

When Sciences Overlaps: A Matching Game under a Graph Concept

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Abstract Initially, we provide basic knowledge of definitions and concepts related to the concept of matching in the graph. We are studying a model of games based on two players who take turns adding edges to G , this process eventually produces a maximal matching of the graph. We call the first Maximizer and second player Minimizer. The first aims to get a final matching to be large while the second one wants to reduce it. Maximizer wins if he manages a maximal matching while Minimizer wins if he can prevent him from doing this. The matcher number $\alpha_g(G)$ is the number of edges chosen when both players play optimally, while the matching number $\alpha(G)$ is the number of maximum matching edges. In this research we study the relationship between $\alpha_g(G)$ and $\alpha(G)$. And we also prove some results on types of graph.

Keywords Maximal matching, Maximum matching, Matching, Matcher game

1. Introduction

As the title suggests, this research is about games on graphs. At first sight, the topic may suggest lots of fun since games are widely recognized as amusing. But the reader should keep in mind that this is a collection of mathematical results and as such has a moderate fun impact.

Matching in graphs has been an important topic and has a lot of applications for solving problems. It is used to solve Important problems. This can be generalized to solve any problem that can be represented as a graph, we will discuss the general matching idea and Matching in graphs.

The matcher game is related to the growing family of competition parameters or competitive optimization games on graphs and hypergraphs. For example, Phillips and Slater introduced a game where two players alternate adding vertices of G to an independent set until it becomes a maximal independent set. One player wants the final set to be large; the other wants it to be small. More recently, Cranston et al. [1] introduced a game where the players alternate adding edges of G to a matching until it becomes a maximal matching. Theirs is equivalent to the game of played on the line graph and is a very different game to ours. Probably the best-known such parameter is the game chromatic number, which was introduced by Brams for planar graphs (cf. [2]) and independently by Bodlaender [3] for general graphs, and for list colorings by Borowiecki et al. [4]. There is also work

on domination [5,6], transversals in hypergraphs [7,8] and more.

We denote the matching number of graphs G by $\alpha(G)$, and define the lower matching number $\alpha_L(G)$ as the minimum cardinality of a maximal matching of G .

The main goal is to study the values of the game that fall between $\alpha_L(G)$ and $\alpha(G)$. where we have two cases either Minimizer starts and the two values are equal and therefore the matching number is not interesting so our focus in this research is on the Max-start matcher number which we simply call the matcher number. There we show that, while for bipartite graphs the matcher number equals the matching number, in general $\alpha_g(G) \geq \frac{2}{3}\alpha(G)$, and we characterize the graphs G for which $\alpha_g(G) = \frac{2}{3}\alpha(G)$. We also investigate the relationship with the lower matching number, showing that $\alpha_g(G) \geq 2\alpha(G) - 2\alpha_L(G)$. We conclude with some sufficient conditions for $\alpha_g(G)$ to be half the order.

1.1. Introduction of the Matching Game

In this research we introduce the matching game. This is played on undirected graph G by two players, called Maximizer and Minimizer, who take turns in constructing a matching of G . Each round, one player chooses a vertex u_i with at least one neighbor not previously chosen, and the other player chooses a vertex v_i not previously chosen that is a neighbor of u_i . This process continues until no more play is possible; that is, the edges $u_i v_i$ form a maximal matching of G . Maximizer wishes to maximize the number of edges in this matching, while Minimizer wishes to minimize it. The Max-start matcher number $\alpha_g(G)$ of G is the number of edges in the matching when Maximizer starts and both players play optimally, while the Min-start matcher number $\tilde{\alpha}_g(G)$ of G is the number of edges chosen when Minimizer

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starts and both players play optimally.

For notation and graph-theory terminology not defined herein, we in general follow [9]. We denote the degree of a vertex v in a graph G by $d(v)$. The minimum degree among the vertices of G is denoted by $\delta(G)$.

