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On the Krasnoselskii's Fixed Point Theorem and the Existence of Periodic Solution for a Damped and Forced Duffing Oscillator

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Abstract This paper is devoted to study the existence of periodic solution for a damped and forced Duffing oscillator using the Krasnoselskii's fixed point theorem in Banach space. As an application, uniqueness and compactness of solution of Duffing oscillator was achieved using Gronwall's Inequality and Eberlein Simultan theorem which extends some results in literature.

Keywords Krasnoselskii's Fixed Point Theorem, Banach Space, Compactness, Analytic Semigroup, Duffing Oscillator

1. Introduction

The aim of this paper is to study existence of periodic solution for a damped and forced Duffing oscillator of the form

$$\ddot{x} + c\dot{x} + ax + bx^2 + \beta x^3 = h(t) \quad (1.1)$$

with boundary conditions

$$x(0) = x(2\pi)$$
$$\dot{x}(0) = \dot{x}(2\pi) \quad (1.2)$$

In equation (1.1) a, b, c, β are real constants and $h(t)$ is continuous. Also, $h: [0\ 2\pi] \to \mathbb{R}^+$ is periodic in $t \in \mathbb{R}^+$. Duffing oscillator is a second order nonlinear differential equation used to model dynamics of special types of mechanical and electrical systems. This differential equation has been named after the studies of Duffing in [1] which has a cubic nonlinearity and describes an oscillator. It is the simplest oscillator displaying catastrophic jumps of amplitude and phase when the frequency of the forcing term is taken as a gradually changing parameter. The main application have been in electronics and biology. For example, the brain is full of oscillators at micro and macro levels [2]. Several techniques have been used by many authors to study the existence of periodic solution of the Duffing type of equation (1.1) such as polar coordinates, the method of upper and lower solution, coincidence degree theory and a series of existence results of nontrivial solution of equation (1.1). We refer to [3-5] and reference therein. However, some methods of proving existence have some limitations and in fact for practical purposes serious difficulties arise frequently in the search for fixed point of Duffing equation with cubic nonlinearity.

In this paper, we chose another strategy of proof which rely essentially on a fixed point theorem due to Krasnoselskii for a set that is closed, bounded and convex subset of a Banach space [6]. This result has been extensively employed in the related literature in the study of several kinds of separated boundary value problems (see for instance in [7, 8, 9, 10, 11] and their references); while for the periodic problem, it is more difficult to find references [12]. The reason for this contrast may be the fact that in order to apply this fixed point theorem, it is necessary to study the semigroup operator for linear equation, contraction and compactness of solution which are relatively difficult to study. To overcome this problem, Gronwalls inequality and Eberlein Simultian theorem were employed to obtain uniqueness and compactness of solution of Duffing equation.

2. Preliminaries

Definition 2.1. (Boundedness of a function): A function $f(x)$ is bounded if $\exists\ \ M > 0\ such\ that\ \|f(x)\| \le M \Rightarrow -M \le f(x) \le M \Rightarrow f(x) \in [-M, M]$.

Definition 2.2. (Convex Set): Suppose X is a vector space. A subset $C \subset X$ is said to be convex if whenever $a, b \in C$ and $r \in [0,1]$, it follows that $ra + (1-r)b \in C$. The closure of a set is again convex.

Definition 2.3. (Hilbert Space): A pre-Hilbert space which is complete (considered as a normed linear space) is called Hilbert space.

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Definition 2.4. (Pre-Hilbert Space): A linear space X is said to be pre-Hilbert space if for every ordered pair of elements (x, y) $x, y \in X$, there is associated real number where X is a real linear space and complex number where X is a complex linear space such that

(i) $\langle x.x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff$x = 0$
(ii) For $\alpha \in H$, then $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$
(iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ | $\overline{\langle y, x\rangle}$ denotes the complex conjugate of $\langle y, x\rangle$ and
(iv) $\langle x + y, w\rangle = \langle x, w\rangle + \langle y, w\rangle$ for $x, y, w \in X$.

