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The Topp-Leone Rayleigh Distribution with Application

Fatoki Olayode

Department of Statistics, Ogun State Institute of Technology, Igbesa, Ogun State

Abstract We introduce a new lifetime distribution which is the generalization of Rayleigh distribution using the Topp-Leone generated family of distributions proposed by Rezaei et al. The new distribution is called the Topp-leone Rayleigh (TLR) distribution. The new distribution was found to be more flexible in modeling data that exhibits increasing, decreasing, non-monotone failure rate. Expressions for several probabilistic measures were provided, such as probability density function, hazard function, moments, quantile function, mean, variance and median, moment generating function, orders statistics etc. Inference is maximum likelihood based and tractability of model was shown by its application to a real data set.

Keywords Moment generating function, Rayleigh distribution, Survival function, Order statistics

1. Introduction

Rayleigh distribution was first introduced by Rayleigh [6], Siddiqui [8] discussed theorigin and properties of the Rayleigh distribution also Lalitha et al. [4] and Abd Elfattah et al. [3]. Faton Merovci [5], generalizes the Rayleigh distribution using the quadratic rank transmutation map which was introduced by Shaw et al. [7] and named it Transmuted Rayleigh distribution. Inference for Rayleigh model has been considered by Sinha and Howlader [9]. Ahmad et al. [2] developed the Transmuted Inverse Rayleigh distribution and discussed its properties, Afaq Ahmad et al [1], examined the characterisation of the transmuted Rayleigh distribution. The probability density function (pdf) of Rayleigh distribution is given as:

$$f(x;\eta) = 2\eta x e^{-\eta x^2} \tag{1}$$

For real positive values of the variable x and a real positive parameter η. It is named after the British physicist Lord Rayleigh (1842–1919). parameter η is simply a scale factor.

Integrating the equation (1) above will yield the cumulative density function of Rayleigh distribution given as

$$F(x;\eta) = 1 - e^{-\eta x^2} \tag{2}$$

2. Topp-leone Rayleigh Distribution

We shall refer to the new distribution using (3) and (4) as the Topp-Leone Rayleigh (TLR) distribution using the Topp-Leone generated (TLG) family of distributions which was introduced by Rezaei, S et al. The pdf and cdf of the TLG family of distributions are given by

$$G(x;\zeta) = F(x;\zeta)^{\alpha}(2 - F(x;\zeta))^{\alpha}, \tag{3}$$

And

$$g(x;\zeta) = 2\alpha f(x;\zeta)\big(1 - F(x;\zeta)\big)F(x;\zeta)^{\alpha-1}\big(2 - F(x;\zeta)\big)^{\alpha-1} \tag{4}$$

Where $\alpha > 0$ is the shape parameter and ζ is the parameter vector of the baseline distribution G inserting equation (2) into (3), we have the cumulative density function of Topp-Leone Rayleigh distribution given as

$$F(x) = \left(1 - e^{-\eta x^2}\right)^{\alpha}\left(1 + e^{-\eta x^2}\right)^{\alpha} \quad x > 0, \eta, \alpha > 0 \tag{5}$$

The graph of the cumulative density of the TLR distribution is given below for various values of the parameters.

By differentiating the equation (5) above, we obtain the Pdf of TLR distribution given as

$$f(x) = 4\alpha\eta x e^{-2\eta x^2}\left(1 - e^{-\eta x^2}\right)^{(\alpha-1)}\left(1 + e^{-\eta x^2}\right)^{(\alpha-1)}; \quad x > 0, \eta, \alpha > 0 \tag{6}$$

Using both the Taylor series and Prudnikov series expansion which is given respectively as follows

$$(1 - m)^z = \sum_{j=1}^{\infty}\binom{z}{j}(-1)^j m^j \tag{7}$$

$$(1 + m)^z = \sum_{k=0}^{\infty}\frac{\Gamma(z)}{\Gamma(z-k+1)}\frac{m^k}{k!} \tag{8}$$

