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Predictive Analyses of Logarithmic Non – Homogeneous Poisson Process in Software Reliability Using Bayesian Approach with Informative Priors

Nickson Cheruiyot*, Luke Akong'o Orawo, Ali Salim Islam

Department of Mathematics, Egerton University, Egerton, Kenya

Abstract Musa – Okumoto (1984) non-homogeneous Poisson Process (NHPP) software reliability model also known as logarithmic NHPP model is one of the widely used reliability model. The model is based on the assumptions that failures are observed during execution time caused by remaining faults in the software; whenever a failure is observed, an instantaneous effort is made to find what caused the failure and the faults are removed prior to future tests and whenever a repair is done it reduces the number of future faults not like other models. The failure intensity function of this model reduces exponentially with time and the expected number of failures has logarithmic function. The predictive analysis of software reliability model is of great importance for modifying, debugging and determining when to terminate software development testing process. This paper presents some results about predictive analyses for the Musa – Okumoto (1984) NHPP model. Four issues in single-sample prediction associated closely with development testing program are addressed. Bayesian approach based on informative prior was adopted to develop explicit solutions to the problems which arise during software development testing process. Developed methodologies were illustrated using real data in form of time between failures.

Keywords Bayesian approach, Intensity function, Informative prior, Software reliability model

1. Introduction

Developing a reliable software is a challenging task facing software industry. This therefore calls for a method for checking whether the developed software is reliable or not. To determine when to terminate development process of a software there is need to carry out predictive analyses. Bayesian predictive analyses using various software reliability growth models has attracted a number of researchers. For instant, predictive analyses for the power law process (PLP) was developed, where most problems that relates to development process of software were solved using Bayesian approach [1]. [2] also solved the issues related to software development process by conducting a Bayesian predictive analyses for Goel- Okumoto software reliability growth model. Both models assume that failures are finite and that a software can be free of errors at a given time when all faults have been removed which might not happen at a real situation. Predictive analyses for Musa – Okumoto software reliability growth model has not been developed and the model assume failures to be infinite, that is there is no point in time a software will have zero faults, which is true in real situation. The model assumes that the earlier faults that are removed have great impact than the remaining faults. The Musa – Okumoto software reliability model is one of non-homogeneous Poisson process software model with the intensity function given by;

$$\lambda(t) = \frac{\alpha\beta}{1+\beta t} \tag{1}$$

The model is based on the assumptions that failures are observed during execution time caused by remaining faults in the software; whenever a failure is observed, an instantaneous effort is made to find what caused the failure and the faults are removed prior to future tests and whenever a repair is done it reduces the number of future faults not like other models. The model must remain stable during the entire testing period for any particular testing environment and a reasonably accurate prediction of reliability must be provided by the model. These are the two main aspects of a good reliability model [3]. The Musa – Okumoto (1984) model has been used in various testing environment and in many instances, it provides good estimation and prediction of software reliability. Compared to other models when used in testing industrial data set, Musa- Okumoto model is the best performer in terms of fitting and predictive capability to the data [4].

* Corresponding author:
mutaicher21@gmail.com (Nickson Cheruiyot)
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Bayesian reliability modeling is one of the best methods in predictive analysis. Development of reliability posterior distribution from which predictive inference is made is the main thing required in Bayesian reliability model. The reliability posterior distribution is usually constructed using prior distribution for the parameters of the software reliability model and the likelihood function based on the observed data. The advantage of using Bayesian approach is that it allows prior information such as engineering judgments and test results to be combined with more recent information from test or field data. This is vital since it helps software developers to arrive at a prediction of reliability based upon a combination of all available information. This information includes; the environment under which the software will work, previous tests on the software and even intuition based upon experience [5]. This paper present single – sample prediction analyses for Musa – Okumoto model using Bayesian approach with informative priors.

2. Bayesian Method

Computer Software is an important complex intellectual product that has become driver of almost everything in the 21st century. During its development testing, developers and statisticians are interested on some prediction problems that are believed to be helpful in modifying the development testing program. In this section we present four issues A, B, C and D in single – sample prediction associated closely to development testing program. The four issues that were addressed are outline as propositions and their proof given in the appendix. Predictive distributions were derived using Bayesian method with informative priors. In this paper, it is assumed that a reliability growth testing is performed on a computer software system and the number of failures in the time interval $(0,t]$, denoted by $N(t)$ is observed. It is also assumed that $\{N(t), t>0\}$ follows the NHPP with intensity given in equation (14). Let $0<t_1<t_2<\cdots$ be the successive failure times. When testing stops after a pre-determined n number of failures is observed, the failure data is said to be failure-truncated. We denote the n failures time by $Y_{obs}^{f}=[t_i]_{i=1}^{n}$ where $0<t_1<t_2<\cdots<t_n$. a time-truncated data is when testing is observed for fixed time t. We denote the corresponding observed data by $Y_{obs}^{t}=\{n,t_1,\cdots,t_n;t\}$, where $0<t_1<\cdots<t_n\le t$.