The following are some important concepts:

- A matching M : is a set of edges in G no two of which are adjacent.
- A perfect matching M : if every vertex of G is incident to an edge of M .
- Maximal matching: if no other edges of G can be added to M .
- Maximum matching: are largest maximal matching.
- Matching number $\alpha(G)$: is number of edges in the maximum matching of G .
- Lower matching number $\alpha_L(G)$: is the minimum cardinality of a maximal matching of G .
- If M is a matching in G , then a vertex is M -matched or covered by M if it is incident with an edge of M ; otherwise, the vertex is M -unmatched.
- A path or cycle is M -alternating if its edges are alternately in and not in M .
- Max start matcher number $\alpha_g(G)$: is the number of edges in the matching when Maximizer starts and both players play optimally
- Min start matcher number $\hat{\alpha}_g(G)$: is the number of edges chosen when Minimizer starts and both players play optimally.

The following figure shows the difference between Maximal matching and Maximum matching.

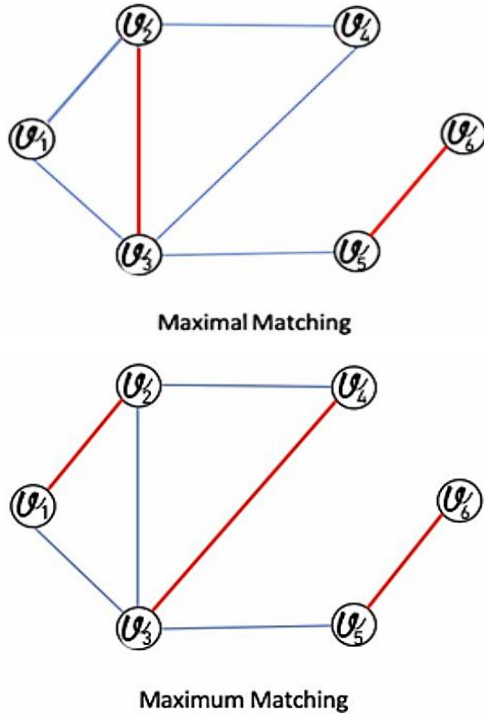


Figure 1. The difference between Maximal matching and Maximum matching

2. The Minimizer Starts the Game

We start by showing that the Min-start matcher number of a graph is precisely the matching number of the graph. We shall need the following trivial well-known preliminary lemma.

Lemma 1.

For every graph G , every non-isolated vertex is incident with an edge that belongs to some maximum matching of G [10].

In the following graph we note that every vertex is incident with an edge belong to a maximum matching

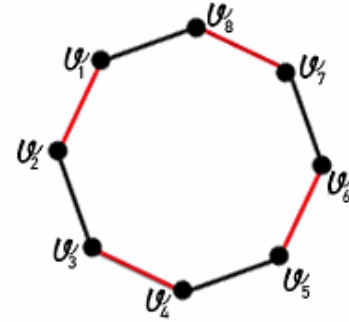


Figure 2. Example of Lemma 1

Theorem 1.

If G is a graph, then have $\hat{\alpha}_g(G) = \alpha(G)$.

Proof.

We show that Maximizer has a strategy that guarantees that the Min-start matcher game always finishes with a maximum matching, implying that $\hat{\alpha}_g(G) = \alpha(G)$. Suppose that Minimizer chooses vertex u . By Lemma 1, the vertex u is incident with an edge, say uv , that belongs to some maximum matching, M say, in G . So, Maximizer chooses vertex v . Then let $\tilde{G} = G - \{u, v\}$. Note that $\alpha(\tilde{G}) = \alpha(G) - 1$, while by induction we have $\alpha_g(\tilde{G}) = \alpha(\tilde{G})$, then $\hat{\alpha}_g(\tilde{G}) = \alpha(G) - 1$, and from it $\hat{\alpha}_g(\tilde{G}) + 1 = \alpha(G) - 1 + 1$ and then $\hat{\alpha}_g(G) = \alpha(G)$.

3. The Maximizer Starts the Game

For the remainder of the paper we study the Max-start matcher game. Here is an example:

Lemma 2.

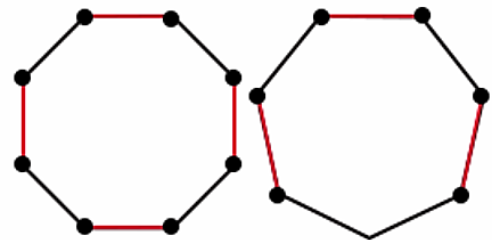


Figure 3. Max start matcher number for the cycle C_8 and C_7

For the path P_n with $n \geq 1$ it holds that $\alpha_g(P_n) = \lfloor n/2 \rfloor$. For the cycle C_n with $n \geq 3$ it holds that $\alpha_g(C_n) = \lfloor n/2 \rfloor$ [10].