Remark: If we defined $\|x\| = \sqrt{\langle x, y\rangle}$, a pre-Hilbert space becomes a normed linear space.

Definition 2.5. (Banach space): A normed linear space is called Banach space if it is complete in the sense of a metric given by the norm. Completeness means that every Cauchy sequence is convergent. Let $\{x_k\}_{k=1}^{\infty} \subset X$ be any Cauchy sequence that is a sequence $\{x_k\}_{k=0}^{\infty}$ for which $\|x_m - x_k\| \to 0$ as $m, k \to \infty$ independently, then $\exists$ an element $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x\| = 0. \quad (1.3)$$

Definition 2.6. Let (X, d) be a complete metric space. Then $T: X \to X$ is called a contraction mapping if there exists a constant $\alpha < 1$ such that

$$d(T(x), T(y)) \leq \alpha d(x, y) \quad (1.4)$$

for each $x, y \in X$ and where $d(x, y) = \|x - y\|$

Theorem 2.9. (Contraction Mapping Principle) Let (X, d) be a complete metric space and let $T: X \to X$ be a contraction, then T has a unique fixed point x^*. Furthermore, for each $x \in M$

$$\lim_{n\to\infty} T_n(x) = x^* \quad (1.6)$$

From this, one draws three conclusion in which this paper is written

(i) T has a unique fixed point, say x^*
(ii) For each $x \in X$, the picard sequence $\{T_n(x)\}$ converges to x^* and also sequentially compact and converges to Tx^*
(iii) The convergence is uniform if X is bounded.

Theorem 2.10. (Gronwalls-Bellman's Inequality) Let f and g be continuous real-valued functions on some interval $[a, b]$, then

$$f(t) \leq K + \int_a^b f(s)g(s)ds \quad (1.7)$$

for some $K > 0$ implies that

$$f(t) \leq Kexp[\int_a^b g(s)ds] \quad (1.8)$$

Proof. Multiplying both sides of equation (2.9) by $g(t)$ we have

$$f(t)g(t) \leq g(t)[K + \int_a^b f(s)g(s)ds].$$

By hypothesis $\frac{dA}{dt} \leq g(t)A(t)$ then

$$f(t) \leq A(t) \leq A(a)exp[\int_a^b g(s)ds]$$

$$\leq Kexp[\int_a^b g(s)ds] \text{ where } A(a) = K.$$

Hence, $f(t) \leq Kexp[\int_a^b g(s)ds]$

Remark: There is a generalization of this inequality. Its statement and proof is all about technicality.

Definition 2.11. (Compactness) The subset A of a topological space X i.e $A \subset X$ is said to be compact if every open cover of A has a finite subcover.

Note: A subset $A \subset X$ is pre-compact if $\bar{A}$ is compact.

Definition 2.12. (Contraction Semigroup) Let E be a Banach space. A one-parameter family $\{T_t\}_{t\geq 0}$ of bounded linear operators on E into itself is called a contraction semigroup of class (C_0) or simply a contraction semigroup if it satisfies the following conditions

(i) $T_{t+s} = T_t.T_s$ $t, s \geq 0$
(ii) $\lim_{t\to 0}\|T_t x\| = x$ for $x \in E$
(ii) For $t \geq 0$, $\|T_t\| \leq 1$

Note: (i) is called the semigroup property.

Examples:

(1) $U(s + t) = \exp(t + s)$ is a semigroup.
(2) Let $X = L^p(\mathbb{R}^n)$, with $p \in [1, \infty)$ and $K_t(x) = \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-|k|^2}{4t}\right), t > 0$ is a semigroup.

Theorem 2.13. (Eberlein-$\hat{S}$mulian) A subset of a Banach space X is relatively weak compact if only if it is relatively weakly sequentially compact. In particular, a subset of a Banach space X is weakly compact if and only if it is weakly sequentially compact.

