

# Boundedness and Stability of Periodic Solutions of a Hill's Equation with Six Independent Arbitrary Parameters

E. O. Eze\*, U. E. Obasi, S. I. Ezeh

Department of Mathematics, Michael Okpara University of Agriculture Umudike, Umuahia, Abia State, Nigeria

**Abstract** In this paper necessary and sufficient conditions that guaranteed the boundedness and stability of periodic solution of a Hill's equation with six independent arbitrary parameters were investigated using a combination of Simpson's method and Lyapunov direct methods. Regions of stable and unstable points were identified, which extended some results in literature.

**Keywords** Boundedness, Stability, Lyapunov Method, Simpson's Technique, Hill's Equation

## 1. Introduction

The purpose of this paper is to extend the results obtained in [8] and [10].

Consider the second order linear differential equation of Hill's type of the form

$$\ddot{Z} + (A + B \cos t + C \cos 2t + D \cos 3t + E \cos 4t + F \cos 5t)Z = 0 \quad (1)$$

where  $\ddot{Z}$  is the second order derivative with respect to time,  $A, B, C, D, E, F$  are independent arbitrary parameters and dots represent differentiation with respect to time. For a given  $A, B, C, D, E, F$  the points  $(A, B, C, D, E, F)$  is said to be stable if all solutions of (1) above are bounded for all  $t > 0$  and unbounded if any unbounded solution exists. Equation (1) is a linear differential equation with periodic coefficients.

Various researchers have worked on Hill's equation with very great results. See for instance [11, 12, 13]. [8] investigated the stability of Hill's equation with four independent parameters using Floquent theory and perturbation.

In [9] the stability of Hill's equation with three arbitrary parameters was also investigated using Fourier analysis. The Fourier analysis method employed did not include explicit algebraic expressions for the regions of stability. [6] also investigated the stability of Hill's equation with independent small parameters by the method of perturbations. However, the method of investigating stability and boundedness of equation (1) using a

combination of numerical integration and Lyapunov functions has been rare in literature to the best of our knowledge.

Equation (1) has a lot of physical applications especially in the areas of Genetic regulatory circuit and has been widely used in Physics, Chemistry and Biology [3] and other physical phenomenon. Furthermore, the significance are found in amplitude distortion in moving coil of loud speakers, frequency modulation, dynamical systems and vibration of stretched strings, also for scatter theory and wave mechanics and for relativistic oscillators. Due to the importance of Hill's equation in real world problems, the study of boundedness and stability of the equation has continued to attract the attention of many researchers see for instance [1, 4, 9]. Other researchers like [2], [5], [14] and [7] have investigated the stability and boundedness of linear and nonlinear differential equations.

## 2. Main Body

### 2.1. Preliminaries

**Definition 2.1:** Simpson's rule is a method of numerical integration that provides an approximation of a definite integral over the interval  $[a, b]$  using parabola. Furthermore, the interval of a function  $f(x)$  over the interval  $[a, b]$  with subintervals

$a = x_0 < x_1 < x_2 \dots < x_n = b$  and subintervals length  $n = b - a / h$  can be approximated as

$$\int_a^b f(x)dx \approx \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_{n-1}) + f(x_n)] \quad (2)$$

as long as  $n$  is even. Let  $a = x_0, x_1, x_2, \dots, x_n = b$  be a regular partition of  $[a, b]$  into an even number of subintervals (so  $n$  must be even) and assume that  $f(x)$  is a continuous function on  $[a, b]$ , then  $\int_a^b f(x) dx \approx S_n$  where

\* Corresponding author:

obinwanneze@gmail.com (E. O. Eze)

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$$S_n = b - a/3n[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \quad (3)$$

**Definition 2.2:** Assume that  $x_0 = 0$  otherwise the point  $x = x_0$  is a singular point of

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0 \quad (4)$$

and that  $Q_1(x) = \frac{xQ(x)}{P(x)}$  and  $Q_2(x) = \frac{x^2R(x)}{P(x)}$  are analytic at  $x = 0$ , then they will have Maclaurin series expansion

$$f(x) = f(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \quad (5)$$

with radius of convergence  $r_1 > 0$  and  $r_2 > 0$  respectively. That is

$$Q_1(x) = \frac{xQ(x)}{P(x)} = \sum_{n=0}^{\infty} P_n X^n$$

which converges for  $|X| < r_2$ . Then the point  $x_0 = 0$  is called a regular singular point of (4).