A prediction interval is an interval estimate for a future observation or a function of some future observations (Jun – Wu *et al.*, 2007). Specifically, a double-sided (bilateral) prediction interval for x_{n+k} a future failure time with confidence level γ is defined by $[X_{n+k,l(\gamma)}, X_{n+k,u(\gamma)}]$ where $X_{n+k,l(\gamma)}$ and $X_{n+k,u(\gamma)}$ are the lower and upper prediction limits respectively such that $Pr\{X_{n+k,l(\gamma)}\le x_{n+k}\le X_{n+k,u(\gamma)}\}=\gamma$.

Similarly, a single-sided (unilateral) lower or upper prediction limit for x_{n+k} with level γ is defined by $X_{n+k,L(\gamma)}$ (or $X_{n+k,U(\gamma)}$), which satisfies $Pr\{X_{n+k,L(\gamma)}\le x_{n+k}\}=\gamma$ or $Pr\{x_{n+k}\le X_{n+k,U(\gamma)}\}=\gamma$. Both lower and upper prediction limits, $X_{n+k,L(\gamma)}$ and $X_{n+k,U(\gamma)}$ respectively, depend only on a single sample (or a single software) and are called single-sample prediction limits. Prediction limits involving two samples (or two software) can be defined similarly and are called two-sample prediction limits.

2.1. Predictive Issues

Here, we consider one software and assume that its cumulative time between failure times obey Musa – Okumoto software reliability growth model with observed data as either Y_{obs}^{f} or Y_{obs}^{t}. Based on Y_{obs}^{f} or Y_{obs}^{t}, we are interested in the following problems:

A: What is the probability that at most k software failure will occur in the future time period $(T,\tau]$ with $\tau>T$?

B: Given that the pre-determined target value λ_{tv} for the failure rate of the software undergoing development testing is not achieved at time T, what is the probability that the target value λ_{tv} will be achieved at time $\tau, \tau>T$?

C: Suppose that the target value λ_{tv} for the software failure rate is not achieved at time T, how long will it take so that the software failure rate will be attained at λ_{tv}?

D: What is the upper prediction limit (UPL) of $\lambda_\tau=\alpha\beta/(1+\beta\tau)$ with level γ. τ being a pre-determined value greater than T?

2.2. Posterior and Predictive Distribution

Let Y_{obs} represent Y_{obs}^{f} or Y_{obs}^{t}. The joint density distribution of Y_{obs} is therefore [6]:

$$f(Y_{obs}/\alpha,\beta)=(\alpha\beta)^n\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{(-\alpha\ln(1+\beta T))}$$

$$\alpha>0,\beta>0 \tag{2}$$

Case 1: When β the shape parameter is known, we adopt the following an informative prior for α, that is $\alpha\sim Gamma(a,b)$, where a and b are known

$$\pi(\alpha)\propto\alpha^{a-1}e^{-b\alpha}. \tag{3}$$

The posterior of α is thus obtained as

$$h(\alpha/Y_{obs})=\frac{f(Y_{obs}/\alpha,\beta)\pi(\alpha)}{\int_0^\infty f(Y_{obs}/\alpha,\beta)\pi(\alpha)d\alpha} \tag{4}$$

Substituting equation (2) and equation (3) into equation (4) we have

$$h(\alpha/Y_{obs})=[\Gamma(n+a)]^{-1}\alpha^{n+a-1}[\ln(1+\beta T)+b]^{n+a}e^{-\alpha[\ln(1+\beta T)+b]}. \tag{5}$$

Let $\tilde{y}$ be the random variable being predicted. The predictive density of $\tilde{y}$ is;

$$f(\tilde{y}/Y_{obs}) = \int_0^\infty f(\tilde{y}/Y_{obs})h(\alpha/Y_{obs})d\alpha \tag{6}$$

