(x))$, then

$\exists\, \varepsilon > 0$ such that $B_{X^*}(0, \varepsilon) \subset \partial f(x)$,

where

$$B_{X^*}(0, \varepsilon) = \{x^* \in X^*: \|x^*\| < \varepsilon\}$$

Let $d \in X\backslash\{0\}$ and consider the linear mapping $\ell_d(x^*) = \langle x^*, d\rangle$, for $x^* \in X^*$.

By open mapping theorem [4, Pseudo 8], we

$$B_{X^*}(0, \varepsilon) = \langle \partial f(x), d\rangle.$$

Since f is quasi-convex, then ∂f is quasi-monotone. By Definition 2.1 of [16], the multivalued operator $\partial f_{x,d}$ defined by

$$\partial f_{x,d}(\lambda) = \langle \partial f(x + \lambda d), d\rangle,$$

is quasi-monotone. And then,

$$\langle \lambda d, \partial f(x + \lambda d)\rangle \subset \mathbb{R}_+,$$

for all $\lambda \in \mathbb{R}$ and $d \in X\backslash\{0\}$. Thus, ∂f satisfies (10).

5. Conclusions

We have studied optimizations under a weak condition of convexity, called quasi-convexity in infinite dimensional spaces. Although many theorems involving the characterizations of quasi-convex functions and optimizations in finite dimensional spaces appear in the literature, very few results exist on the characterizations of quasi-convex functions in infinite dimensional spaces which involve a generalized derivatives of quasi-convex functions. Although the condition $0 \in \partial f(x)$ for $x \in X$, is known to be necessary optimality condition for existence of a minimizer in quasi-convex programming for some sub-differentials, it is not a sufficient condition. This study is an extension of the study of [2-4] and [21] by using some variational inequalities approach instead of the normal cone approach to obtain a necessary and sufficient condition for $x \in X$ to be either a local minimum or a global minimum.

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