**Definition 2.3:** The functions  $g_1(x)$  and  $g_2(x)$  are analytic at  $x = x_0$  if they have Taylor series expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots \quad (6)$$

with radius of convergence  $r_1 > 0$  and  $r_2 > 0$  respectively. That is

$$g_1(x) = \frac{Q(x)}{P(x)} = \sum_{n=0}^{\infty} P_n(x-x_0)^n \text{ which converges for } |x-x_0| < r_1 \text{ and } g_2(x) = \frac{R(x)}{P(x)} = \sum_{n=0}^{\infty} q_n(x-x_0)^n \text{ which converges for } |x-x_0| < r_2.$$

**Definition 2.4:** Frobenius method of solving differential equation is a method that assumes that  $x_0 = 0$  is a regular singular point of the differential equation.

$$P(x)\ddot{y}(x) + Q(x)\dot{y}(x) + R(x)y(x) = 0 \quad (7)$$

The Frobenius series of the form  $y(x) = x^r \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} C_n x^{n+r}$  can be used to solve the differential equation (7). The parameter  $r$  must be chosen so that when the series is substituted into the differential equation the coefficient of the smallest power of  $x$  is zero. This is called the indicial equation. Also, a recursive equation for the coefficient is obtained by setting the coefficient of  $x^{n+r}$  equal to zero.

**Definition 2.5:** Let  $E_s$  be the error in Simpson's rule for a particular  $n$  i.e,  $E_s = \int_a^b f(x) dx - S_n$ . Then if  $|f^{(4)}(x)| \leq k$  on  $a \leq x \leq b$  (for some constant  $k$ ), then  $|E_s| \leq \frac{k(b-a)^5}{180n^4}$ .

**Definition 2.6:** Let

$$\dot{x} = f(t, x) \quad (8)$$

A solution  $\bar{x}(t)$  of (8) is Lyapunov stable if for each  $\varepsilon > 0$  and  $t_0 \in \mathbb{R} \exists \delta = \delta(\varepsilon, t_0) > 0$  such that if  $x \in t_0$  is a solution of (2.4) and  $|x(t_0) - \bar{x}(t_0)| < \delta$  then  $|x(t) - \bar{x}(t)| < \varepsilon$  for all  $t \geq t_0$ .

**Definition 2.7:** A solution  $\bar{x}(t)$  of (8) is asymptotically stable if it is Lyapunov stable and if for every  $t_0 \in \mathbb{R}$

$|x(t) - \bar{x}(t)| > 0$  as  $t \rightarrow \infty$ .

**Definition 2.8:** Consider a real valued function  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  which is continuously differentiable with  $V(0) = 0$ .  $V$  is said to be

- (i) Positive definite if  $V(x) > 0$  for all  $x \neq 0$  and  $V(0) = 0$ .
- (ii) Negative definite if  $V(x) < 0$  for all  $x \neq 0$  and  $V(0) = 0$ .
- (iii) Negative semi-definite if  $V(x) \leq 0$  and it can also vanish for some  $x \neq 0$ .
- (iv) Positive semi-definite if  $V(x) \geq 0$  and it can vanish for some  $x \neq 0$ .

**Definition 2.9:** Let the origin  $x = 0 \in D \subset \mathbb{R}^n$  be an equilibrium point for  $\dot{x} = f(x)$ . Let  $V: D \rightarrow \mathbb{R}$  be a continuously differentiable function such that,  $V(0) = 0$  and  $V(x) > 0 \forall x \in D \setminus \{0\}$ ,  $\dot{V}(x) \leq 0 \forall x \in D$  then  $x = 0$  is stable. Moreover if  $\dot{V}(x) < 0 \forall x \in D \setminus \{0\}$  then  $x = 0$  is asymptotically stable.