For the cycle C_8 and C_7 , we find $\alpha_g(C_8) = \lfloor 8/2 \rfloor = 4$ and $\alpha_g(C_7) = \lfloor 7/2 \rfloor = 3$ as shown in Figure 3.

The above case is an example where the matcher number equals the matching number. For an example where the matcher and matching numbers are different is the following:

Lemma 3.

If G is a graph with minimum degree at least two that contains a unique maximum matching, then $\alpha_g(G) < \alpha(G)$.

Proof.

Let M be the unique maximum matching in G . By Lemma 1, every vertex is incident with M ; that is, M is a perfect matching. Let u be the first vertex chosen by Maximizer. Since u has degree at least two, Minimizer can respond by choosing a vertex v such that uv is not in M ; thus, the maximal matching that is constructed will not be M . It follows that $\alpha_g(G) < \alpha(G)$.

Definition: We define a vertex v of a graph G as liberal if it is not isolated and every edge incident with v belongs to some maximum matching in G .

An example in the following figure we call the vertexes v_2 and v_3 liberal vertex, while the vertexes v_1 and v_4 are not.

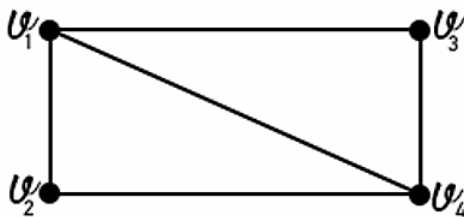


Figure 4. Graph that contains liberal vertexes

The usefulness of the idea can be expressed in the following observation. Recall that a family of graphs is hereditary if it is closed under vertex removal.

Lemma 4.

Let \mathcal{G} be a hereditary graph family such that every nonempty graph in \mathcal{G} has a liberal vertex. Then for all $G \in \mathcal{G}$ it holds that $\alpha_g(G) = \alpha(G)$.

Proof.

The proof is by induction. If G has no edges, then $\alpha_g(G) = \alpha(G) = 0$. If G is nonempty, then by assumption there is a liberal vertex in G , say u . Maximizer chooses this as his first move. Let v be the response of Minimizer and let $\tilde{G} = G - \{u, v\}$. Note that $\alpha(\tilde{G}) = \alpha(G) - 1$, while by induction we have $\alpha_g(\tilde{G}) = \alpha(\tilde{G})$ since $\tilde{G} \in \mathcal{G}$, then $\alpha_g(\tilde{G}) = \alpha(G) - 1$, and from it $\alpha_g(\tilde{G}) + 1 = \alpha(G) - 1 + 1$ and then $\alpha_g(G) = \alpha(G)$.

Note from Figure 4 for liberal vertex v_2 we have $\alpha_g(G) = \alpha(G) = 2$.

For example, it is immediate that any end-vertex of a

graph is liberal; thus trees/forests are examples of the above situation. We show next that isolate-free graphs without perfect matchings always have a liberal vertex (see [11]).

Lemma 5.

If u is a non-isolated vertex and there exists a maximum matching M where u is M -unmatched, then u is liberal.

Proof.

As in the proof of Lemma 1, every neighbor w of u is incident with an edge of M , say f_w , and $(M \setminus \{f_w\}) \cup \{uw\}$ is a maximum matching in G containing uw . That is, u is liberal.

3.1. Bipartite Graphs

We conclude below some properties to see the relation between $\alpha_g(G)$ and $\alpha(G)$.

Lemma 6.

If G is an isolate-free bipartite graph, then each partite set contains a liberal vertex.

Proof.

Color red all edges of G that belong to no maximum matching. Let M be a fixed maximum matching of G , and color blue all edges of M . Thus, each vertex of G is incident with at most one blue edge.

Let us say a path is happy if the edges alternate colors. Let $p : v_1 v_2 \dots v_k$ be a longest happy path in G . As in the proof of Lemma 5, an M -unmatched vertex is not incident with any red edge. So if the first edge of p is red, vertex v_1 must be incident with a blue edge; by the maximality of p , this blue edge must join v_1 to a vertex of p ; indeed, this blue edge must join v_1 to v_k , since all other vertices of p are already incident with a blue edge. But then, if we take M and remove the (blue) edges of p and add the red edges of p , we get another maximum matching, contradicting the claim that the edges are red. It follows that both the first and last edge of the happy path p are colored blue. In particular this implies that k is even.