Theorem 2.14. (Krasnoselskii's Fixed Point Theorem) Assume that F is a closed bounded convex subset of a Banach space X. Futhermore, assume that u_1 and u_2 are mappings from F into X such that the following conditions hold:

(i) $u_1 + u_2 \in F$
(ii) u_1 is a contraction.
(iii) u_2 is continuous and compact

then $u_1 + u_2$ has a fixed point in F.

Theorem 2.15. If $\{S(t); t \geq 0\}$ is a $C_o - semigroup$ then $\exists\, M \geq 1$ and $\alpha \in \mathbb{R}$ such that $\|S(t)\|_{L(X)} \leq Mexp(\alpha t)$ for each $t \geq 0$.

Proof: Since $\{S(t); t \geq 0\}$ is continuous. Suppose by contraction, Let there exist a sequence $(\alpha_n)_{n\in\mathbb{R}} \subset \mathbb{R}^+$ such that $\alpha_n \to 0\; as\; n \to \infty$, $\|S(t)\|_{L(X)} \to \infty\; as\; n \to \infty$ then by the uniform boundedness principle, there exist $x \in X$ such that $(\|S(\alpha_n)x\|)_{n\in\mathbb{N}}$ is unbounded contradicting the fact that $S(t)$ is continuous at $t = 0$.

3. Main Result

We consider the Duffing equation of the form

$$\ddot{x} + c\dot{x} + ax + bx^2 + \beta x^3 = p(t) \quad (1.9)$$

Equation (1.9) can be re-written in the following form

$$\ddot{x} + c\dot{x} + ax = p(t) - bx^2 - \beta x^3 \quad (1.10)$$

Vectorization of equation (1.10) is as follows

Let $x = x_1$, $\dot{x}_1 = x_2, \ddot{x}_1 = \dot{x}_2$ then equation (1.10) gives

$$\ddot{x}_1 + c\dot{x}_1 + ax_1 = p(t) - bx_1^2 - \beta x_1^3 \qquad (1.11)$$

The equivalent system of equation (1.11) is given by

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1 - cx_2 + p(t) - bx_1^2 - \beta x_1^3 \end{aligned}\right\} \qquad (1.12)$$

In matrix form equation (1.12) is written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & -c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ p(t) - bx_1^2 - \beta x_1^3 \end{pmatrix}$$

The above is of the form of non-autonomous equation given by

$$\dot{\underline{x}}(t) = A\underline{x}(t) + g\underline{x}(t) \qquad (1.13)$$

where $A = \begin{pmatrix} 0 & 1 \\ -a & -c \end{pmatrix}$, $g\underline{x}(t) = \begin{pmatrix} 0 \\ p(t) - bx_1^2 - \beta x_1^3 \end{pmatrix}$ and $\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

General solution of equation (1.13) is

$$x(t) = \underline{C_0}\exp(At) + \int_{t_0}^{t}\exp((t-s)A)\ g(s)ds \ \forall t \in \mathbb{R} \qquad (1.14)$$

or $x(t) = \underline{C_0}\,S(t) + \int_{t_0}^{t} s(t-u)g(u)du$ for $u \in \mathbb{R}$ (1.15)

where $S(t)$ is a semigroup operator.

To generate sequence of solutions in equation (1.14) we have that for $t = t_1, t_2, \ldots, t_k\ \forall t_k \in \mathbb{R}$ we have

$$x(t_1) = \underline{C_0}\exp(At_1) + \int_{t_0}^{t_1}\exp((t_1-s)A)\ g(s)ds \qquad (1.16)$$

$$x(t_2) = \underline{C_0}\exp(At_2) + \int_{t_0}^{t_2}\exp((t_2-s)A)\ g(s)ds$$

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$$x(t_k) = \underline{C_0}\exp(At_k) + \int_{t_0}^{t_k}\exp((t_k-s)A)\ g(s)ds$$

Now assuming that $u_1(t)$ and $u_2(t)$ are solutions of equation (1.14) then we have that

$$u_1(t) = \underline{C_0}\exp(At_1) + \int_{t_0}^{t_1}\exp((t_1-s)A)\ g(s)ds \qquad (1.17)$$

$u_2(t) = \underline{C_0}\exp(At_2) + \int_{t_0}^{t_2}\exp((t_2-s)A)\ g(s)ds$ for $t_1t_2 \in \mathbb{R}$

Furthermore, we will show that $u_1(t) = u_2(t)$ that is uniqueness.

