American Journal of Mathematics and Statistics 2019, 9(5): 199-202
DOI: 10.5923/j.ajms.20190905.03

Contraction Conditions in Probabilistic Metric Space

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Abstract The classical Banach contraction principle in metric space is one of the fundamental results in metric space with wide applications. And the probabilistic metric space is one of the important generalizations of metric space introduced by Austrian mathematician **Karl Menger** in 1942. The purpose of this article is to describe different contraction conditions in Probabilistic Metric Space. Also, mention the generalized contraction conditions and interrelationships between contraction conditions.

Keywords Fixed Point, t-norm, Probabilistic metric space, Contraction condition

1. Introduction and Preliminaries

In mathematics, analysis plays an important role in the development of mathematics. Among several branches of analysis, functional analysis which deals with the study of several functions, come under Functional Analysis. It describes two types of functional analysis one is linear and another is non-linear functional analysis.

Fixed point theory is one of the most important topics of non-linear functional analysis since 1960. It has wide applications to the numerous fields of mathematics as well as outside mathematics such as differential equations, integral equations, variational problems, optimization problems, game theory, graph theory, image and signal processing, economics, and many more.

The notion of distance later known as metric space, introduced by M. Frechet in 1906, furnishes the common idealization of a large number of mathematical, physical and other scientific constructs in which the distance of a 'distance' appears. The objects under consideration may be most varied. They may be points, functions, sets, and even the subjective experiences of sensation. What matters is the possibility of associating a non-negative real number with each ordered pair of elements of a certain set, and that the number associated with pairs and triples of such elements satisfy certain conditions. However, in numerous instances in which the theory of metric spaces is applied, this very association of a single number with a pair of elements is, realistically speaking, an over idealization. This is so even in the measurement of an ordinary length, where the number given as the distance between two points is often not the result of a single measurement, but the average of a series of measurements. Indeed, in this and many similar situations, it is appropriate to look upon the distance concept as a statistical rather a determinate one. More precisely, instead of associating a number - the distance $d(p,q)$ - with every pair of elements p,q, one should associate a distribution function Fpq any for any positive number x, interpret $Fpq(x)$ as the probability that the distance from p to q less than x. When this is done one obtains a generalization of the concept of metric space - a generalization which was first introduced by Austrian Mathematicians Karl Menger in 1942 and following him, is called a statistical metric space [8].

In this paper, we analyze the different contraction conditions in probabilistic metric space and their inter-relationships.

Definition 1.1: **Metric space** is a pair (S,d), where S is a non-empty set and d is a distance function or metric of the space defined by $d: S\times S \to [0,\infty)$, satisfies the following conditions:

(i) $d(p,q)=0$ if $p=q$ (Indiscrinibles)
(ii) $d(p,q)>0$ if $p\neq q$ (Positivity)
(iii) $d(p,q)=d(q,p), \forall\, p,q\in S$ (Symmetry)
(iv) $d(p,r)\leq d(p,q)+d(q,r), \forall\, p,q,r\in S$ (Triangle Inequality)

Example 1.1: Let X be a non-empty set. For $x,y\in X$, we define

$$d(x,y)=\begin{cases}0 & if\ x=y\\ 1 & if\ x\neq y\end{cases}$$

Then, d is discrete metric and the space (X,d) is discrete metric space.

Definition 1.2: Let $f: X\to X$ be a map. Then, an element $x\in X$ is said to be **fixed point** of f if $f(x)=x$.

Example 1.2: Let $y=f(x)=x^3-4x^2+x+6=0$, cubic equation.

Then, it can be transferred to as

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Published online at http://journal.sapub.org/ajms

$$x = f(x) = \frac{x^3 + 6}{4x - 1}$$

Here, $f(-1) = -1, f(2) = 2 \;\&\; f(3) = 3$.

So, by definition $x = -1, x = 2 \; and \; x = 3$ are fixed points of f.

Definition 1.3: Let (X, d) be a metric space and let $f: X \to X$ be a mapping. Then, f is called contraction if there exists a fixed constant $h \in [0,1)$, such that

$$d\big(f(x), f(y)\big) \le hd(x, y), \forall\, x, y \in X$$

Example 1.3: Let $f[0,2] \to [0,2]$ be defined by,

$$f(x) = \begin{cases} 0 & x \in [0,1] \\ 1 & x \in (1,2] \end{cases}$$

Then, $f^2(x) = 0$ for all $x \in [0, 2]$. So, f^2 is a contraction on $[0,2]$. But f is not continuous and thus not a contraction map.

Definition 1.4: For the set $\mathbb{R}$ of real numbers, a function $F: \mathbb{R} \to [0,1]$ is called a **distribution function** if

(i) F is non-decreasing,
(ii) F is left continuous, and
(iii) $\inf_{x\in\mathbb{R}} F(x) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$.