Suppose that v_1 is not liberal. That is, it is incident with at least one red edge. It follows by the maximality of p that $v_1 v_i$ is a red edge for some $i \in \{3, \dots, k\}$. Since G is bipartite, the cycle $C : v_1 v_2 \dots v_i v_1$ is an even cycle whose edges alternate between blue and red edges. Replacing the (blue) edges of M that belong to C with the (red) edges of C not in M and leaving all other edges of M unchanged, produces a new maximum matching of G that contains some red edges, contradicting the definition of a red edge [10].

And by this we have proven the vertex v_1 is liberal and in the same way we prove that v_k is liberal, then every partite sets have liberal vertex.

Theorem 2.

If G is a bipartite graph, then $\alpha_g(G) = \alpha(G)$.

Proof.

It is produced from Lemmas 4 and 6 [10].

An example of this, in the following figure we have $\alpha_g(G) = \alpha(G) = 4$.

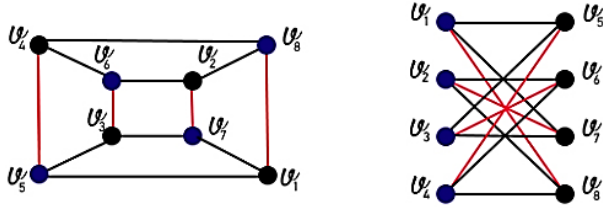


Figure 5. Example of bipartite graph has $\alpha_g(G) = \alpha(G)$

Theorem 3.

If G is a complete multipartite graph, then $\alpha_g(G) = \alpha(G)$.

Proof.

Assume G is a complete multipartite graph with partite sets V_1, \dots, V_k where $|V_1| = \max \{|V_1|, \dots, |V_k|\}$. By Lemma 4 it is sufficient to show that if G is nonempty then it contains a liberal vertex. By Lemma 5 we may assume that G has a perfect matching, say M .

We claim that every vertex $u \in V_1$ is liberal. For, suppose there is an edge uv incident with u that is not in any perfect matching; say $v \in V_2$. By symmetry, no edge between V_1 and V_2 is in a perfect matching. Since M is perfect, it contains edges incident with u and v ; say uu' and vv' . If u' and v' are adjacent, then $(M \setminus \{uu', vv'\}) \cup \{uv, u'v'\}$ is a perfect matching that contains the edge uv , a contradiction. So, assume that u' and v' are in the same partite set, say V_3 (as observed earlier, $u' \notin V_2$ and $v' \notin V_1$).

Since $|V_1| \geq |V_3|$, there is a vertex, w say, in V_1 that is not M -matched to a vertex of V_3 . Hence, the vertex w is incident with an edge of M , say ww' , where $w' \notin V_1 \cup V_2 \cup V_3$. But then $(M \setminus \{vv', ww'\}) \cup \{vw, v'w'\}$ is a perfect matching that contains the edge vw , a contradiction. Thus, every vertex of V_1 is liberal [10].

An example of this, in the following figure the complete multipartite graph $k_{2,2,2}$ we have $\alpha_g(G) = \alpha(G) = 3$.

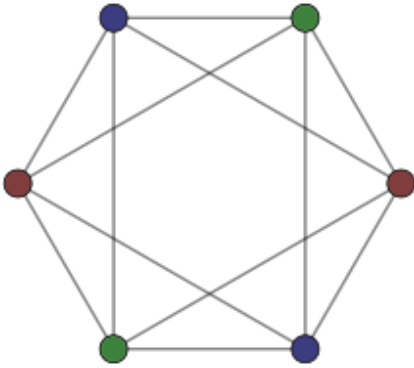


Figure 6. Complete multipartite graph $k_{2,2,2}$

4. Lower Bounds

We next consider lower bounds for the matcher game, especially in relation to the matching and lower matching numbers. We will need the following lemma.

Lemma 7.

If G is a graph and A is an independent set of vertices such that there is a matching that covers A , then $\alpha_g(G) \geq |A|$.

Proof.