Suppose that $F \subset X$ where X is a Banach space and F is a subspace and consider convexity of its solutions. For $\alpha \in F$ say, then

Claim 1: $\alpha u_1(t) + (1-\alpha)u_2(t) \in X$ is convex, $\alpha \in [0,1]$

Proof:

Case I

Take $\alpha = 0$, then $0.\ u_1(t) + (1-0)u_2(t) = u_2(t) \in X$

Case II

Take $\alpha = 1$, then $1.u_1(t) + (1-1)u_2(t) = u_1(t) \in X$

Case III, take $\alpha \in (0,1)$ we have

$$\alpha u_1(t) + (1-\alpha)u_2(t) \in X$$

$\|\alpha u_1(t) + (1-\alpha)u_2(t)\|$

$\leq \|\alpha u_1(t)\| + \|(1-\alpha)u_2(t)\|$

$= \alpha\|u_1(t)\| + (1-\alpha)\|u_2(t)\|$ since $\alpha \geq 0$.

Claim 2: Is $\|u_1(t)\| \leq 1$?

Proof:

Define a ball, $B = \{u_1(t) \in X : \|u_1(t)\| \leq 1\}$ is convex. Then

$$\|\alpha u_1(t) + (1-\alpha)u_2(t)\| \leq \alpha + (1-\alpha) = 1 \in B \qquad (1.18)$$

Hence the solution of the equation is convex.

Next, we verified the boundedness property of our solution.

Let $u_1u_2 \in X$, we show that $u_1 + u_2 \in F$

$$u_1(t_1) + u_2(t_2) = \underline{C_0}\,S(t_1) + \int_{t_0}^{t_1} S(t_1-u)g(u)du + \underline{C_0}\,S(t_2) + \int_{t_0}^{t_2} S(t_2-u)g(u)du$$

This is true by uniform boundedness principle. Applying theorem 2.15 we have

$$u_1(t_1) + u_2(t_2) = \underline{C_0}\,(S(t_1) + S(t_2)) + \int_{t_0}^{t_1} S(t_1-u)g(u)du + \int_{t_0}^{t_2} S(t_2-u)g(u)du \qquad (1.19)$$

$$\|u_1(t_1) + u_2(t_2)\| = \left\|\underline{C_0}\,(S(t_1) + S(t_2)) + \int_{t_0}^{t_1} S(t_1-u)g(u)du + \int_{t_0}^{t_2} S(t_2-u)g(u)du\right\|$$

$$\leq \underline{C_0}\|(S(t_1) + S(t_2)\| + \left\|\int_{t_0}^{t_1} S(t_1-u)g(u)du + \int_{t_0}^{t_2} S(t_2-u)g(u)du\right\|$$

By semigroup property,

$$S(t_1-u) = S(t_1-t_2)\,.S(t_2-u) \text{ and } \|S(t_1-t_2)\| \leq 1. \qquad (1.20)$$

We have

$$\|u_1(t_1) + u_2(t_2)\| \leq \underline{C_0}\|(S(t_1) + S(t_2)\| + \left\|\int_{t_0}^{t_1} S(t_1-t_2).S(t_2-u)g(u)du + \int_{t_0}^{t_2} S(t_2-u)g(u)du\right\|$$

$$\leq \underline{C_0}\|(S(t_1) + S(t_2)\| + \|S(t_1-t_2)\|\int_{t_2}^{t_1}\|S(t_2-u)\|\ \|g(u)\|du + \int_{t_0}^{t_2}\|S(t_2-u)\|\ \|g(u)\|du$$