If X is a non-empty set, $F: X \times X \to \Delta$ is called probabilistic distance on X and $F(x, y)$ is usually denoted by F_{xy}. We will denote by Δ the family of all distribution function on $(-\infty, \infty)$ and Δ^+ on $[-0, \infty)$.

Example 1.4: Let H is a maximal element for Δ^+ then, distribution function H is defined by

$$H(x) = \begin{cases} 0, & if\; x \le 0, \\ 1, & if\; x > 0. \end{cases}$$

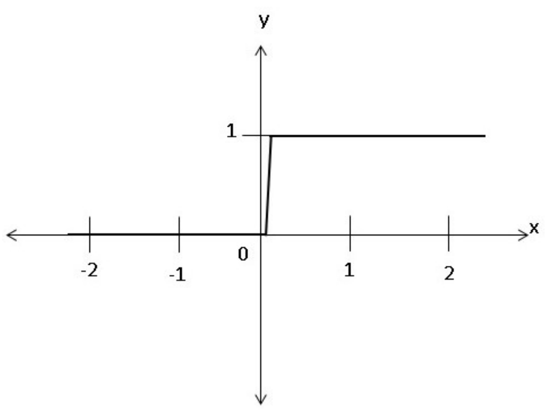

Figure 1. Distribution Function

Definition 1.5: [13] A **probabilistic metric space** (brief, PM-space) is an order pair (X, F) where X is a non-empty set and F is a function defined by $F: X \times X \to \Delta^+$ (the set of all distribution functions) that is F associates a distribution function $F(p, q)$ with every pair (p, q) of points in X. The distribution function $F(p, q)$ is denoted by $F_{p,q}$, whence the symbol $F_{p,q}(x)$ will represent the value of $F_{p,q}$ at $x \in \mathbb{R}$. And the function $F_{p,q}, p, q \in X$ are assumed to satisfy following conditions:

(i) $F_{p,q}(0) = 0$; (ii) $F_{p,q} = F_{q,p}$, (iii) $F_{p,q}(x) = 1$, for every $x > 0 \Leftrightarrow p = q$.

(iv) For every $p, q, r \in X$ and for every

$$x, y > 0, F_{p,q}(x) = 1, F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x + y) = 1$$

The interpretation of $F_{p,q}(x)$ as the probability that the distance from p to q is less than x, it is clear that PM condition (iii), (i) and (ii) are straight forward generalizations of the corresponding metric space conditions (i), (ii) and (iii). The PM condition (iv) is a 'minimal' generalization of the triangle inequality of metric space condition (iv). If it is certain that the distance of p and q is less than x, and like wise certain that the distance of q and r is less than y, then it is certain that the distance of p and r is less than $x + y$. The PM condition (iv) is always satisfied in metric spaces, where it reduces to the ordinary triangle inequality.

Definition 1.6: [6] A mapping $T: [0,1] \times [0,1] \to [0,1]$ is called a **triangular norm** (shortly t-norm) if for all $a, b, c, d, \in [0,1]$ the following conditions are satisfied:

(i) $T(a, 1) = a$ for every $a \in [0,1]$, (Neutral Element 1)
(ii) $T(a, b) = T(b, a)$ for every $a, b \in [0,1]$, (Commutativity)
(iii) $T(a, b) \le T(c, d)$ whenever $a \le c \; and \; b \le d$ (Monotonicity)
(iv) $T\big(a, T(b, c)\big) = T(T(a, b), c))(a, b, c \in [0,1])$. (Associativity)

Example 1.5 of t-norms
$T(a, b) = \max\{(a + b) - 1, 0\}$ and
$T(a, b) = \min\{a, b\}$

The four basic standard t-norms are:

(i) The minimum t-norm, T_M, is defined by $T_M(x, y) = \min\{x, y\}$,
(ii) The product t-norm, T_p, is defined by $T_p(x, y) = x, y$,
(iii) The Lukasiewicz t-norm, T_L, is defined by $T_L(x, y) = \max\{x + y - 1, 0\}$,
(iv) The weakest t-norm, the drastic product, T_D, is defined by

$$T_D(x, y) = \begin{cases} \min(x, y) \; if & \max(x, y) = 1, and \\ 0 & otherwise \end{cases}$$

With references to the point wise ordering, we have the following inequalities

$$T_D < T_L < T_P < T_M.$$

Definition 1.7: [8] A **Menger probabilistic metric space** (briefly, Menger PM-space) is a triple (S, F, T), where (S, F) is a probabilistic metric space, T is a triangular norm and also satisfies the following conditions, for all $x, y, z \in X$ and $t, s > 0, (v) F_{xy}(t + s) \ge T(F_{xz}(t), F_{zy}(s))$. This is the extension of triangle inequality. This inequality is called Menger's triangle inequality.