Let H be the bipartite subgraph of G with vertex set $A \cup N(A)$ and edge-set the edges incident with A . By Lemma 6 there is a vertex v of A that is liberal in H . Maximizer chooses this vertex. Say Minimizer chooses w in response. Since v is liberal in H , the edge vw is in a maximum matching N of H . The matching $N - \{vw\}$ covers $A - \{v\}$ in $G - \{v, w\}$. Thus the result follows by induction.

For example, the above lemma shows that if a graph G has a perfect matching, then $\alpha_g(G) \geq \alpha(G)$, here $\alpha(G)$ is the independence number of G . We next use the above lemma to provide a lower bound on the matcher number in terms of the lower matching number.

Lemma 8.

If G is a graph, then $\alpha_g(G) \geq 2\alpha(G) - 2\alpha_L(G)$.

Proof.

Consider a maximum matching M and a minimum maximal matching N . Then the vertices of $V(G) - V(N)$ form an independent set. Further, at most $n - 2|M|$ of them are not covered by M , where n is the order of G . That is, there is a subset of $V(G) - V(N)$ of at least $(n - 2|N|) - (n - 2|M|) = 2|M| - 2|N|$ vertices, all of whose vertices are M -matched. By Lemma 7, it follows that $\alpha_g(G) \geq 2|M| - 2|N|$.

For example, if the lower matching number is half the matching number, then the matcher and matching numbers are equal.

As another consequence of Lemma 8 and the earlier trivial observation that $\alpha_g(G) \geq \alpha_L(G)$, we obtain the following lower bound:

Theorem 4.

If G is a graph then $\alpha_g(G) \geq 2\alpha(G)/3$.

Proof.

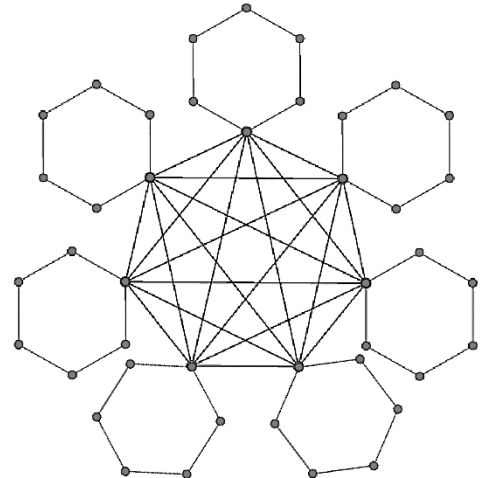


Figure 7. Depiction of $\text{cor}(K_7, C_6)$

As long as an edge remains, Max plays an edge belonging to a maximum matching; this reduces the matching number by 1. When $\alpha(G) \geq 3$, the edge played by Min in response is incident to at most two edges of a maximum matching and hence reduces the matching number (of the residual graph) by at most 2. Hence a round reduces α by at most 3 while adding 2 to the number of edges played. When $\alpha(G) = 3$ and Max starts, two more edges will be played.

For graph G and integer $k \geq 3$, define the generalized corona $\text{cor}(G, C_k)$ as follows.

For each vertex v of G introduce a disjoint k -cycle and identify one vertex of the cycle with v ; we call the vertex v the root of the cycle. For example, here is a depiction of $\text{cor}(K_7, C_6)$.

Lemma 9.

If graph G has order n and $k \geq 3$ is odd, then $\alpha_g(\text{cor}(G, C_k)) \leq n(k-1)/2$.

Proof.

In each copy of C_k , color red both edges incident with the root and then every alternate edge. Note that every root is incident with exactly two red edges and every other vertex is incident with exactly one red edge. The strategy for Minimizer is to prefer red edges. That is, if possible, she chooses a neighbor connected by a red edge to Maximizer's choice, otherwise she chooses any neighbor.

We claim that Minimizer's strategy means that every root will be matched to a vertex inside its copy of C_k . Focus on some copy D , and let F be the set consisting of the root and its two neighbors in D . The strategy for Minimizer means that if Maximizer chooses a vertex in $D \setminus F$, then so does Minimizer. Thus, Maximizer must be the first to choose a vertex in F ; and when he does so, Minimizer chooses its neighbor in F . Thus the root is matched within D .

It follows that at the end of the game, at least one vertex from each copy of C_k is unmatched. That is, the maximal matching obtained has size at most $(nk - n)/2$.