* Corresponding author:
fatoki_olayode@yahoo.com (Fatoki Olayode)
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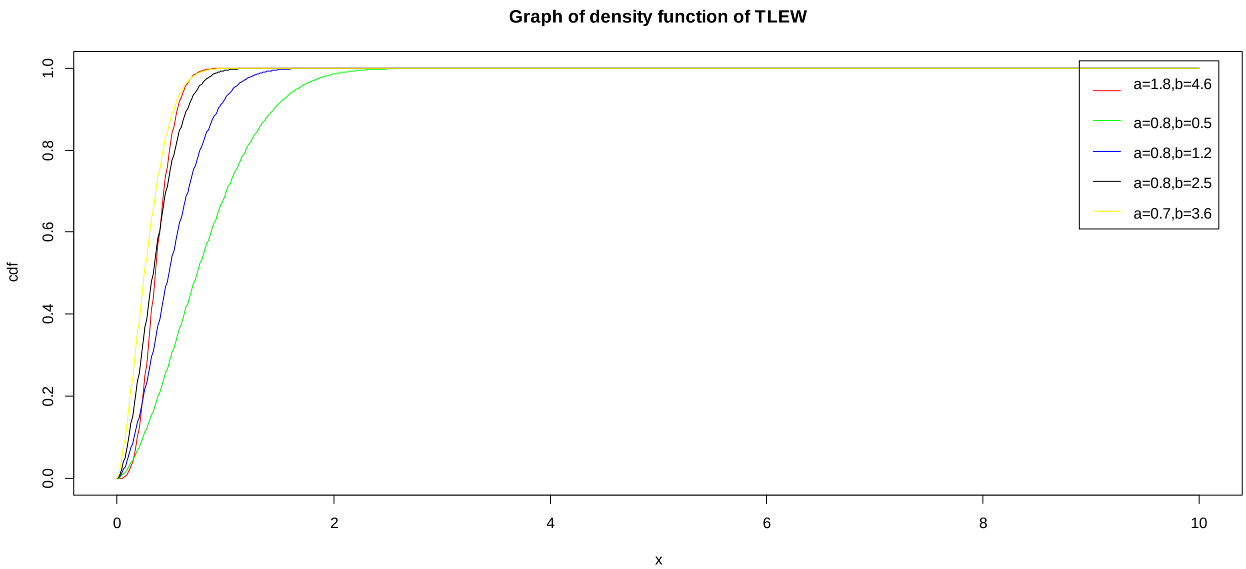

Figure 1. The figure above shows the pdf of TLR distribution is a proper cdf

On the equation (6) above then we have

$$f(x) = 4\alpha\eta \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\binom{\alpha+1}{j}(-1)^j \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)}\frac{xe^{-\eta x^2(j+k+1)}}{k!} \quad (9)$$

The graph of the probability density function for the TLR distribution is given below for various values of the parameters.