Hence, the Bayesian UPL of $\tilde{y}$ with level γ, denoted as $y_U^{(\beta)}$, must satisfy

$$\gamma = \int_{-\infty}^{y_U^{(\beta)}} f(y/Y_{obs})dy \tag{7}$$

Case 2: Shape parameter β is unknown, we assume the informative priors for α and β as $\alpha \sim Gamma(a,b)$ and $\beta \sim Gamma(c,d)$. This implies that $\pi(\alpha) \propto \alpha^{a-1}e^{-b\alpha}$ and $\pi(\beta) \propto \beta^{c-1}e^{-d\beta}$. Since α and β are independent the joint prior density $\pi(\alpha,\beta)$ is given as $\pi(\alpha,\beta) \propto \pi(\alpha).\pi(\beta)$. Implying that

$$\pi(\alpha,\beta) \propto \alpha^{a-1}e^{-b\alpha}\beta^{c-1}e^{-d\beta}. \tag{8}$$

The joint posterior density of α and β is thus

$$h(\alpha,\beta/Y_{obs}) = \left[p\Gamma(n+a)\right]^{-1}\alpha^{n+a-1} \beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}e^{-\alpha\left[\ln(1+\beta T)+b\right]}. \tag{9}$$

where $p = \int_0^\infty \frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{\left[\ln(1+\beta T)+b\right]^{n+a}}d\beta$

Equation (9) is similar to equation (5), let $\tilde{y}$ be the random variable predicted. The predictive density of $\tilde{y}$ is;

$$f(\tilde{y}/Y_{obs}) = \int_0^\infty\int_0^\infty f(\tilde{y}/Y_{obs},\alpha,\beta)h(\alpha,\beta/Y_{obs})d\alpha d\beta \tag{10}$$

and the Bayesian UPL denoted by y_U of $\tilde{y}$ with level γ similar to equation (6) is;

$$\gamma = \int_{-\infty}^{y_U} f(\tilde{y}/Y_{obs})dy \tag{11}$$

3. Main Results for Prediction Using Informative Priors

In this section we address the four issues stated in section 2.1 using the Bayesian approach. The main results are presented as propositions and their proofs given in the appendix. Below, we use $\chi^2(n;\gamma)$ to represent γ the percentage point of the chi-square distribution with n degrees of freedom such that $\Pr\{\chi^2(n) \le \chi^2(n;\gamma)\} = \gamma$, and define the Poisson mass function as $(h/\theta) = \theta^h e^{-\theta}/h$ and gamma density function as $(x/n,\lambda) = \lambda^n x^{n-1}e^{-\lambda n}/\Gamma(n)$. The prior is assumed to be equation (3) and (8) in all subsequent propositions.

Preposition 1 (Issue A)

The probability that at most k failures will occur in the interval (T,τ) with $\tau > T$ is

$$\gamma_k = \begin{cases} \left[\frac{\ln(1+\beta T)+b}{\ln(1+\beta\tau)+b}\right]^a \frac{\left[\ln(1+\beta T)+b\right]^n}{\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^n}\sum_{j=n}^{n+k}\binom{j+a-1}{n+a-1}\frac{\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^j}{\left[\ln(1+\beta\tau)+b\right]^j} & if\,\beta \quad is \quad known \\ \sum_{j=n}^{n+k}\frac{\Gamma(j+a)}{(j-n)!\,p\Gamma(n+a)}\int_0^\infty\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{\left[\ln(1+\beta\tau)+b\right]^{j+a}}d\beta & if\,\beta \quad is \quad unknown \end{cases} \tag{12}$$

Preposition 2 (Issue B)

The probability that the target value λ_{tv} will be achieved at time τ $\tau > T$ is

$$\gamma = \begin{cases} 1-\sum_{h=0}^{n+a-1}\frac{\left[\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]^h}{h!}e^{-\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]} & if\,\beta \quad is \quad known \\ 1-\frac{1}{p}\sum_{h=0}^{n+a-1}\int_0^\infty\frac{\left[\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]^h}{h!}e^{-\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{\left[\ln(1+\beta T)+b\right]^{n+a}}d\beta & if\,\beta \quad is \quad unknown \end{cases} \tag{13}$$

Preposition 3 (Issue C)

For a given level γ, the time τ^* required to attain λ_{tv}

$$\tau^{*}=\begin{cases}\left[\frac{\chi^{2}(2n;\gamma)}{2\lambda_{tv}\left[\ln(1+\beta T)+b\right]}-\frac{1}{\beta}\right]-T \; if\, \beta \quad is \quad known \\ \tau-T \; if\, \beta \quad is \quad unknown\end{cases} \tag{14}$$