**Corollary 2.10:** Let  $x = 0$  be an equilibrium point of  $\dot{x} = f(x)$ . Let  $V: D \rightarrow \mathbb{R}$  be a  $C^1$  positive definite function containing the origin  $x = 0$  such that  $\dot{V} \leq 0$  in  $D$ . Let  $S = \{x \in D: V = 0\}$  and suppose that no solution can stay identically in  $S$ , other than the trivial solution  $x(t) = 0$ , then the origin is asymptotically stable.

## 2.2. Results and Discussion

### 2.2.1. Numerical Integration Approach

We consider a differential equation of the form

$$\ddot{Z} + (A + B \cos t + C \cos 2t + D \cos 3t + E \cos 4t + F \cos 5t) Z = 0 \quad (9)$$

where

$$A + B \cos t + C \cos 2t + D \cos 3t + E \cos 4t + F \cos 5t = \alpha(t).$$

Then equation (9) becomes

$$\ddot{Z} + \alpha(t)Z = 0 \quad (10)$$

Assume that equation (6) has a solution of the form

$$Z(t) = \sum_{n=0}^{\infty} a_n t^{n+r} \quad (11)$$

Finding the derivative of  $Z(t)$  term by term gives

$$\dot{Z}(t) = \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1} \quad (12)$$

$$\ddot{Z}(t) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2} \quad (13)$$

Substituting for  $Z(t)$  and  $\ddot{Z}(t)$  in equation (10) we have,

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2} + \alpha(t) \left( \sum_{n=0}^{\infty} a_n t^{n+r} \right) = 0 \quad (14)$$

When  $n = 0, 1, 2, \dots$  respectively we have

$$\begin{aligned} & a_0(r)(r-1)t^{r-2} + \alpha(t)a_0t^r + a_1(r+1)(r)t^{r-1} + \\ & \alpha(t)a_1t^{r+1} + a_1t^{r+1} + a_2(r+2)(r+1)t^r + \\ & \alpha(t)a_2t^{r+2} + a_3(r+3)(r+2)t^{r+1} + \alpha a_3t^{r+3} + \\ & a_4(r+4)(r+3)t^{r+2} + \alpha a_4t^{r+4} + \dots \\ & + a_k(k+r)(k+r-1)t^{k+r-2} + \alpha a_k t^{r+k} + \dots = 0 \end{aligned}$$

For a power series to vanish identically over the interval, the coefficient must be zero.

For  $t^{r-2}$

$$a_0(r)(r-1) = 0, \quad a_0 \neq 0 \quad (15)$$

$$r^2 - r = 0 \Rightarrow r = 0, r = 1, \text{ i.e. } r_1 = 0, r_2 = 1$$

Hence  $r_1 - r_2 = \text{integer}$ .

For  $t^{r-1}$  we have

$$a_1(r+1)(r) = 0 \quad (16)$$

When  $r_2 = 1$ , then  $a_1 = 0$

$r_1 = 0$  in (15) implies that  $a_1$  is intermediate

For  $t^r$  we have

$$a_2(r+2)(r+1) + \alpha(t)a_0 = 0 \quad (17)$$

$$a_2 = \frac{-\alpha(t)a_0}{(r+2)(r+1)} \quad (18)$$

For the general term  $x^{k+r}$ , we have

$$a_{k+2} = \frac{-\alpha(t)a_k}{(r+k+1)(r+k+2)} \quad (19)$$

which gives

$$a_{k+2}(r+k+1)(r+k+2) + \alpha a_k = 0 \quad (20)$$

From the indicial equation  $r = 0$  implies that  $a_1$  is intermediate.