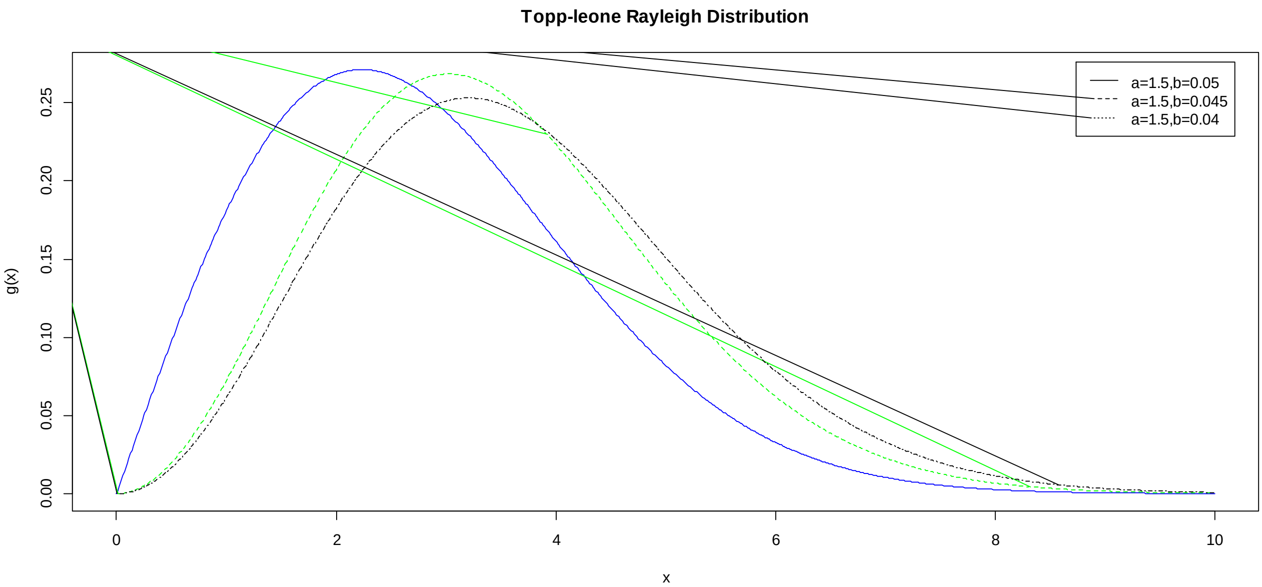

Figure 2. The graph of the pdf of TLR distribution

✓ The graph of the pdf of ECTW distribution drawn above indicates that the distribution is positively skewed and unimodal

3. Reliability Function

The reliability function or the survival function of a random variable X is defined by $R(x) = P(X > x) = 1 - F(x)$. It could be interpreted as the probability of a system not failing before some specified time t. The reliability function of the Topp-leone Rayleigh is given by

$$R(x) = 1 - \left(1 - e^{-\eta x^2}\right)^{\alpha}\left(1 + e^{-\eta x^2}\right)^{\alpha} \quad (10)$$

The graph of the Reliability function of Topp-leone is drawn below for various values of the parameters.

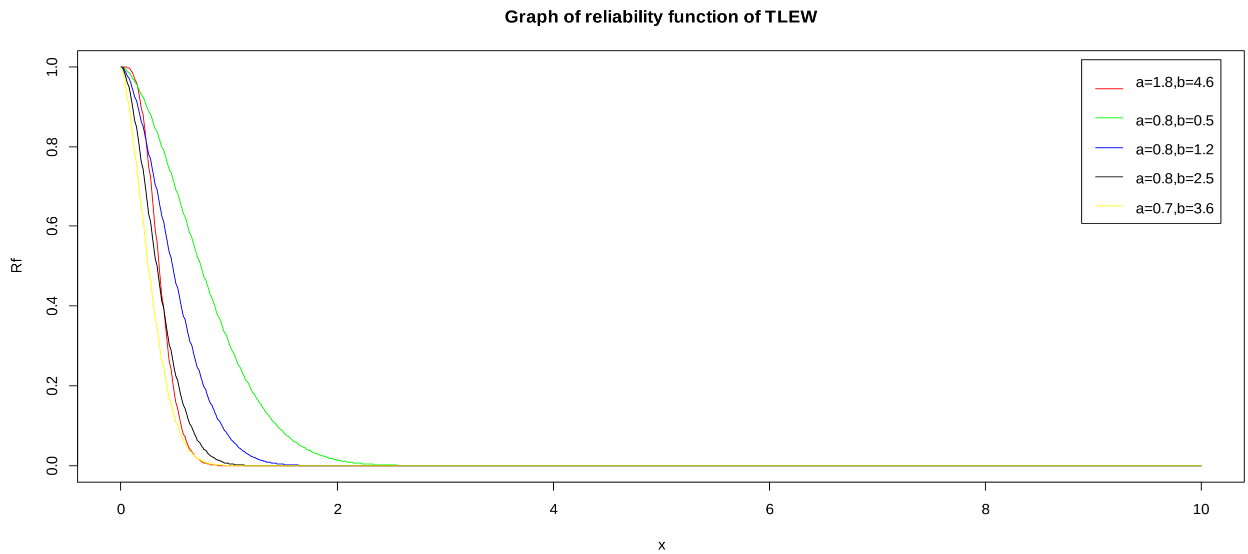

Figure 3. The graph of the survival function of TLR distribution

✓ The graph of the reliability function drawn above shows that as the time increases the reliability probability decreases