Remark 1: For the second part of equation (14), τ is the solution to the equation

$$\gamma=1-\frac{1}{p}\sum_{h=0}^{n+a-1}\int_{0}^{\infty}\frac{\left[\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]^{h}}{h!}e^{-\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_{i})^{-1}e^{-d\beta}}{\left[\ln(1+\beta T)+b\right]^{n+a}}d\beta . \tag{15}$$

Preposition 4 (Issue D)

The Bayesian UPL of $\lambda_{\tau}=\frac{\alpha\beta}{(1+\beta\tau)}$ with level γ is

$$\lambda_{U}^{(\beta)}(\tau)=\begin{cases}\frac{\beta\chi^{2}(2n;\gamma)}{2(1+\beta\tau)\left[\ln(1+\beta T)+b\right]} \; if\, \beta \quad is \quad known \\ \lambda_{tv}^{*} \; if\, \beta \quad is \quad unknown\end{cases} \tag{16}$$

Remark 2: The second part of equation (16) is such that λ_{tv}^{*} is the solution to

$$\gamma=1-\frac{1}{p}\sum_{h=0}^{n+a-1}\int_{0}^{\infty}\frac{\left[\lambda_{tv}^{*}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]^{h}}{h!}e^{-\lambda_{tv}^{*}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_{i})^{-1}e^{-d\beta}}{\left[\ln(1+\beta T)+b\right]^{n+a}}d\beta . \tag{17}$$

4. Real Example

We have used the time between failures data described in [7] to illustrate the developed methodologies for the single-sample Bayesian predictive analysis. We conducted the goodness of fit test using Laplace statistics as presented in [8] and found that the data obey the Musa – Okumoto process. From the given data the maximum likelihood estimates for the parameters β and α of Musa – Okumoto growth model were obtained numerically by Newton Raphson method as $\hat{\beta}=0.008282448$ and $\hat{\alpha}=15.285550499$ respectively. In this paper we have used gamma priors for both parameters. The values of the parameters of informative priors $Gamma(a,b)$ and $Gamma(c,d)$ are chosen arbitrarily as $a=2, b=1/2, c=2$ and $d=1/2$.

(A) Suppose we are interested in the probability γ_{k} that at most k will occur in a future time period $(T,\tau]=(180,250]$. Considering the case when β is known (i.e, $\beta=0.008282448$), using the first formula in equation (12) we have γ_{0}=0.01202933, γ_{1}=0.06169460, γ_{2}=0.16742455, γ_{3}=0.32202724, γ_{4}=0.49656381, γ_{5}=0.65870012, γ_{6}=0.78770083, γ_{7}=0.87805309, γ_{8}=0.93488273, γ_{9}=0.96747044, γ_{10}=0.98470892, γ_{11}=0.99320106, γ_{12}=0.99712719, γ_{13}=0.99884169, γ_{14}=0.99955271, γ_{15}=0.99983403

Figure 1 shows the graph of probabilities that at most k failures will occur in the time interval $(180,250]$ for β known for both informative and non-informative priors. From the graph it can be seen that the probabilities for informative prior is high as compared to that of non-informative. This is more seen at issue C, where there high reduction of time required to achieve a predetermined target value in informative prior.

(B) Suppose the target value is given by $\lambda_{tv}=0.03$. At the time $T=182.21$, the MLE of the achieved failure rate for this software is $\hat{\lambda}(182.21)=\hat{\alpha}\hat{\beta}\big/(1+182.21\hat{\beta})=0.05045615$, which is greater than λ_{tv} thus it cannot be achieved at time $T=182.21$ and development testing will continue. Suppose we want to find the probability that the target value λ_{tv} will be achieved at the time $\tau=277.83h$. (i) When β is known (say, $\beta=0.008282448$), from the first formula in equation (13), we obtain $\gamma=0.0007319763$. In this case also as that of non-informative prior, the target value is unlikely to be achieved. (ii) When β is unknown, from the second formula in equation (13), we obtain $\gamma=0.08730647$ where the Monte Carlo sample size $L=1000$.