From equation (20)  $k = 0$ ,  $a_2 = \frac{-\alpha(t)a_0}{2}$

$$k = 1, \quad a_3 = \frac{-\alpha(t)a_1}{2.3}$$

$$k = 2, \quad a_4 = \frac{-\alpha(t)a_2}{3.4} = \frac{-\alpha(t)}{3.4} \left( \frac{-\alpha(t)a_0}{2} \right) = \frac{\alpha^2(t)a_0}{2.3.4}$$

Hence, one solution is

$$Z_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots \quad (21)$$

$$Z_1(t) = a_0 + a_1(t) + \left( \frac{-\alpha(t)a_0 t^2}{2!} \right) + \left( \frac{-\alpha(t)a_1 t^3}{3!} \right) + \frac{\alpha^2(t)}{4!} a_0 t^4 + \dots$$

$$= a_0 - \frac{\alpha(t)a_0 t^2}{2!} + \frac{\alpha^2(t)a_0 t^4}{4!} + a_1 t - \frac{\alpha(t)a_1 t^3}{3!} + \dots$$

$$= a_0 \left( 1 - \frac{\alpha(t)t^2}{2!} + \frac{\alpha^2(t)t^4}{4!} + \dots \right) + a_1 \left( t - \frac{\alpha(t)t^3}{3!} + \frac{\alpha^2(t)t^5}{5!} - \dots \right) \quad (22)$$

Since  $a_0$  and  $a_1$  are arbitrary constants, we have

$$Z_1(t) = G \left( 1 - \frac{\alpha(t)t^2}{2!} + \frac{\alpha^2(t)t^4}{4!} + \dots \right) + H \left( t - \frac{\alpha(t)t^3}{3!} + \frac{\alpha^2(t)t^5}{5!} - \dots \right) \quad (23)$$

Similarly when  $r = 1$ ,  $a_1 = 0$

$$k = 0, \quad a_2 = -\frac{\alpha(t)a_0}{2.3} = \frac{-\alpha(t)a_0}{3!}$$

$$k = 1, \quad a_3 = -\frac{\alpha(t)a_1}{3.4} = \frac{0}{3.4} = 0$$

$$k = 2, \quad a_4 = \frac{-\alpha(t)a_2}{4.5} = \frac{-\alpha(t)}{4.5} \left( \frac{-\alpha(t)a_0}{3!} \right) = \frac{\alpha^2(t)a_0}{5!}$$

$$k = 3, \quad a_5 = \frac{-\alpha(t)a_3}{6.7} = \frac{0}{6.7} = 0$$

Then

$$\begin{aligned} Z_2(t) &= a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + a_5 t^6 \\ &= a_0 t + \left( -\frac{\alpha(t)a_0}{3!} \right) t^3 + \frac{\alpha^2(t)a_0 t^5}{5!} + \dots \\ &= a_0 \left( t - \frac{\alpha(t)t^3}{3!} + \frac{\alpha^2(t)t^5}{5!} + \dots \right) \end{aligned} \quad (24)$$

Since  $a_0$  is an arbitrary constant, we have that

$$Z_2(t) = I \left( t - \frac{\alpha(t)t^3}{3!} + \frac{\alpha^2(t)t^5}{5!} + \dots \right) \quad (25)$$

Hence, the general solution is

$$\begin{aligned} Z(t) &= G \left( 1 - \frac{\alpha(t)t^2}{2!} + \frac{\alpha^2(t)t^4}{4!} + \dots \right) \\ &+ H \left( t - \frac{\alpha(t)t^3}{3!} + \frac{\alpha^2(t)t^5}{5!} + \dots \right) \\ Z(t) &= G + Ht - \frac{G\alpha(t)t^2}{2!} - \frac{H\alpha(t)t^3}{3!} \\ &+ \frac{G\alpha^2(t)t^4}{4!} + \frac{H\alpha^2(t)t^5}{5!} + \dots \end{aligned} \quad (26)$$

Since  $G\alpha(t) = H\alpha(t) = \alpha(t)$  then

$$\begin{aligned} \alpha^2(t) &= A + 2B \cos t + 2c \cos t \cos 2t \\ &+ 2D \cos t \cos 3t + 2E \cos t \cos 4t + 2F \cos t \cos 5t \\ &+ 2D \cos 2t \cos 3t + 2E \cos 2t \cos 4t + 2F \cos 2t \cos 5t \\ &+ 2E \cos 3t \cos 4t + 2F \cos 3t \cos 5t + 2E \cos 4t \cos 5t. \end{aligned} \quad (27)$$