Example 1.6: Let $X = \mathbb{R}, a * b = \min(a, b) \, \forall\, a, b \in (0,1)$ and

$$f_{u,w}(x) = \begin{cases} H(x) & for\; u \ne v \\ 1 & for\; u = v \end{cases}$$

where

$$H(x) = \begin{cases} 0 & if\ x \le 0 \\ x & if\ 0 \le x \le 1 \\ 1 & if\ x > 0 \end{cases}$$

then $(X, F, *)$ is Menger Space.

Definition 1.8: [4] Let (X, F, T) be a **Menger Space** and T be a continuous t-norm (1) A sequence $\{x_n\}$ in X is said to be **converge** to a point x in X (written $x_n \to x$) iff for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \ge N$.

(2) A sequence $\{x_n\}$ in X is called a **Cauchy** if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \ge N$.

(3) A Menger space in which every Cauchy sequence is convergent is said to be **Complete Menger Space**.

Banach Contraction Condition in Metric Space: The most basic fixed-point theorem is analysis known as the Banach Contraction Principle (BCP). It is due to S. Banach [1] and appeared in his Ph.D. thesis (1920, published in 1922). The BCP was first stated and proved by Banach for the Contraction maps in setting of complete normed linear spaces. At about the same time the concept of an abstract metric space was introduced by Hausdorff for the set valued mappings, which then provided the general framework for the principle for contraction mappings in a complete metric space. The BCP can be applied to mappings which are differentiable, or more generally, Lipschitz continuous.

Theorem 1.1: Let (X, d) be a complete metric space, then each contraction map $f: X \to X$ has a **unique fixed point**.

Example 1.7: $T: \mathbb{R} \to \mathbb{R}, T(x) = \frac{x}{2} + 3, x \in \mathbb{R}$. Obviously T is a Banach contraction and

$\mathrm{Fix}(T) = \{6\}$ where $\mathrm{Fix}(T)$ denotes the fixed point of the mapping T.

2. Contraction Conditions in Probabilistic Metric Space

2.1. V.M. Seghal and A.T. Bharucha-Reid (B) Contraction Conditions in PM Space

The following definition of a contraction mapping was suggested and studied by V.M. Seghal and A.T. Bharucha-Reid in 1972, which is very natural probabilistic version of the notion of Banach contraction in metric space.

Definition 2.1.1: [12] The following definition of a contraction mapping was suggested and studied by V.M. Seghal and A.T. Bharucha-Reid in 1972, which is very natural probabilistic version of the notion of Banach contraction in metric space.

Let (X, F) be a probabilistic metric space. A mapping $T: X \to X$ is a **contraction mapping** (or a **SB - Contraction mapping or B-contraction**) on (X, F) if and only if there is a $k \in (0,1)$ such that

$$F_{Tp,Tq}(t) \ge F_{p,q}(t/k), \tag{2.1}$$

where $p, q \in X$ and $t > 0$. It is also known as probabilistic ***k*-contraction**.

The geometrical interpretation expression (2.1) is that the probability that the distance between the image points F_p, F_q being less than kx, is at least equal to the probability that the distance between p, q that is less than x.

Dentition 2.1.2: [2] Let (X, F) be a probabilistic metric space. A mapping $f: X \to X$ is a probabilistic q-contraction $(q \in (0,1))$ if

$$F_{f_{P1},f_{P2}}(x) \ge F_{P1,P2}(x/q) \tag{2.2}$$

for every $p_1, p_2 \in X$ and every $x \in \mathbb{R}$

It is obvious that $f: X \to X$ is a probabilistic q-contraction if and only if for every $p_1, p_2 \in X$ and every $x \in \mathbb{R}$ the following implication holds

$$(\forall \alpha \in (0,1))(F_{P1,P2}(x) > 1 - \alpha \Rightarrow F_{f_{P1},f_{P2}}(qx) > 1 - \alpha). \tag{2.3}$$

The inequality (2.2) is a generalization of inequality.

$$d(f_{p1}, f_{p2}) \le q d(p_1, p_2),$$

where $f: X \to X$ and (X, d) is a metric space. In order to prove that (2.3) implies (2.2) recall that every metric space (X, d) is also a Menger space (X, F, T_X), if F is defined in the following way:

$$F_{p1,p2}(x) = \begin{cases} 1 & if\ d(p_1, p_2) < x, \\ 0 & if\ d(p_1, p_2) \ge x\ for\ x\ in\ \mathbb{R} \end{cases} \tag{2.4}$$

Suppose that $f: X \to X$ is such that (2.3) holds and prove that (2.2) is satisfied i.e.,

that for every $x > 0$, we have

$$F_{p1,p2}\left(\frac{x}{q}\right) = 1 \Rightarrow F_{f_{p1},f_{p2}}(x) = 1$$

If $F_{p1,p2}\left(\frac{x}{q}\right) = 1$, then $d(p_1, p_2) < \frac{x}{q}$ and (2.3) implies

$$d(f_{p1}, f_{p2}) < q\frac{x}{q} = x,$$

which means that

$$F_{f_{p1},f_{p2}}(x) = 1$$

2.2. Hick's Contraction (C) in PM Space

Definition 2.2.1: [7] T.L. Hicks in 1996, defined the following C-contraction mapping in PM space.