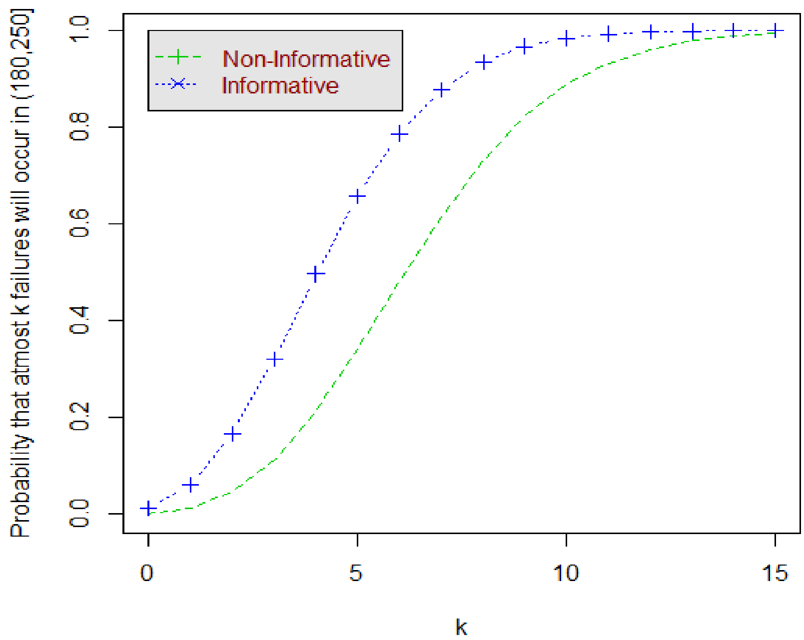

Figure 1. The graph of the probabilities γ_k that at most k failures will occur in the time interval (180, 250] for the cases of β known for informative and non-informative prior

(C) Since the target value $\lambda_{tv} = 0.03$ is not achieved at time $T = 182.21$. It is interesting now to know how long it will take in order to achieve the desired target value. (i) When β is known (say, $\beta = 0.008282448$), using the first formula in equation (14) and letting $\gamma = 0.90$, we obtain $\tau^* =$ 242.3671h . Thus, it will take another 242.3671 hours in order to achieve the target value which is a significant reduction from the value obtained for the case of non-informative prior. (ii) When β is unknown, from the second formula in equation (14) we have $\tau^* = 147$h . It will take another 147 hours for the desired target value to be achieved when β is unknown. . (D) Given $\tau = 900h$, when β is known (i.e, $\beta = 0.008282448$), from first formula in equation (16) the Bayesian UPL of $\lambda_\tau = \dfrac{\alpha\beta}{1+\beta\tau}$ with level $\gamma = 0.90$ is given by $\lambda_U^{(\beta)}(\tau) = 0.01602707$.

5. Conclusions

Reliable software has been the main goal of any software developer. This is because non- reliable software means that the customers will be dissatisfied with the product thus loss of market shares and significant cost to the supplier. For critical applications such as banking or health monitoring, non-reliability can lead to great damage not only to the consumer but also to the developer. Due to the above reasons, there is need to develop reliable software. There are many software reliability growth models that have been used in analyzing software reliability data. Musa- Okumoto is one of the software reliability models which best performed in fitting industrial failure data set. In this paper, explicit solution to predictive issues that may arise during development process were derived using Bayesian approach. These solutions are helpful to software developers in many instances such as resource allocation, when to terminate the testing process, modification needed in the software before termination.

The study used informative to derived explicit solutions for predictive issues that may arise during software development process. In all the cases when the shape parameter was known, solutions to posterior and predictive distributions had closed forms while when it is unknown, solutions had no closed forms and the study used Markov Chain Monte Carlo (MCMC). Bayesian approach was used as it is advantageous over classical approach. Bayesian approach is available for small sample sizes and allows the input of prior information about reliability growth process and provides full posterior and predictive distributions [1].

Appendix: Proof of preposition 1 – 4

We first state the following identity without proof: That is

$$\int_{D(m;a,b)} dF(t_1)\cdots dF(t_m) = \left[F(a) - F(b)\right]^m / m! \tag{A.1}$$

where m is any positive integer, a and b are two real numbers such that $a < b$, $F(t)$ is an increasing and differentiable function and $D(m;a,b) \triangleq \{(t_1, \cdots t_m) : a < t_1 < \cdots < t_m < b\}$.