Let

$$\begin{aligned} \beta &= A + 2B \cos t \\ &+ 2C \cos t \cos 2t \\ &+ 2D \cos t \cos 3t + 2E \cos t \cos 4t \\ &+ 2F \cos t \cos 5t + 2D \cos 2t \cos 3t \\ &+ 2E \cos 2t \cos 4t + 2F \cos 2t \cos 5t \\ &+ 2E \cos 3t \cos 4t + 2F \cos 3t \cos 5t \\ &+ 2E \cos 4t \cos 5t \end{aligned}$$

Then,

$$Z(t) = G + Ht - \frac{\alpha t^2}{2!} - \frac{\alpha t^3}{3!} + \frac{\beta}{4!} + \dots \quad (28)$$

$$Z(t) = G + Ht - \frac{\alpha t^2}{2!} - \frac{\alpha t^3}{3!} + \frac{\alpha^2 t^4}{4!} + \dots \quad (29)$$

Applying Simpson's Integration formula on  $[a, b] = [0, 1]$  with step size  $h = 0.25$

$$\int_a^b Z(t) dt = \frac{h}{3} [Z_0 + 4Z_1 + 2Z_2 + 4Z_3 + Z_4] \quad (30)$$

Let  $[0, 1]$  be the interval then,

$$t_0 = 0 \quad Z_0 = G$$

$$t_1 = 0.25Z_1 = G + 0.25H - 0.03385A + 0.00076276B$$

$$t_2 = 0.50Z_2 = G + 0.50H - 0.14583A + 0.0026B$$

$$t_3 = 0.75Z_3 = G + 0.75H - 0.35156A + 0.01318B$$

$$t_4 = 1.00Z_4 = G + H - 0.6667A + 0.04167B$$

$$\begin{aligned} \int_0^1 Z(t)dt &= \frac{0.25}{3} [G \\ &+ 4(G + 0.25H - 0.03385A \\ &+ 0.00076276B) \\ &+ 2(G + 0.50H - 0.14583A \\ &+ 0.0026B) \\ &+ 4(G + 0.75H - 0.35156A \\ &+ 0.01318B) + G + H - 0.6667A) \\ &+ 0.04167B = 0.083] [G + 4G + H \\ &- 0.1354A + 0.00305B + 2G + H \\ &- 0.29166A + 0.00536B + 4G + 3H \\ &- 1.40624A + 0.05272B + G + H \\ &- 0.6667A + 0.04167B] \end{aligned}$$

$$\begin{aligned} \int_0^1 Z(t)dt &= 0.996G + 0.498H - 0.2075A \\ &+ 0.0085324B. \end{aligned} \quad (31)$$

### 2.2.2. Stability Analysis

Consider

$$\ddot{Z} + \left( \begin{array}{c} A + B \cos t + C \cos 2t \\ +D \cos 3t + E \cos 4t + F \cos 5t \end{array} \right) Z = 0 \quad (32)$$

Let  $Z = Z_1$ ,  $\dot{Z}_1 = Z_2$  and  $\ddot{Z}_1 = \dot{Z}_2$

The first equivalent systems of (32) is

$$\dot{Z}_1 = Z_2 \quad (33)$$

$$\dot{Z}_2 = - \left( \begin{array}{c} A + B \cos t + C \cos 2t + D \cos 3t \\ +E \cos 4t + F \cos 5t \end{array} \right) Z_1 \quad (34)$$

(33) and (34) can be written as

$$\begin{aligned} \dot{Z}_1 &= Z_2 \\ \dot{Z}_2 &= -\alpha(t)Z_1 \end{aligned}$$

where  $\alpha(t) = A + B \cos t + C \cos 2t + D \cos 3t + E \cos 4t + F \cos 5t$ . In matrix form we have

$$\begin{pmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha(t) & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \quad (35)$$

(35) can be written as  $\dot{Z} = AZ$  where A is the matrix.