It follows that $\text{cor}(G, C_3)$ is an example of equality in Theorem 4 whenever G has a perfect matching.

4.1. Odd Girth

We now consider a generalization of Theorem 4. Recall that the odd-girth of a nonbipartite graph G is the shortest length of an odd cycle in G .

Theorem 5.

If graph G has odd girth g , then $\alpha_g(G) \geq \frac{g-1}{g} \alpha(G)$.

Proof.

The proof is by induction on the order. If there is an isolated vertex, then we can just discard it and induct; so, assume that G is isolated-free. If there is a liberal vertex u in G , then Maximizer chooses that vertex and we induct after Minimizer's response. So, we may assume there is no liberal vertex. In particular, by Lemma 5, the graph has a perfect matching, say M .

As usual we color red each edge that is in no perfect matching.

Say Maximizer starts with vertex u and Minimizer responds with vertex v . If the edge uv is in some perfect matching, then again, we can just induct on $G - \{u, v\}$. So we may assume that the edge uv is red. Let Q_1 be the path uv . Now the initial strategy of Maximizer will be to do the following as long as it is possible:

choose the M -partner of one end of the current path, where by M -partner of vertex w we mean the vertex w' such that ww' is in M . So, we obtain a series of M -alternating paths Q_1, Q_2, \dots such that each Q_i has $2i$ vertices and starts and finishes with an edge not in M .

If such a choice were always valid, then Maximizer would match off the entire graph, a contradiction of the fact that uv is red. So, Maximizer becomes stuck somewhere; that is, he cannot choose the M -partner of either end of the current path since neither has an unchosen neighbor. Say we have path Q_k with ends x and y . Let x' be the M -partner of x and y' the M -partner of y . Since Maximizer is stuck, all neighbors of x' and y' are on Q_k .

Suppose that $k < g - 1$. By assumption, vertex x' is not liberal and so is incident to a red edge, say $x'x^*$. If the edge $x'x^*$ creates an even cycle with the path Q_k , then we have a contradiction, since every edge in such an M -alternating cycle is in a perfect matching. So, the edge $x'x^*$ must create an odd cycle with Q_k . Similarly, y' is joined by a red edge to some vertex y^* on Q_k , that creates an odd cycle. By the odd-girth condition, x^* is closer to y than y^* is along the path Q_k . See figure.

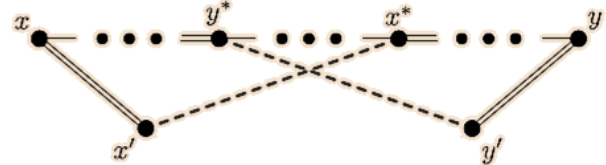


Figure 8. M -alternating cycle formed by $x'x^* \dots yy'y^* \dots xx'$

But then there is an M -alternating cycle formed by $x'x^* \dots yy'y^* \dots xx'$. It follows that the edge $x'x^*$ is in a perfect matching, a contradiction of the claim that it is red.

That is, it must be the case that $k \geq g - 1$. Then the remaining graph $G' = G \setminus (V(Q_k) \cup \{x', y'\})$ has a perfect matching, and we can apply induction to G' . The final matching is the union of a matching that has size k from $Q_k \cup \{x', y'\}$ and a matching that has size at least $(g-1)/g$ of $\alpha(G)$. The lower bound follows since $k/(k+1) \geq (g-1)/g$.

Note that for k odd, the generalized corona $\text{cor}(H, C_k)$ has a perfect matching if and only if the graph H has a perfect matching. By Lemma 9, it follows that $\text{cor}(H, C_k)$ is an example of equality in Theorem 5 for all odd k and graphs H with a perfect matching and girth at least k .

Using the ideas in the above proof, one can characterize the graphs that achieve equality in Theorem 4.

Theorem 6.

If G is an isolate-free graph, then $\alpha_g(G) = 2\alpha(G)/3$ if and only if it is isomorphic to $\text{cor}(H, C_3)$ for some graph H with a perfect matching.

Proof.

Suppose that $\alpha_g(G) = 2\alpha(G)/3$. Consider the proof of Theorem 5 for the case that $g = 3$. To obtain equality, it is necessary that every vertex is incident with a red edge and that there is a perfect matching M . Further, in applying his strategy, Maximizer must get stuck with $k = g - 1$. So we have an alternating path Q_2 with 4 vertices. Let B denote the subgraph induced by $V(Q_2) \cup \{x', y'\}$. See figure.