4. Hazard Function

The hazard rate function $h(x)$ or the instantaneous failure rate of a random variable X is the probability that a system fails given that it has survived up to time t and is given by

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{R(x)} \tag{11}$$

Then the hazard rate function of TLR distribution is given as

$$h(x) = \frac{4\alpha\eta x e^{-2\eta x^2}\left(1 - e^{-\eta x^2}\right)^{(\alpha-1)}\left(1 + e^{-\eta x^2}\right)^{(\alpha-1)}}{1 - \left(1 - e^{-\eta x^2}\right)^{\alpha}\left(1 + e^{-\eta x^2}\right)^{\alpha}} \tag{12}$$

The graph of the hazard function for various values of the parameters is given as

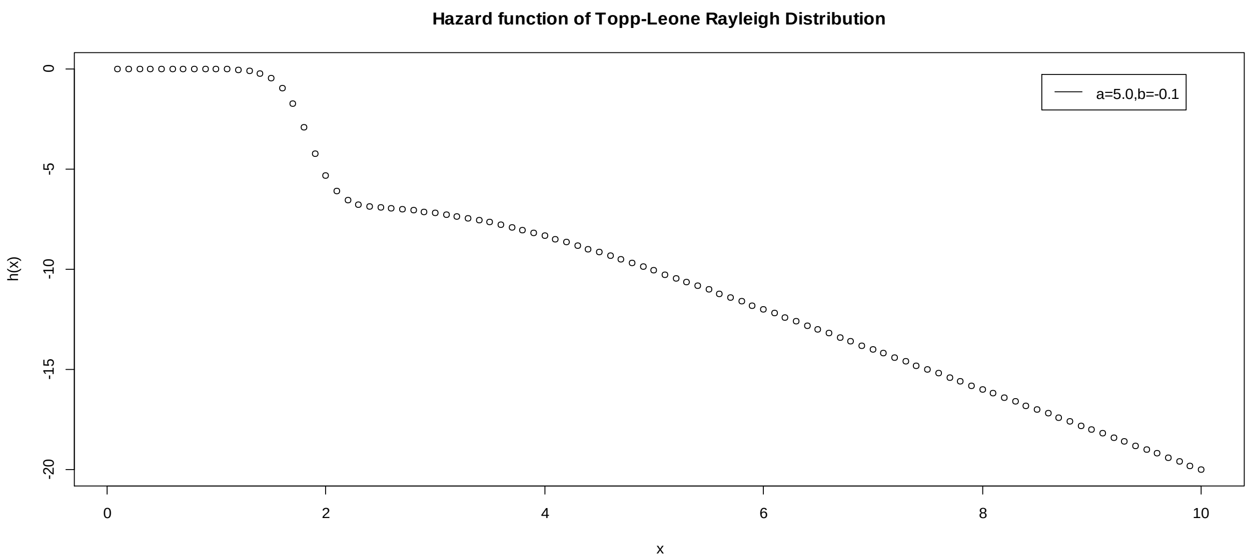

Figure 4. The graph of the hazard function of TLR distribution

5. Moments, Mean, Variance, Kurtosis and Skewness of (TLR) Distribution

In this section we shall present the moments and qunatiles for the (TLR) distribution. The k^{th} order moments, for (TLRD) can be obtained as follows for a random variable X,

$$E(X)^r = \int_{-\infty}^{\infty} x^r f(x)dx \tag{13}$$

Putting equation (7) in (13) we have

$$E(X)^r = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\binom{\alpha+1}{j}(-1)^j \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)k!}\int_{-\infty}^{\infty} x^{r+1}\, e^{-\eta x^2 (j+k+1)} dx \tag{14}$$