$$\leq \underline{C_0}\|(S(t_1) + S(t_2)\| + M\int_{t_0}^{t_1}\|S(t_2-u)\|\,du + M\int_{t_0}^{t_2}\|S(t_2-u)\|\,du$$

$$\leq \underline{C_0}\|t_1 + t_2\| + M\left(\int_{t_0}^{t_1}\|S(t_2-u)\| + \int_{t_0}^{t_2}\|S(t_2-u)\|\right)du$$

$$\leq \underline{C_0}\|t_1 + t_2\| + M\left(\int_{t_0}^{t_1} \exp\big(\alpha(t_1 - u)\big) + \int_{t_0}^{t_2} exp\big(\alpha(t_2 - u)\big)\right) du \text{ by theorem 2.15}$$

$$= \underline{C_0}\|t_1 + t_2\| + MK = \beta \tag{1.21}$$

where $K = Sup\left(\int_{t_0}^{t_1} \exp\big(\alpha(t_1 - u)\big) + \int_{t_0}^{t_2} exp\big(\alpha(t_2 - u)\big)\right) du$

$\|u_1(t_1) + u_2(t_2)\| \leq \beta$. So bounded for each $t_0, t_1, t_2 \in [0, \alpha]$ where $\alpha = \frac{1}{n} \to 0$ as $n \to \infty$.

Next, we show that $u_1 = u_2$ that is Uniqueness of Solution.

But $u_1(t) = \underline{C_0}S(t) + \int_{t_0}^{t} S(t-u)g(u)du$ where $S(t)$ is a semigroup

$$u_1(t) - u_2(t) = \underline{C_0}S(t) + \int_{t_0}^{t} S(t-u)g(u)du - (\underline{C_0}S(t) + \int_{t_0}^{t} S(t-u)y(u)du)$$

$$= \int_{t_0}^{t} S(t-u)g(u)du - \int_{t_0}^{t} S(t-u)y(u)du$$

$$= \int_{t_0}^{t} S(t-u)[g(u) - y(u)]du \tag{1.22}$$

Hence,

$$\|u_1(t) - u_2(t)\| = \left\|\int_{t_0}^{t} S(t-u)[g(u) - y(u)]du\right\| \tag{1.23}$$

$$\leq \int_{t_0}^{t} \|S(t-u)[g(u) - y(u)]du\|$$

$$\leq \int_{t_0}^{t} \|S(t-u)\|\,\|g(u) - y(u)\|du$$

by theorem 2.15 and theorem 2.10 we have

$$\|u_1(t) - u_2(t)\| \leq 0 \tag{1.24}$$

Therefore $u_1(t) = u_2(t)$ and since uniqueness of solution are satisfied, the closure is trivial. Hence $u_1 + u_2 \in F$.

Furthermore, we will show that $u_1(t)$ is a contraction that is

$$\|u_1(t_1) - u_2(t_2)\| \leq K\|t_1 - t_2\| \text{ for } K \in [0,1), t_1, t_2 \in [0, \alpha]$$

From equation (1.15) we have that

$$u(t) = \underline{C_0}\, S(t) + \int_{t_0}^{t} S(t-u)g(u)du \tag{1.25}$$

where $S(t)$ is a semigroup operator, $\underline{C_0} > 0$, $S(t) = \exp(At)$ and $\|S(t_1 - t_2)\| \leq 1$

Claim: Equation (1.25) is a contraction

Proof:

$$u_1(t_1) - u_1(t_2) = \underline{C_0}\, S(t_1) + \int_{t_0}^{t_1} S(t_1-u)g(u)du - \underline{C_0}\, S(t_2) + \int_{t_0}^{t_2} S(t_2-u)g(u)du \tag{1.26}$$

$$\|u_1(t_1) - u_1(t_2)\| \leq \left\|\underline{C_0}\, S(t_1) - \underline{C_0}\, S(t_2)\right\| + \left\|\int_{t_0}^{t_1} S(t_1-u)g(u)du - \int_{t_0}^{t_2} S(t_2-u)g(u)du\right\|$$

$$= \underline{C_0}\, \|S(t_1) - S(t_2)\| + \left\|\int_{t_0}^{t_1} S(t_1-u)g(u)du - \int_{t_0}^{t_2} S(t_2-u)g(u)du\right\|$$

$$\|u_1(t_1) - u_1(t_2)\| \leq \underline{C_0}\, \|t_1 - t_2\| + \left\|\int_{t_0}^{t_1} S(t_1-u)g(u)du - \int_{t_0}^{t_2} S(t_2-u)g(u)du\right\| \tag{1.27}$$