For the characteristics polynomial

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 0 - \lambda & 1 \\ -\alpha(t) & 0 - \lambda \end{vmatrix} = 0$$

$$= \begin{vmatrix} -\lambda & 1 \\ -\alpha(t) & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + \alpha(t) = 0 \Rightarrow \lambda^2 = -\alpha(t)$$

$$\Rightarrow \lambda = \pm \sqrt{-\alpha(t)} = \pm \alpha(t)i \quad (36)$$

Hence the general solution is

$$Z(t) = e^{\gamma z} (A \sin \alpha(t)z + B \cos \alpha(t)z) \quad (37)$$

which can be written as

$$Z(t) = A \sin \alpha(t)z + B \cos \alpha(t)z \quad (38)$$

Equation (38) shows that solution of Hill's equation is periodic with respect to the independent parameters.

### 2.2.3. Lyapunov Direct Method

Consider the equation

$$\ddot{Z} + \alpha(t)Z = 0 \quad (39)$$

At fixed point  $\dot{Z}_1 = \dot{Z}_2 = 0$

Using  $\dot{Z}_2 = -\alpha(t)Z_1$  we have  $Z_1 = 0$

Hence at fixed point we have (0,0) as the only equilibrium point of the system.

For the Lyapunov function, we multiply equation (39) by  $\dot{Z}$  which gives

$$\dot{Z}\ddot{Z} + Z\alpha(t)\dot{Z} = 0 \quad (40)$$

Integrating equation (40) we have

$$\int \frac{1}{2} \frac{d}{dt} \dot{Z}^2 dt + \int \dot{Z}\alpha(t)Z dt = \int 0 dt \quad (41)$$

which gives

$$\frac{1}{2} \dot{Z}^2 + \int \alpha(t)Z dt = c \quad (42)$$

The energy function  $H(Z, \dot{Z}) = \frac{1}{2} \dot{Z}^2 + \int \alpha(t)Z dt$

But  $Z = Z_1$ ,  $\dot{Z}_1 = Z_2$ ,  $\dot{Z}_2 = -\alpha(t)Z$

Hence the Lyapunov function is given by

$$V(Z_1, Z_2) = \frac{1}{2} \dot{Z}_2^2 - Z_2 \quad (43)$$

Applying definition (2.9) to equation (43) we have  $V(0,0) = 0$  and  $V(Z_1, Z_2) > 0$

Differentiating equation (43) we have

$$\dot{V}(Z_1, Z_2) = \frac{\partial V}{\partial Z_1} \cdot \frac{dZ_1}{dt} + \frac{\partial V}{\partial Z_2} \cdot \frac{dZ_2}{dt} \quad (44)$$

$$\begin{aligned} &= (\dot{Z}_2 - 1)(-\alpha(t)Z_1) \\ &= (\alpha(t))^2 Z_1^2 + \alpha(t)Z_1 > 0 \end{aligned} \quad (45)$$

Hence the equilibrium point is unstable.

### 2.2.4. Numerical Solution of Hill's Equation

$$\underline{A} := 1 \quad \underline{B} := 4 \quad \underline{C} := 5 \quad \underline{D} := 6 \quad \underline{E} := 7 \quad \underline{F} := 8$$

Define a function that determines a vector of derivative values at any solution point (t,Y):

$$D(t, X) := \begin{bmatrix} X_1 \\ \left( \begin{array}{c} A + B \cdot \cos(t) + C \cdot \cos(2t) \\ +D \cdot \cos(3t) + E \cdot \cos(4t) \\ +F \cdot \cos(5t) \end{array} \right) \cdot X_0 \end{bmatrix}$$

Define additional arguments for the ODE solver:

$t_0 := 0$  Initial value of independent variable

$t_1 := 150$  final value of independent variable

$X_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  Vector of initial function values