By letting $w = \eta x^2 (j + k + 1)$, finally we have

$$E(X)^r = \mu_r' = 4\alpha\eta \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\binom{\alpha+1}{j}(-1)^j \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)k!}\left[\frac{1}{\alpha(j+k+1)}\right]^{\frac{r+2}{2}} \Gamma\left(\frac{r+k}{2}\right) \tag{15}$$

Using eq. (15), we obtain the $1^{st}, 2^{nd}, 3^{rd}, and\ 4^{th}$ moment for $r = 1, 2, 3, 4$.

$$\mu_1' = 4\alpha\eta \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\binom{\alpha+1}{j}(-1)^j \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)k!}\left[\frac{1}{\alpha(j+k+1)}\right]^{\frac{3}{2}} \Gamma\left(\frac{1+k}{2}\right) \tag{16}$$

$$\mu_2' = 4\alpha\eta \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\binom{\alpha+1}{j}(-1)^j \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)k!}\left[\frac{1}{\alpha(j+k+1)}\right]^{2} \Gamma\left(\frac{2+k}{2}\right) \tag{17}$$

$$\mu_3' = 4\alpha\eta \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\binom{\alpha+1}{j}(-1)^j \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)k!}\left[\frac{1}{\alpha(j+k+1)}\right]^{\frac{3}{2}} \Gamma\left(\frac{3+k}{2}\right) \tag{18}$$

$$\mu_4' = 4\alpha\eta \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\binom{\alpha+1}{j}(-1)^j \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)k!}\left[\frac{1}{\alpha(j+k+1)}\right]^{3} \Gamma\left(\frac{4+k}{2}\right) \tag{19}$$

The mean of TLR distribution is the first moment about the origin (μ_1') which corresponds to equation (16). It then follows that the variance (μ_2), the coefficient of variation (ρ), the coefficient of skewness (γ_1), and the coefficient of kurtosis (γ_2) of the TLR distribution are respectively, obtained as

$$\mu_2 = \mu_2' - (\mu_1')^2 \tag{20}$$

$$\rho = \frac{\sqrt{\mu_2}}{\mu_1'} = \frac{\sqrt{\mu_2' - (\mu_1')^2}}{\mu_1'}, \tag{21}$$

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^2}{\{\mu_2' - (\mu_1')^2\}^{\frac{3}{2}}} \tag{22}$$

$$\gamma_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3(\mu_1')^2}{\{\mu_2' - (\mu_1')^2\}^2} \tag{23}$$

6. Moment Generating Function of TLR Distribution

The moment generating function of a random variable x is defined by

$$M_t(x) = \int_{-\infty}^{\infty} e^{tx} f(x) \tag{24}$$

The above expression can further be simplify as

$$M_t(x) = \sum_{k=0}^{\infty}\frac{t^r}{r!}\int_{-\infty}^{\infty} x^r f(x)dx \tag{25}$$

Since,

$$e^{tx} = \sum_{r=0}^{\infty}\frac{t^r x^r}{r!} \tag{26}$$

Inserting equation (15) in equation (26) we have

$$M_t(x) = 4\alpha\eta \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\binom{\alpha+1}{j}(-1)^j \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)k!}\left[\frac{1}{\alpha(j+k+1)}\right]^{\frac{r+2}{2}}\Gamma\left(\frac{r+k}{2}\right) \quad (27)$$

7. Quantile Function, Median and of the TLR Distribution

The quantile function x_q of the TLR distribution can be obtained as the inverse of Equation (6) as

$$x_q = \left\{-\frac{ln\left[1-u^{\frac{1}{\alpha}}\right]}{2\eta}\right\}^{\frac{1}{2}} \quad (28)$$

And the median can be obtained as

$$x_{0.5} = \left\{-\frac{ln\left[1-u^{\frac{1}{\alpha}}\right]}{2\eta}\right\}^{\frac{1}{2}} \quad (29)$$

The lower quartile and upper quartile can also be derived from Equation (28) when q = 0.25 and q = 0.75 respectively.