Using equation (1.20) which have the same idea with Banach-Mazur distance in supper multiplicative metric space we have

$$\|u_1(t_1) - u_1(t_2)\| \leq \underline{C_0}\, \|t_1 - t_2\| + \left\|\int_{t_0}^{t_1} S(t_1-t_2)S(t_2-u)g(u)du - \int_{t_0}^{t_2} S(t_2-u)g(u)du\right\|$$

$$= \underline{C_0}\, \|t_1 - t_2\| + \left\|\int_{t_0}^{t_1} S(t_1-t_2)S(t_2-u)g(u)du + \int_{t_2}^{t_0} S(t_2-u)g(u)du\right\|$$

$$= \underline{C_0}\, \|t_1 - t_2\| + \left\|\int_{t_2}^{t_1} [S(t_1-t_2)S(t_2-u) + S(t_2-u)]g(u)du\right\|$$

$$\leq \underline{C_0}\, \|t_1 - t_2\| + \int_{t_2}^{t_1} \|S(t_1-t_2)S(t_2-u) + S(t_2-u)]\|\,\|g(u)\|du$$

$$\leq \underline{C_0}\, \|t_1 - t_2\| + M\int_{t_2}^{t_1} \|S(t_1-t_2)S(t_2-u) + S(t_2-u)]\|\,du \tag{1.28}$$

where $\|g(u)\| \leq M$

By theorem 2.15

$$\|u_1(t_1) - u_1(t_2)\| \leq \underline{C_0}\, \|t_1 - t_2\| + 2M\int_{t_2}^{t_1} \|S(t_2-u)\|\,du \tag{1.29}$$

By theorem 2.10 equation (1.25) becomes

$$\|u_1(t_1) - u_1(t_2)\| \le \underline{C_0}\,\|t_1 - t_2\| \exp 2M\|t_1 - t_2\|$$

where $t_1, t_2 \in [0, \alpha]$

$$= \underline{C_0}\,\|t_1 - t_2\|$$

$$\|u_1(t_1) - u_1(t_2)\| \le \underline{C_0}\,\|t_1 - t_2\| \qquad (1.30)$$

Now take $\underline{C_0} = K \in [0,1)$ and equation (1.30) becomes

$$\|u_1(t_1) - u_1(t_2)\| \le K\|t_1 - t_2\| \qquad (1.31)$$

Hence equation (1.31) is a contraction.

To show that u_2 is a continuous, we proceeds as follows. Recall that $u_1 = u_2$ that is uniqueness of solution, then u_2 is also a contraction.

Given any $\varepsilon > 0 \ \exists\ \delta(\varepsilon)$ and $\|t_2 - t_1\| < \delta, \forall\, t_1, t_2 \in X$ such that $\|u_2(t_2) - u_2(t_1)\| < K\delta$. If $\delta = \frac{\varepsilon}{K+1}$ then

$$\|u_2(t_2) - u_2(t_1)\| \le \frac{K\varepsilon}{K+1} < \varepsilon \qquad (1.32)$$

By equation (1.32) $u_2(t_2)$ is continuous.

Finally we use theorem 2.13 to establish the compactness of u_2. Assume that we are in finite dimensional space, Hein-Borel guarantees our result that is since u_2 is closed and bounded hence compact. But since we are in infinite dimensional space (Banach space), by theorem 2.13, u_2 is sequentially compact hence compact. Therefore $u_1 + u_2$ has a fixed point in F.

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