$N := 2000$  Number of solution values on  $[t_0, t_1]$

Solution matrix:

$$S := \text{Rkadapt}(x_0, t_0, t_1, N, D)$$

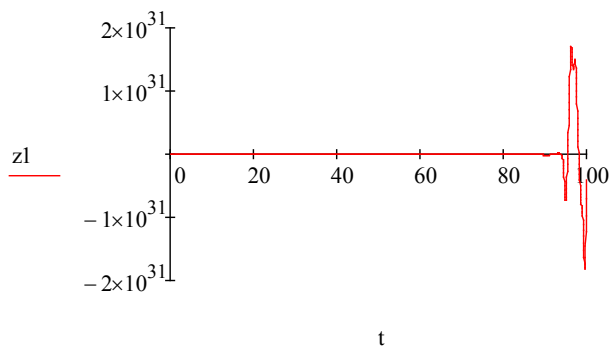
$t := S^{(0)}$  Independent variable values

$z1 := S^{(1)}$  First solution function values

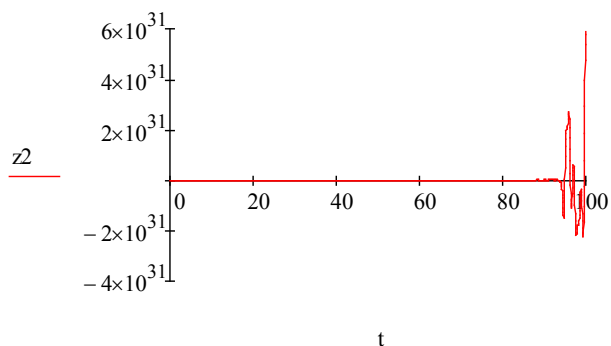
$z2 := S^{(2)}$  Second solution function values

**Table 1.** Table of values for the independent variables

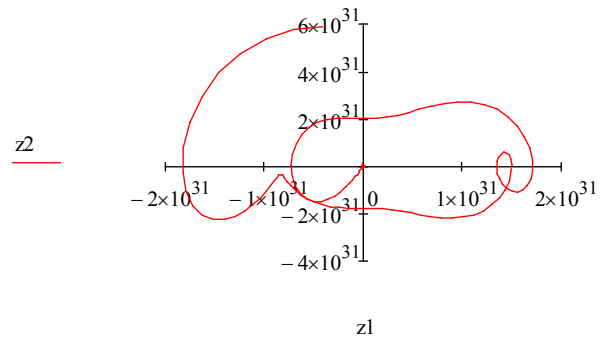
	0	1	2
0	0	0	1
1	0.075	0.077	1.087
2	0.15	0.167	1.343
3	0.225	0.282	1.747
4	0.3	0.432	2.243
5	0.375	0.618	2.723
6	0.45	0.836	3.024
7	0.525	1.063	2.968
8	0.6	1.269	2.43
9	0.675	1.415	1.409
10	0.75	1.472	0.058
11	0.825	1.422	-1.362
12	0.9	1.273	-2.583
13	0.975	1.044	-3.435
14	1.05	0.767	-3.89
15	1.125	0.469	...



**Figure 1.** The relation between first solution function values and independent variable values



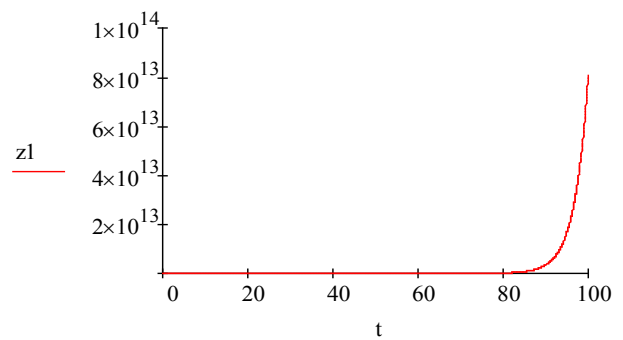
**Figure 2.** The relation between second solution function values and independent variable values