Proof of preposition 1

The probability that at most k failures will occur in the interval (T, τ) is $\gamma_k = \Pr\left[N(\tau) \le n + k / Y_{obs}\right]$, when β is known, we have

$$\gamma_k = \int_0^\infty \Pr\left[N(\tau) \le n + k / Y_{obs}, \alpha\right] h(\alpha / Y_{obs}) d\alpha\ . \tag{A.2}$$

where $h(\alpha / Y_{obs})$ is given by equation (5) and

$$\Pr\left[N(\tau) \le n + k / Y_{obs}, \alpha\right] = \sum_{j=n}^{n+k} f(Y_{obs}, N(\tau) = j/\alpha) / f(Y_{obs} / \alpha)\ . \tag{A.3}$$

From equation (2) $f(Y_{obs} / \alpha) = \alpha^n \beta^n \prod_{i=1}^{n} (1 + \beta t_i)^{-1} e^{-\alpha \ln(1+\beta T)}$

and

$$f(Y_{obs}, N(\tau) = j/\alpha) = \int_{D(j-n;T,\tau)} f(Y_{obs}, x_{n+1}, \ldots, x_j, N(\tau) = j) \prod_{\ell=n+1}^{j} dx_\ell$$

$$f(Y_{obs}, N(\tau) = j/\alpha) = \int_{D(j-n;T,\tau)} \alpha^j \beta^j e^{-\alpha \ln(1+\beta T)} \prod_{i=1}^{j} (1 + \beta t_i)^{-1} \prod_{\ell=n+1}^{j} dt_\ell$$

$$= \alpha^j \beta^j \prod_{i=1}^{n} (1 + \beta t_i)^{-1} e^{-\alpha \ln(1+\beta \tau)} \int_{D(j-n;T,\tau)} \prod_{\ell=n+1}^{j} (1 + \beta t_\ell)^{-1} \prod_{\ell=n+1}^{j} dt_\ell\ . \tag{A.4}$$

Solving the integral part in equation (A.4), we proceed as follows:

$\int_0^t (1 + \beta t)^{-1} = \frac{1}{\beta} \ln(1 + \beta t)$. Substituting the limits T and τ we have $\frac{1}{\beta}\ln(1+\beta\tau) - \frac{1}{\beta}\ln(1+\beta T)$ which reduces to $\frac{1}{\beta}\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)$. Therefore the integral part of equation (A.4) becomes

$$\int_{D(j-n;T,\tau)} \prod_{\ell=n+1}^{j} (1 + \beta t_\ell)^{-1} \prod_{\ell=n+1}^{j} dt_\ell = \frac{1}{\beta^{j-n}} \frac{\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^{j-n}}{(j-n)!}\ . \tag{A.5}$$

Substituting equation (A.5) to equation (A.4) we have

$$f(Y_{obs}, N(\tau) = j/\alpha) = \alpha^j \beta^j \prod_{i=1}^{j} (1 + \beta t_i)^{-1} e^{-\alpha \ln(1+\beta\tau)} \frac{1}{\beta^{j-n}} \frac{\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^{j-n}}{(j-n)!}$$

From equation (A.3), we obtain

$$f(Y_{obs}, N(\tau)=j/\alpha)\,/\,f(Y_{obs}/\alpha)=\frac{\alpha^j\beta^j\prod_{i=1}^{n}(1+\beta t_i)^{-1}\frac{1}{\beta^{j-n}}\frac{\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^{j-n}}{(j-n)!}}{\alpha^n\beta^n\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-\alpha\ln(1+\beta T)}}$$

$$=\frac{\alpha^{j-n}\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^{j-n}e^{-\alpha\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)}}{(j-n)!}$$

Thus equation (A.3) becomes

$$\Pr\left[N(\tau)\le n+k/Y_{obs},\alpha\right]=\sum_{j=n}^{n+k}\alpha^{j-n}\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^{j-n}e^{-\alpha\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)}\frac{1}{(j-n)!}. \quad (A.6)$$

and equation (A.2)

$$\gamma_k=\int_0^\infty\sum_{j=n}^{n+k}\alpha^{j-n}\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^{j-n}\frac{e^{-\alpha\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)}}{(j-n)!\Gamma(n+a)}.\alpha^{n+a-1}\left[\ln(1+\beta T)+b\right]^{n+a}e^{-\alpha\left[\ln(1+\beta T)+b\right]}d\alpha \quad (A.7)$$

The integral part of equation (A.7) integrates to 1 since it is a gamma distribution with parameters $j+a$ and $\ln(1+\beta T)+b$.

On re-arranging equation (A.7), it becomes

$$\gamma_k=\left[\frac{\ln(1+\beta T)+b}{\ln(1+\beta\tau)+b}\right]^a\frac{\left[\ln(1+\beta T)+b\right]^n}{\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^n}\sum_{j=n}^{n+k}\binom{j+a-1}{n+a-1}\frac{\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^j}{\left[\ln(1+\beta\tau)+b\right]^j}. \quad (A.8)$$

This implies the first formula of equation (12).