8. Maximum Likelihood Estimators

If $x_1, x_2, \ldots, x_n$ is a random sample from Topp-Leone Rayleigh distribution given by (6), then the Likelihood function (L) becomes:

$$L = \prod_{i=1}^{n} f(x_i, \alpha, \beta) \quad (30)$$

By substituting from equation (6) into Equation (30), we get

$$L = \prod_{i=1}^{n} 4\alpha\eta x_i e^{-2\eta x_i^2}\left(1-e^{-\eta x_i^2}\right)^{(\alpha-1)}\left(1+e^{-\eta x_i^2}\right)^{(\alpha-1)} \quad (31)$$

Then the log – likelihood (l) function becomes

$$l = \ln(\alpha) + \ln(\eta) + \sum_{i=1}^{n}\ln(x_i) - 4\eta\sum_{i=1}^{n}\ln(x_i) + (\alpha-1)\sum_{i=1}^{n} ln\left(1-e^{-\eta x_i^2}\right) + (\alpha-1)\sum_{i=1}^{n} ln\left(1+e^{-\eta x_i^2}\right) \quad (32)$$

The Normal equation become

$$\frac{dl}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln\left(1-e^{-\eta x_i^2}\right) + \sum_{i=1}^{n} ln\left(1+e^{-\eta x_i^2}\right) \quad (33)$$

$$\frac{dl}{d\eta} = \frac{n}{\eta} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\eta x_i^2}}{\left(1-e^{-\eta x_i^2}\right)} - (\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\eta x_i^2}}{\left(1+e^{-\eta x_i^2}\right)} \quad (34)$$

Table 1. Descriptive Statistics of the failure data

Min	Q_1	Median	$mean$	Q_3	Max	$kurtosis$	Range	Skewness
1.00	2.75	54.00	92.07	118.00	603.00	5.20	600	2.17

Table 2. MLEs (standard error in parenthesis) and the statistics $l(\hat{\theta})$, AIC, BIC and $HQIC$

Distributions	$Parameter\ Estimates$			l	AIC	BIC	$HQIC$
$TESSG\ II$ (α, η, λ)	1.4004 (0.0254)	0.0188 (0.0020)	11.2907 (1.0062)	−260.78	527.56	542.28	533.15
$G\ II$ (α, η)	1.0287 (0.0179)	0.2854 (0.0131)	–	−448.68	901.37	911.18	905.10

The MLE of $\alpha, and\, \eta$ can be obtain by solving the eq. (33) and (34), using $\frac{dl}{d\alpha} = 0, and\ \frac{dl}{d\eta} = 0$.

9. Application

We consider the number of failures for the air conditioning system of jet airplanes. These data were reported by Cordeiro and Lemonte (2011) and Huang and Oluyede (2014): 194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18,209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220,120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182,71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46,230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71. The data was observed to possess unimodal failure rate.

Table 1 gives the exploratory data analysis of the data, Table 2 provides the maximum likelihood estimate of the unknown parameters (and the corresponding standard errors in parentheses) and the measure of goodness-of-fit tests that was used to verify which distribution fits better to the data set between the Topp-Leone Rayleigh distribution and the Rayleigh distribution. We consider the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and the Hanna Quinn Information Criteria (HQIC) as the selection criteria.

10. Conclusions

The smaller the values of the statistics which measures the goodness of fits the better the model in fitting the data. Since the Topp-leone Rayleigh distribution possesses the smallest Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and the Hanna Quinn Information Criteria (HQIC), it can be considered to be the better model than the Rayleigh in modeling the data.

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