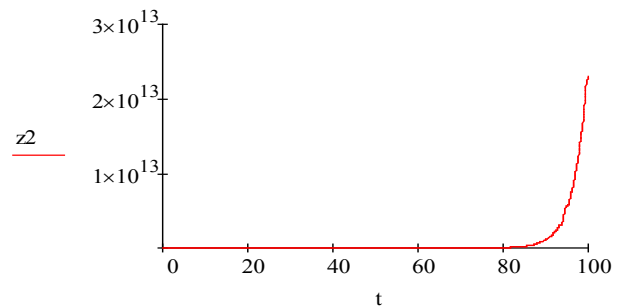
**Figure 3.** Phase portrait of Hill's equation showing instability of the solution as a spiral source

$$A := 0.1 \quad B := 0.01 \quad C := 0.02 \quad D := 0.03$$

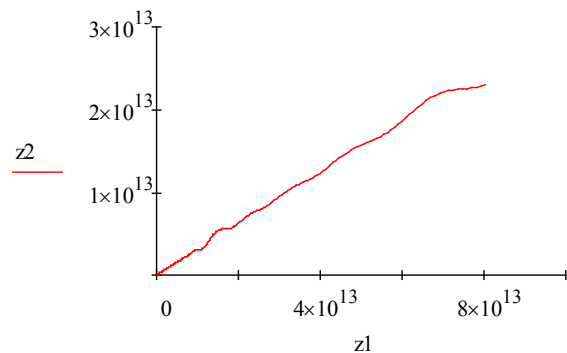
$$E := 0.04 \quad F := 0.05$$



**Figure 4.** The relation between the first solution function values and the independent variable values

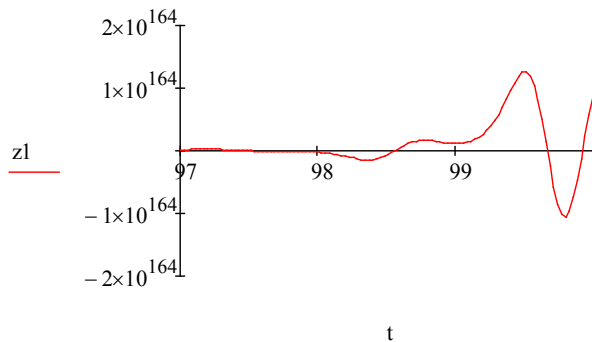


**Figure 5.** The relation between the second solution function values and the independent variable values

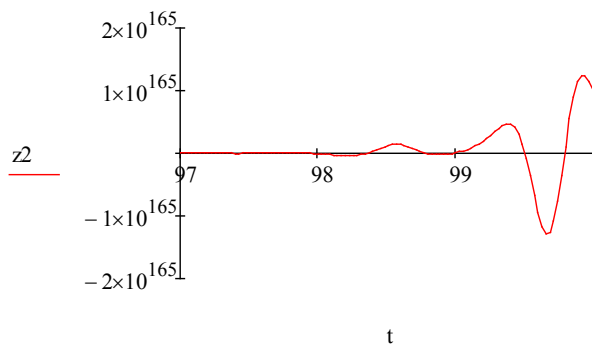


**Figure 6.** The relation between first solution function values and second solution function values showing instability of Hill's equation

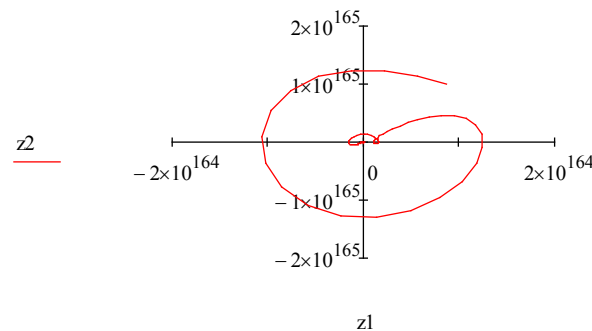
$A := 10$   $B := 50$   $C := 60$   $D := 70$   $E := 80$   
 $F := 100$



**Figure 7.** Trajectory profile of first solution function values and independent variable values



**Figure 8.** Trajectory profile of second solution function values and independent variable values



**Figure 9.** Phase portrait of Hill's equation of first solution function values and second solution function values showing instability around the origin

### 3. Conclusions

From our result, it was observed that regions of instability were basically around the equilibrium point and stable otherwise.

We also observed that this region of instability remained radially unbounded. Although, the total derivative existed, the region is unbounded, which implies that the existence of total derivative does not necessarily implied boundedness using Lyapunov methods.

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