Figure 9. Alternating path Q_2 with 4 vertices

Note that vertices x' and y' have degree 2. Since Maximizer can start with any vertex, it follows that every vertex has a neighbor of degree 2. In particular, the degree-2 vertices come in pairs.

For equality, one needs that the graph $G' = G \setminus B$ is also an example of equality. By induction, the graph G' is isomorphic to $\text{cor}(H', C_3)$ where H' is a graph with a perfect matching. Since the degree-2 vertices form a paired dominating set, it follows that in G the only edges between B and $\text{cor}(H', C_3)$, if any, are between the vertices that already have degree at least 3 in that subgraph. Thus, the original graph G has the desired structure.

In general, it seems likely that the only graphs that achieve equality in Theorem 5 are the ones already noted; that is, $\text{cor}(H, C_k)$ for k odd and H a graph with a perfect matching and girth at least k . As in the proof of Theorem 6, one can readily show that graphs with equality in Theorem 5 have $\text{cor}(H, C_g)$ as a spanning subgraph, and by the odd-girth condition, the copies of C_g are induced. What seems harder to resolve is the absence or existence of other edges between the cycles.

5. Dense Graphs

We consider some dense graphs and show that their matcher number is half their order. It is well-known that if the minimum degree is at least half the order, and the order is even, then the graph has a perfect matching. (For example, this follows from Dirac's theorem [12].)

It follows from Theorem 3 that graphs of even order n with minimum degree at least $n - 2$ have matcher number $n/2$.

(Since such minimum degree implies the graph is complete multipartite.) Here is a tiny improvement on this.

Lemma 10.

If graph G has even order $n \geq 6$ and minimum degree $\delta(G) \geq n - 3$, then $\alpha_g(G) = n/2$.

Proof.

The strategy for Maximizer is simply to choose any vertex of minimum degree.

Consider the situation when four vertices remain. The current graph, say G_4 , has minimum degree at least 1. Suppose it does not have a perfect matching. Then G_4 is (isomorphic to) the star with three edges. Let A denote the set of the three end-vertices in G_4 . Now, go back to the graph, say G_6 , with six vertices remaining. That graph G_6 has minimum degree at least 3. So all three vertices of A are adjacent in G_6 to the two vertices matched in G_6 ; this means that in G_6 each vertex in A has degree 3 while the vertices matched each have degree at least 4. This is a contradiction of Maximizer's strategy.

Thus, it follows that G_4 has a perfect matching. It is easily checked that the strategy of choosing a vertex of minimum degree ensures that the resultant maximal matching is perfect.

One can improve the above lemma slightly. Computer search shows that all graphs of order 8 and minimum degree at least 4 have matcher number 4. It follows that if graph G has even order $n \geq 8$ and minimum degree at least $n - 4$, then the matcher number is $n/2$. (Maximizer plays arbitrarily until eight vertices remain, and then uses the optimal strategy.)

Lemma 11.

If two graphs have the same order m , then the matcher number of their join is m .

Proof.

The vertex set of the join can be partitioned into sets A and B of size m such that every edge between A and B is present. If both A and B are independent sets, then the graph is $K_{m,m}$ and we are done. So, assume there is some edge e with both ends in A .

Then Maximizer starts by choosing any vertex in B . If Minimizer takes a vertex of A , then the result follows by induction (as the remaining graph is the join of two graphs of order $m - 1$). So, assume that she chooses a vertex of B .

If only two vertices of A remain, then we are done, since they are joined by edge e and every other vertex has been matched. So, assume more than two vertices remain in A . Then Maximizer chooses a vertex in A but avoiding both ends of e . If Minimizer chooses a vertex in A , we are back to having equal number of vertices in the two sets, and so can induct.

So, assume that she chooses a vertex of B . But then Maximizer chooses another vertex in A , avoiding the ends of e , and the argument repeats.

6. Conclusions

We provided bounds on the matcher games. It seems that the Max-start matcher game might be interesting for special families of graphs. For example, rooks' graphs or cartesian products in general seem worthy of consideration, as do cubic graphs or chordal graphs. In another direction, a

natural question is the complexity of the game.

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