When β is unknown, from equation (9) and equation (A.6) we have

$$\gamma_k=\int_0^\infty\int_0^\infty\Pr\left[N(\tau)\le n+k/Y_{obs},\alpha,\beta\right]h(\alpha,\beta/Y_{0bs})d\alpha d\beta$$

$$=\int_0^\infty\int_0^\infty\sum_{j=n}^{n+k}\frac{\alpha^{j-n}\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^{j-n}}{(j-n)!}e^{-\alpha\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)}\frac{\alpha^{n+a-1}\beta^{n+c-1}}{p\Gamma(n+a)}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}e^{-\alpha\left[\ln(1+\beta T)+b\right]}d\alpha d\beta$$

$$=\int_0^\infty\sum_{j=n}^{n+k}\frac{\left[\ln\left(\frac{1+\beta\tau}{1+\beta T}\right)\right]^{j-n}\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{(j-n)!\,p\Gamma(n+a)}\left\{\int_0^\infty\alpha^{j+a-1}e^{-\alpha\left[\ln(1+\beta T)+b\right]}d\alpha\right\}d\beta$$

$$=\sum_{j=n}^{n+k}\frac{\Gamma(j+a)}{(j-n)!\,p\Gamma(n+a)}\int_0^\infty\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{\left[\ln(1+\beta\tau)+b\right]^{j+a}}d\beta. \quad (A.9)$$

Equation (A.9) implies the second formula of equation (12).

Proof of Preposition 2

Let $f(\lambda_\tau/Y_{obs})$ denote the posterior of $\lambda_\tau=\alpha\beta/(1+\beta\tau)$. Hence, the probability that the target value λ_{tv} will be achieved at time τ is given by $\gamma=\Pr\left\{\lambda_\tau\le\lambda_{tv}/Y_{obs}\right\}=\int_0^{\lambda_{tv}}f(\lambda_\tau/Y_{obs})d\lambda_\tau$. When β is known, making transformation

$\lambda_\tau = {}^{\alpha\beta}\!/_{(1+\beta\tau)}$, we have $\alpha = \lambda_\tau \dfrac{(1+\beta\tau)}{\beta}$ and $\dfrac{d\alpha}{d\lambda_\tau} = \dfrac{(1+\beta\tau)}{\beta}$. Consequently, the posterior density of λ_τ is

$$f(\lambda_\tau / Y_{obs}) = h(\alpha / Y_{obs})\left|\frac{d\alpha}{d\lambda_\tau}\right|$$

$$=\left[\frac{\lambda_\tau(1+\beta\tau)}{\beta}\right]^{n+a-1}\frac{1}{\Gamma(n+a)}\left[\ln(1+\beta T)+b\right]^{n+a} e^{-\lambda_\tau\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}\frac{(1+\beta\tau)}{\beta}$$

$$=\frac{\lambda_\tau^{n+a-1}}{\Gamma(n+a)}\left[\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]^{n+a} e^{-\lambda_\tau\left[\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]}. \quad (A.10)$$

Equation (A.10) follows a gamma distribution with parameters $n+a$ and $\left[\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]$. From the relationship of gamma and Poisson distribution $\dfrac{b^a}{\Gamma(a)}\displaystyle\int_0^\lambda x^{a-1}e^{-bx}dx = 1-\sum_{h=0}^{a-1}\frac{(b\lambda)^h}{h!}e^{-b\lambda}$.

Thus, we have

$$\gamma = 1-\sum_{h=0}^{n+a-1}\frac{\left[\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]^h}{h!}e^{-\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}. \quad (A.11)$$

Equation (A.11) implies the first formula of equation (13).

When β is unknown, making transformation on $\lambda_\tau = {}^{\alpha\beta}\!/_{(1+\beta\tau)}$ and $\beta=\beta$, we obtain $\alpha = \lambda_\tau\frac{(1+\beta\tau)}{\beta}$ and $\beta=\beta$. Note that the Jacobian is ${}^{\partial(\alpha,\beta)}\!/_{\partial(\lambda_\tau,\beta)} = \dfrac{(1+\beta\tau)}{\beta}$. From equation (9), the joint posterior density of (λ_τ,β) is

$$f(\lambda_\tau,\beta / Y_{obs}) = h(\alpha,\beta / Y_{obs})\left|\frac{d(\alpha,\beta)}{d(\lambda_\tau,\beta)}\right|$$

$$=\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}}{P\Gamma(n+a)}\left[\frac{\lambda_\tau(1+\beta\tau)}{\beta}\right]^{n+a-1}e^{-d\beta}e^{-\lambda_\tau\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}.\frac{(1+\beta\tau)}{\beta}$$

$$=\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{p\left[\ln(1+\beta T)+b\right]^{n+a}}\frac{\left[\frac{(1+\beta\tau)}{\beta}\left(\ln(1+\beta T)+b\right)\right]^{n+a}}{\Gamma(n+a)}\lambda^{n+a-1}e^{-\lambda_\tau\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}. \quad (A.12)$$