Let (X, T) be a probabilistic metric space and $T: X \to X$. The mapping T is called **Hicks C-contraction (or, C-contraction)** if there exists $k \in (0,1)$ such that the following implication holds for every $p, q \in X$: and for every $t > 0$

$$T_{pq}(t) > 1 - t \Rightarrow T_{T(p)T(q)}(kt) > 1 - kt.$$

Definition 2.2.2: [9] **D.Mihet** in 2005, introduced the weak- hicks contraction in PM Space as follows:

Let S be a nonempty set and F be a probabilistic distance on S. A mapping $f: S \to S$ is said to be **weak - Hicks contraction (w-H contraction)** if there exists $k \in (0,1)$ such that, for all $p, q \in S$.

$$(w-H): t \in (0,1), F_{pq}(t) > 1-t \Rightarrow F_{f(p)f(q)}(kt) > 1-kt.$$

Example 2.2.1: Let $X=[0,\infty)$ and

$$F_{xy}(t) = \frac{\min(x,y)}{\max(x,y)}, \forall\, t \in (0,\infty), \forall\, x,y \in X, x \neq y.$$

It is known ([10], [11]) that (X,F,T) is a complete Menger space under the triangular norm $T=T_p > T_L$. Also, it can easily be seen that the mapping $g: X \to X$,

$$g(x) = \begin{cases} 0 & if\ x=0 \\ 1 & if\ x>0 \end{cases}$$

is a w-H contraction for every $k \in (0,1)$.

2.3. Generalization of Bharucha (B)-Contraction

As a generalization of the notion of a probabilistic B-contraction, we shall introduce the notion of a probabilistic (m,k) - B-contraction where $m \geq 1$ and $k \in (0,1)$.

Definition 2.3.1: [6] If (S,F) is a PM - space, $m \geq 1$ and $k \in (0,1)$, a function $f: S \to S$ is called probabilistic (m,k)-B-contraction if for any $p,q \in S$ there is an i with $1 < i < m$ such that for every $t > 0$,

$$F_{f^i,f^i q}(k^i t) \geq F_{q,q}(t).$$

If $m=1$ and $k \in (0,1)$ then a probabilistic $(1-k)$-B-contraction f is a probabilistic B-contraction.

As a generalization of C-contraction, we have

Definition 2.3.2: [6] If (S,φ) is a PM - space, $m \geq 1$ and $k \in (0,1)$, a function $f: S \to S$ is called a (m,k)-C-contraction if for any $p,q \in S$ there is an i with $l < i < m$ such that for every $t > 0$.

$$F_{p,q}(t) > 1-t \Rightarrow F_{f^i,f^i q}(k^i t) > 1-k^i t.$$

If $m=1$ and $k \in (0,1)$ then a probabilistic $(1,k)$-C-contraction f is a probabilistic C-contraction.

2.4. Probabilistic G-contraction Mapping

Definition 2.5.1: [5] g-contraction mapping is the generalization of Hick's C-contraction in Probabilistic Metric Space. Let f,g be two mappings defined on a Menger space (S,F,T) with values into itself and let us suppose that g is bijective. The mapping f is called a probabilistic g-contraction with a constant $k \in (0,1)$ if

$t > 0$ and $F_{g(x),g(y)}(t) > 1-t$ impies $F_{f(x),f(y)}(kt) > 1-kt$.

The notion of g-contraction is justified because the images of two points x,y under the function f are nearer than images of the same points under the function g.

3. Conclusions [3]

The Probabilistic **g**-contraction is Hicks **C**-contraction when **g** = **I**, an identity mapping. Since **H**-contraction need not be **B**-contraction. So, Probabilistic **g**-contraction need not be **B**-contraction. Moreover, **C**-contraction is an extension of Banach contraction in Probabilistic Metric Space.

It is clear that

(i) **(m-k)** contraction ⇒ **C**-contraction ⇒ **B**-contraction ⇒ Banach contraction
(ii) **g**-contraction ⇒ **C**-contraction ⇒ **B**-contraction
(iii) **C**-contraction ⇒ **(w-H)** contraction

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