We obtain,

$$\gamma = \frac{1}{p}\int_0^\infty\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{\left[\ln(1+\beta T)+b\right]^{n+a}}\left\{1-\sum_{h=0}^{n+a-1}\frac{\left[\lambda_\tau\frac{(1+\beta\tau)}{\beta}\left(\ln(1+\beta T)+b\right)\right]^h}{h!}e^{-\lambda_\tau\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}\right\}d\beta$$

$$=1-\frac{1}{p}\sum_{h=0}^{n+a-1}\int_0^\infty\frac{\left[\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]^h}{h!}e^{-\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]}\frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{\left[\ln(1+\beta T)+b\right]^{n+a}}d\beta. \quad (A.13)$$

Equation (A.13) implies the second formula of equation (13).

Proof of Preposition 3

For given level γ, the time required to attain the target value λ_{tv} is $\tau^* = \tau - T$, where τ satisfies

$\gamma = \Pr\left\{\lambda_\tau \le \lambda_{tv} / Y_{obs}\right\} = \int_0^{\lambda_{tv}} f(\lambda_\tau / Y_{obs}) d\lambda_\tau$. When β is known, from equation (A.10), it can easily been seen that $2\left[\frac{(1+\beta\tau)}{\beta}\left\{\ln(1+\beta T)+b\right\}\right]\lambda_\tau$ follows a chi-square distribution with $2n$ degrees of freedom. Thus we have

$$2\left[\frac{(1+\beta T)}{\beta}\left\{\ln(1+\beta T)+b\right\}\right]\lambda_{tv} = \chi^2(2n;\gamma). \tag{A.14}$$

Hence $\tau^* = \tau - T$ is given as

$$\tau^* = \left[\frac{\chi^2(2n;\gamma)}{2\lambda_{tv}\left\{\ln(1+\beta T)+b\right\}} - \frac{1}{\beta}\right] - T. \tag{A.15}$$

When β is unknown, the time required to attain the target value λ_{tv} with level γ is $\tau^* = \tau - T$. Where τ is the solution to

$$\gamma = 1 - \frac{1}{p}\sum_{h=0}^{n+a-1}\int_0^\infty \frac{\left[\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]\right]^h}{h!} e^{-\lambda_{tv}\frac{(1+\beta\tau)}{\beta}\left[\ln(1+\beta T)+b\right]} \frac{\beta^{n+c-1}\prod_{i=1}^{n}(1+\beta t_i)^{-1}e^{-d\beta}}{\left[\ln(1+\beta T)+b\right]^{n+a}} d\beta. \tag{A.16}$$

Proof of Preposition 4

For a pre-determined $\tau(\tau > T)$, the Bayesian UPL for λ_τ with level γ is $\lambda_U^{(\beta)}(\tau)$ satisfying $\gamma = \Pr(\lambda_\tau \le \lambda_U^{(\beta)} / Y_{obs})$. From $\gamma = \Pr\left\{\lambda_\tau \le \lambda_{tv} / Y_{obs}\right\} = \int_0^{\lambda_{tv}} f(\lambda_\tau / Y_{obs}) d\lambda_\tau$ and equation (A.14) we have

$$\gamma = \int_0^{\lambda_U^{(\beta)}(\tau)} \frac{\left[\frac{(1+\beta\tau)}{\beta}\left(\ln(1+\beta T)+b\right)\right]^{n+a}}{\Gamma(n+a)} \lambda_\tau^{n+a-1} e^{-\lambda_\tau\left[\frac{(1+\beta\tau)}{\beta}\left(\ln(1+\beta T)+b\right)\right]}. \tag{A.16}$$

$$\lambda_U^{(\beta)}(\tau) = \frac{\beta\chi^2(2n;\gamma)}{2(1+\beta\tau)\left[\ln(1+\beta T)+b\right]}. \tag{A.17}$$

Equation (A.17) implies the first formula of equation (16) and the second part follows similarly.

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